Burgers Equation with Affine Linear Noise: Dynamics and Stability

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November 15, 2011

Abstract

We study the dynamics of Burgers equation on the unit interval driven by affine linear noise. Mild solutions of Burgers stochastic partial differential equation generate a smooth perfect and locally compacting cocycle on the energy space. Using multiplicative ergodic theory techniques, we establish the existence of a discrete non-random Lyapunov spectrum for the cocycle. We establish a *local stable manifold theorem* near a hyperbolic stationary point, as well as the existence of *local smooth invariant manifolds* with finite codimension and a countable *global invariant foliation* of the energy space relative to an ergodic stationary point.

AMS Subject Classification: Primary 60H15 Secondary 60F10, 35Q30.

1 Introduction

Our main interest in this article is to study the dynamics and characterize the almost sure asymptotic stability of the equilibrium/stationary point for the following one-dimensional Burgers equation with affine white noise:

$$\left.\begin{array}{l} du(t) = \nu\Delta u\,dt - u\dfrac{\partial u}{\partial \xi}\,dt + \gamma u(t)\,dt + \displaystyle\sum_{k=1}^{\infty}\sigma_k u(t)\,dW_k(t) + \sigma_0(\xi)\,dW_0(t), \quad t>0, \quad \xi\in[0,1], \\ u(t,0)=u(t,1)=0 \quad \text{for all } t>0, \\ u(0,\xi)=f(\xi),\ \xi\in[0,1]. \end{array}\right\} \tag{1.1}$$

In the above stochastic partial differential equation (spde), the noise coefficients σ_k, $k\geq 1$, are constants such that $\sum_{k=1}^{\infty}\sigma_k^2<\infty$; the W_k, $k\geq 0$, are independent standard Brownian

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motions defined on the complete Wiener space $(\Omega, \mathcal{F}, P)$; σ_0 is a smooth function on $[0,1]$; $\gamma u(t)\,dt$ is a deterministic linear drift term with a fixed parameter γ; the positive constant ν is the viscosity coefficient; and $f \in L^2\big([0,1], \mathbf{R}\big)$ is the initial function. Note that the external stochastic forcing in Burgers spde (1.1) is provided by the linear drift term $\gamma u(t)\,dt$, the linear white noise term $\sum_{k=1}^{\infty} \sigma_k u(t)\,dW_k(t)$ and the additive space-time noise term $\sigma_0(\xi)\,dW_0(t)$. The linear noise term may be replaced by a single term $u(t)\,dW(t)$ with $W := \sum_{k=1}^{\infty} \sigma_k W_k$ a Brownian motion independent of W_0. However this replacement does not lend any significant simplification to the computations in this article.

Burgers spde with noise has been studied extensively by many authors, mainly due to its significance in modelling turbulence in physics and engineering. The reader may refer to works by [2], [6], [9], [10], [11], [12], [19], [20] and the references therein.

The main objectives of this article are:

- To describe the stochastic dynamics of Burgers spde (1.1) via a perfect locally compacting smooth cocycle (semiflow) generated by mild solutions of the equation. The construction of the cocycle is described in Sections 2 and 3.
- To characterize the almost sure long-time asymptotics for the cocycle of (1.1) using the Lyapunov spectrum of its linearization along a stationary solution. The Lyapunov spectrum is countable and non-random (Section 4).
- To establish hyperbolicity near a general equilibrium under affine noise in (1.1) (Section 4).
- To establish (when $\gamma = 0$ in (1.1)) the existence of local flow-invariant submanifolds as well as a global invariant foliation through an ergodic equilibrium (Section 4).

2 The Dynamics-Linear Noise

Throughout this article, we will denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ the standard P-preserving ergodic Wiener shift:

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

It is well-known that a unique mild solution to Burgers spde with additive noise exists. See [6] and the references therein.

One of our main objectives in this article is to show that the random field of all mild solutions of (1.1) generates a Fréchet smooth perfect cocycle $U : \mathbf{R}^+ \times L^2([0,1], \mathbf{R}) \times \Omega \to L^2([0,1], \mathbf{R})$. Furthermore, our construction will show that the cocycle is locally compacting in the sense that the map $U(t, \cdot, \omega)$ carries bounded sets in $L^2([0,1], \mathbf{R})$ into relatively compact sets, for each $t > 0$ and almost all $\omega \in \Omega$. The construction also yields Oseledec-type integrability estimates on the cocycle and its Fréchet derivatives (Theorem 2.2).

For simplicity of exposition, we will only consider in this section the zero additive noise case in Burgers spde (1.1). So we will assume for the rest of this section that $\sigma_0(\xi) = 0$ for all $\xi \in [0,1]$; that is, we will consider the following Burgers spde

$$\left.\begin{aligned} &du(t) = \nu\Delta u\,dt - u\frac{\partial u}{\partial \xi}\,dt + \gamma u(t)\,dt + \sum_{k=1}^{\infty}\sigma_k u(t)\,dW_k(t), \quad t>0, \quad \xi\in[0,1],\\ &u(t,0) = u(t,1) = 0, \quad t>0,\\ &u(0,\xi) = f(\xi), \quad \xi\in[0,1]. \end{aligned}\right\} \tag{2.1}$$

A treatment of the general affine (non-zero additive noise) case is outlined in the next section.

Let H^1 denote the Sobolev space of order one, i.e. H^1 is the closure of $C_0^\infty([0,1],\mathbf{R})$ under the norm $\|f\|_{H^1} := \left(\int_0^1 |f'(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$. It is known (see e.g. [10], [11]) that for every initial function $f \in L^2([0,1],\mathbf{R})$, Burgers spde equation (2.1) admits a unique mild solution $u \in C([0,T], L^2([0,1],\mathbf{R}) \cap L^2([0,T],H^1)$ in the sense that

$$u(t,f) = T_t(f) - \int_0^t T_{t-s}\left[u(s,f)\frac{\partial u(s,f)}{\partial\xi}\right]ds + \gamma\int_0^t T_{t-s}(u(s,f))\,ds + \sum_{k=1}^{\infty}\int_0^t \sigma_k T_{t-s}(u(s,f))\,dW_k(s),$$

for all $t\in[0,T]$. In the above equation, $T_t : L^2\big([0,1],\mathbf{R}\big) \to L^2\big([0,1],\mathbf{R}\big)$, $t\geq 0$, is the heat semigroup generated by the Laplacian $\nu\Delta$ with Dirichlet boundary conditions on $[0,1]$: $T_t = e^{t\nu\Delta}$, $t \geq 0$.

For the remainder of the article we will adopt the following convention:

Definition 2.1 (Perfection). A family of propositions $\{P(\omega) : \omega \in \Omega\}$ is said to **hold perfectly in** ω if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$ and $P(\omega)$ is true **for every** $\omega \in \Omega^*$.

In order to study the dynamics of Burgers spde (2.1), our first task in this section is to show that the family of all mild solutions u of (2.1) have a perfect version with a C^∞ cocycle property on $L^2([0,1],\mathbf{R})$. We start with a reduction of Burgers spde (2.1) to a random pde of Burgers type. To do this, let $Q : [0,\infty)\times\Omega \to \mathbf{R}$ be the solution of the one-dimensional linear stochastic ordinary differential equation (sode)

$$\left.\begin{aligned} &dQ(t) = \gamma Q(t)\,dt + \sum_{k=1}^{\infty}\sigma_k Q(t)\,dW_k(t), \quad t\geq 0,\\ &Q(0) = 1. \end{aligned}\right\} \tag{2.2}$$

Using Itô's formula, it follows that

$$Q(t) = \exp\left\{\sum_{k=1}^{\infty}\sigma_k W_k(t) - \frac{t}{2}\sum_{k=1}^{\infty}\sigma_k^2 + \gamma t\right\}, \quad t\geq 0. \tag{2.3}$$

Furthermore, (2.3) implies that

$$E\|Q\|_\infty < \infty,$$

where

$$\|Q\|_\infty \equiv \|Q(\cdot,\omega)\|_\infty := \sup_{0\le t\le T} Q(t,\omega), \quad \omega\in\Omega,$$

for any finite positive T.

We (formally) write each mild solution u of Burgers spde (2.1) in the form

$$u(t,\xi) = V(t,\xi)Q(t), \quad t\ge 0,\ \xi\in[0,1], \tag{2.4}$$

with $V(t,\xi)$ a suitably chosen random field of bounded variation in t. Therefore, by Itô's formula (the product rule), we have

$$du(t) = Q(t)\,dV(t) + V(t)\,dQ(t), \quad t>0. \tag{2.5}$$

Hence, substituting from (2.4) into (2.1) gives the following equalities for $t>0$:

$$\begin{aligned}
&\nu\Delta u\,dt - u\frac{\partial u}{\partial \xi}\,dt + \gamma u\,dt + \sum_{k=1}^{\infty}\sigma_k u(t)\,dW_k(t)\\
&\qquad\qquad = dV(t)\cdot Q(t) + \gamma Q(t)V(t)\,dt + V(t)\sum_{k=1}^{\infty}\sigma_k Q(t)\,dW_k(t),\\
&\left[\nu\Delta\big(V(t)Q(t)\big) - V(t)Q(t)\frac{\partial}{\partial\xi}\big(V(t)Q(t)\big)\right]dt = dV(t)\cdot Q(t),\\
&\nu Q(t)\Delta V(t)\,dt - V(t)Q(t)^2\frac{\partial}{\partial\xi}V(t)\,dt = dV(t)\cdot Q(t).
\end{aligned}$$

The above heuristic argument suggests that V solves the following random Burgers-type pde:

$$\left.\begin{aligned}
\frac{\partial V}{\partial t} &= \nu\Delta V(t) - Q(t)V(t)\frac{\partial V(t)}{\partial \xi}, \quad t>0,\\
V(0,\xi) &= u(0,\xi) = f(\xi), \quad \xi\in[0,1],\\
V(t,0) &= Q(t)^{-1}u(t,0) = 0, \quad t>0,\\
V(t,1) &= Q(t)^{-1}u(t,1) = 0, \quad t>0.
\end{aligned}\right] \tag{2.6}$$

Now let $\phi : \mathbf{R}^+\times L^2\big([0,1],\mathbf{R}\big)\times\Omega \to L^2\big([0,1],\mathbf{R}\big)$ be the perfect linear cocycle generated by the linear spde

$$\left.\begin{aligned}
d\phi(t) &= \nu\Delta\phi(t)\,dt + \gamma\phi(t)\,dt + \sum_{k=1}^{\infty}\sigma_k\phi(t)\,dW_k(t)\\
\phi(0) &= id_{L^2\left([0,1],\mathbf{R}\right)},\ \phi(t)(0) = \phi(t)(1) = 0, \quad \forall\ t>0.
\end{aligned}\right\} \tag{2.7}$$

([15, Theorem 1.2.4]).

Now assume that V is a mild solution of (2.6); that is

$$V(t) = T_t(f) - \int_0^t Q(s) T_{t-s}\left[V(s)\frac{\partial V(s)}{\partial \xi}\right] ds, \quad t \geq 0.$$

Define u by (2.4). Then it is easy to check that u is an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution of the random integral equation

$$u(t,\omega) = \phi(t,f,\omega) - \int_0^t \phi\big(t-s,\cdot,\theta(s,\omega)\big)\left[u(s,\omega)\frac{\partial}{\partial \xi}u(s,\omega)\right] ds, \quad t \geq 0. \tag{2.8}$$

The above relation implies that u is a mild solution of Burgers spde (2.1). The proof of the latter statement follows that of Theorem 1.2.5 ([15]): Interchange Itô and Lebesgue integrals, using the identity

$$\phi(t,f,\cdot) = T_t(f) + \gamma \int_0^t T_{t-s}\phi(s,f,\cdot)\, ds + \sum_{k=1}^{\infty} \int_0^t \sigma_k\, T_{t-s}\, \phi(s,f,\cdot)\, dW_k(s), \quad t \geq 0,\ \omega \in \Omega. \tag{2.9}$$

It is known (via a contraction mapping argument) that, for each $f \in L^2([0,1],\mathbf{R})$, a unique mild solution $U(t,f,\omega)$ of (2.6) exists (cf. [6], p. 262). However, looking beyond existence and uniqueness of the mild solution of (2.1), we need to further establish existence of the cocycle, its Fréchet smoothness in the initial function $f \in L^2([0,1],\mathbf{R})$ and Oseledec integrability estimates on its Fréchet derivatives. To achieve this, we will re-examine the contraction mapping argument being parametrized by the initial function f.

The next proposition gives a priori bounds on solutions of the initial boundary-value problem (2.6). These a priori bounds are needed for the construction of the cocycle (U,θ) for Burgers spde (2.1).

Proposition 2.1. For $f \in L^2\big([0,1],\mathbf{R}\big)$, let $V(t,f,\omega)$ be a mild solution of the initial boundary value problem (2.6) for $0 < t < T$ and some $T > 0$. Then for each $\omega \in \Omega$, the map $[0,T) \ni t \longmapsto \|V(t,f,\omega)\|_{L^2} \in \mathbf{R}$ is decreasing. In particular,

$$\|V(t,f,\omega)\|_{L^2([0,1],\mathbf{R})} \leq \|f\|_{L^2([0,1],\mathbf{R})} \tag{2.10}$$

for all $t \in [0,T)$, and all $\omega \in \Omega$. Also

$$\int_0^T \left\|\frac{\partial V(t,f,\omega)}{\partial \xi}\right\|^2_{L^2([0,1],\mathbf{R})} dt \leq \frac{1}{2\nu}\|f\|^2_{L^2([0,1],\mathbf{R})} \tag{2.11}$$

for all $\omega \in \Omega$.

Proof. Since the apriori bounds depend only on $\|f\|^2_{L^2([0,1],\mathbf{R})}$, it is sufficient to assume that $f \in C_0^\infty\big([0,1],\mathbf{R}\big)$ in (2.6) and $V(t) \equiv V(t,f,\omega)$, $0 < t < T$, is the classical solution of (2.6). We fix and suppress $\omega \in \Omega$ throughout. Multiply both sides of (2.6) by $V(t)$ to get

$$\frac{\partial}{\partial t}V(t)\cdot V(t) = \nu\, V(t)\frac{\partial^2 V(t)}{\partial \xi^2} - Q(t)\, V(t)^2 \frac{\partial V(t)}{\partial \xi}, \quad 0 < t < T. \tag{2.12}$$

Integrate both sides of (2.12) with respect to $\xi \in [0,1]$ to obtain

$$\frac{1}{2}\int_0^1 \frac{\partial V(t)^2}{\partial t} d\xi = \nu \int_0^t V(t)\frac{\partial^2 V(t)}{\partial \xi^2} d\xi - Q(t)\int_0^1 V(t)^2 \frac{\partial V(t)}{\partial \xi} d\xi\,, \quad 0 < t < T, \tag{2.13}$$

because $Q(t)$ is independent of $\xi \in [0,1]$. Using integration by parts and the boundary conditions $V(t)\,|_{\xi=1} = V(t)\,|_{\xi=0} = 0$, we obtain from (2.13):

$$\frac{d}{dt}\|V(t)\|_{L^2}^2 = -2\nu \left\|\frac{\partial V(t)}{\partial \xi}\right\|_{L^2}^2 \leq 0, \quad \text{for all } t \in [0,T). \tag{2.14}$$

Hence the function $[0,T) \ni t \longmapsto \|V(t)\|_{L^2}^2 \in \mathbf{R}$ is non-increasing; i.e.,

$$\|V(t)\|_{L^2}^2 \leq \|V(0)\|_{L^2}^2 = \|f\|_{L^2}^2 \quad \text{for all } t \in [0,T). \tag{2.15}$$

This proves (2.10).

To see (2.11), integrate both sides of (2.14) over $[0,T]$:

$$\|V(T)\|_{L^2}^2 - \|f\|_{L^2}^2 = -2\nu \int_0^T \left\|\frac{\partial V(t)}{\partial \xi}\right\|_{L^2}^2 dt\,.$$

Hence,

$$\begin{aligned}\int_0^T \left\|\frac{\partial V(t)}{\partial \xi}\right\|_{L^2}^2 dt &= \frac{1}{2\nu}\|f\|_{L^2}^2 - \frac{1}{2\nu}\|V(T)\|_{L^2}^2 \\ &\leq \frac{1}{2\nu}\|f\|_{L^2}^2\end{aligned}$$

and (2.11) holds. □

Next, we examine local existence of a unique mild solution of (2.6) and its (Lipschitz) dependence on the initial function f. To do this, we rewrite (2.6) in the mild form

$$V(t) = T_t(f) - \int_0^t Q(s)T_{t-s}\left[V(s)\frac{\partial V(s)}{\partial \xi}\right] ds, \quad t \geq 0 \tag{2.16}$$

and the equivalent integral form:

$$V(t) = T_t(f) + \frac{1}{2}\int_0^t \int_0^1 Q(s)\frac{\partial}{\partial y} p(t-s,\cdot,y)V^2(s)(y)\,dy\,ds, \quad 0 \leq t \leq a. \tag{2.17}$$

In the above expression $p(t,\xi,y)$ denotes the heat kernel for the heat equation

$$\begin{aligned}\frac{\partial u_0}{\partial t} &= \Delta u_0(t), \quad t > 0, \\ u_0(0,\cdot) &= f \in L^2\big([0,1],\mathbf{R}\big), \\ u_0(t,0) &= u_0(t,1) = 0, \quad \text{for all } t \geq 0,\end{aligned}$$

with Dirichlet boundary conditions. Thus,

$$u_0(t,\xi) = \int_0^1 p(t,\xi,y)f(y)\,dy, \quad t > 0,\ \xi \in [0,1]. \tag{2.18}$$

We use a *uniform* contraction mapping argument.

Proposition 2.2 (Local existence and Lipschitz dependence). Let $f \in L^2\big([0,1],\mathbf{R}\big)$. Then, for some $a > 0$, the random integral equation (2.16) has a unique (local) solution $V(f) \in C\big([0,a], L^2\big([0,1],\mathbf{R}\big)\big)$. Furthermore, $V(f)$ is locally Lispchitz in $f \in L^2\big([0,1],\mathbf{R}\big)$.

Proof. Fix $f_0 \in L^2\big([0,1],\mathbf{R}\big)$. Let $a > 0$. Denote by $E := C\big([0,a], L^2\big([0,1],\ \mathbf{R}\big)$, the Banach space of all continuous maps $v : [0,a] \to L^2\big([0,1],\mathbf{R}\big)$ with the usual norm

$$\|v\|_E := \sup_{0\le t\le a} \|v(t)\|_{L^2}\,. \tag{2.19}$$

Fix $\rho > 0$. Denote by $B(f,\rho)$ the closed ball in $L^2\big([0,1],\mathbf{R}\big)$, center f and radius ρ. Let $Y \subset E$ denote the set

$$Y := \big\{v \in E : \|v(t) - f_0\|_{L^2} \le \rho \quad \text{for all } t \in [0,a]\big\}. \tag{2.20}$$

Define the mapping $\psi : B(f_0,\rho_0) \times Y \to E$ by

$$\psi(f,v)(t) := T_t(f) + \frac{1}{2}\int_0^t \int_0^1 Q(s)\frac{\partial}{\partial y}p(t-s,\cdot,y)v^2(s)(y)\,dy\,ds, \quad 0 \le t \le a, \tag{2.21}$$

for all $v \in Y$, $f \in B(f_0,\rho_0)$.

Let $f \in B(f_0,\rho_0)$ and $v \in Y$. Then, for all $s \in [0,a]$,

$$\|v(s)\|_{L^2} \le \|v(s) - f_0\|_{L^2} + \|f_0\|_{L^2} \le \rho + \|f_0\|_{L^2}\ . \tag{2.22}$$

In the computations below, C denotes positive deterministic constants which may change from line to line.

The following estimates on the heat kernel $p(t,\xi,y)$ are well-known:

$$\left|\frac{\partial p(t,\xi,y)}{\partial y}\right| \le \frac{c_1}{t}e^{\frac{-(\xi-y)^2}{2c_2 t}}, \quad t>0,\ \xi,y \in [0,1], \tag{2.23}$$

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2C_2 t}}\,dy \le c_3\sqrt{t}, \quad t > 0. \tag{2.24}$$

where c_1, c_2, c_3 are positive constants.

Using (2.19), (2.20), the estimates (2.22), (2.23) and (2.24), we have

$$\begin{aligned}
\|\psi(f,v)(t) - f_0\|^2_{L^2} &\le 2\|T_t(f) - f_0\|^2_{L^2} + \\
&\quad + \frac{1}{2}\|Q\|^2_\infty \cdot \int_0^1 \left(\int_0^t\int_0^1 \left|\frac{\partial}{\partial y}p(t-s,\xi,y)\right| v^2(s)(y)\,dy\,ds\right)^2 d\xi \\
&\le 2\|T_t(f) - f_0\|^2_{L^2} + C_1\|Q\|^2_\infty \cdot \int_0^1 \Bigg[\int_0^t \frac{1}{(t-s)^{3/8}} \times \\
&\qquad \times \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} v^2(s)(y)\,dy \frac{1}{(t-s)^{1/8}}\,ds\Bigg]^2 d\xi \\
&\le 2\|T_t(f) - f_0\|^2_{L^2} +
\end{aligned}$$

$$
\begin{aligned}
&\quad + C_1\|Q\|_\infty^2 \int_0^1 \Big\{ \int_0^t \frac{ds}{(t-s)^{3/4}} \int_0^t \frac{1}{(t-s)^{1/4}} \times \\
&\qquad\qquad \times \Big(\int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} v^2(s)(y)\, dy \Big)^2 ds \Big\}\, d\xi \\
&\le 2\|T_t(f)-f_0\|_{L^2}^2 + \\
&\quad + C\|Q\|_\infty^2 \cdot t^{1/4} \int_0^1 \int_0^t \frac{1}{(t-s)^{3/4}} \int_0^1 e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} \cdot v^2(s)(y)\, dy \\
&\qquad\qquad \times \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} v^2(s)(y)\, dy\, ds\, d\xi \\
&\le 2\|T_t(f)-f_0\|_{L^2}^2 + \\
&\quad + C\|Q\|_\infty^2 \cdot t^{1/4} \int_0^1 \int_0^t \frac{1}{(t-s)^{3/4}} \Big(\int_0^1 v^2(s)(y)\, dy \Big) \times \\
&\qquad\qquad \times \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} v^2(s)(y)\, dy\, ds\, d\xi \\
&\le 2\|T_t(f)-f_0\|_{L^2}^2 + \\
&\quad + C\|Q\|_\infty^2 \cdot t^{1/4} \sup_{0\le s\le a} \|v(s)\|_{L^2}^2 \int_0^t \frac{1}{(t-s)^{3/4}} \times \\
&\qquad\qquad \times \int_0^1 \Big(\int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} d\xi \Big) \times v^2(s)(y)\, dy\, ds \\
&\le 2\|T_t(f)-f_0\|_{L^2}^2 + \\
&\quad + C\|Q\|_\infty^2 \cdot t^{1/4} \sup_{0\le s\le a} \|v(s)\|_{L^2}^2 \int_0^t \frac{1}{(t-s)^{3/4}} \|v(s)\|_{L^2}^2\, ds \\
&\le 2\|T_t(f)-f_0\|_{L^2}^2 + C\|Q\|_\infty^2 \sup_{0\le s\le a} \|v(s)\|_{L^2}^4 \cdot t^{1/2} \\
&\le 2\|T_t(f)-f_0\|_{L^2}^2 + C\|Q\|_\infty^2 (\rho + \|f_0\|_{L^2})_{L^2}^4 t^{1/2}, \quad 0 \le t \le a,
\end{aligned} \tag{2.25}
$$

where $\|Q\|_\infty := \sup_{0\le t\le a} Q(t)$.

By the strong continuity of the (bounded linear) heat semigroup $T_t : L^2\big([0,1],\mathbf{R}\big) \to L^2\big([0,1],\mathbf{R}\big)$, $t \ge 0$, choose $a \in (0,1)$ and $\rho_0 > 0$ sufficiently small such that

$$
\|T_t(f)-f_0\|_{L^2}^2 < \frac{\rho^2}{4} \qquad \text{and} \qquad C\|Q\|_\infty^2(\rho + \|f_0\|)_{L^2}^4 t^{1/2} < \frac{\rho^2}{2} \tag{2.26}
$$

for all $0 \le t \le a$, and all $f \in B(f_0, \rho_0)$.

Using (2.25) and (2.26), we get

$$
\|\psi(f,v)(t) - f_0\|_{L^2}^2 < \frac{\rho^2}{2} + \frac{\rho^2}{2} = \rho^2, \quad 0 \le t \le a.
$$

Thus

$$
\|\psi(f,v)(t) - f_0\|_{L^2}^2 < \rho
$$

for all $t \in [0,a]$, all $v \in Y$ and all $f \in B(f_0, \rho_0)$. Hence $\psi(f,v) \in Y$ for all $f \in B(f_0,\rho_0)$ and $v \in Y$.

We must show that a and ρ_0 can be chosen sufficiently small so that

$$\begin{aligned} \psi : B(f_0,\rho_0) \times Y &\longrightarrow Y \\ (f,v) &\longmapsto \psi(f,v) \end{aligned}$$

is a uniform contraction on Y. Let $v_1, v_2 \in Y$ and use (2.21) to get

$$\begin{aligned}
\|\psi(f,v_1)(t) - \psi(f,v_2)(t)\|_{L^2}^2 \leq \|Q\|_\infty^2 \int_0^1 \Big(\int_0^t \int_0^1 \Big| \frac{\partial}{\partial y} p(t-s,\xi,y) \Big| \times \\
\times \left| v_1^2(s)(y) - v_2^2(s)(y) \right| dy\, ds \Big)^2 d\xi \\
\leq \|Q\|_\infty^2 \int_0^1 \Big\{ \int_0^t \frac{ds}{(t-s)^{3/4}} \int_0^t \frac{1}{(t-s)^{1/4}} \times \\
\times \Big(\int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} \left| v_1^2(s)(y) - v_2^2(s)(y) \right| dy \Big)^2 ds \Big\} d\xi \\
\leq C\|Q\|_\infty^2 t^{1/4} \int_0^t \frac{1}{(t-s)^{3/4}} \Big\{ \int_0^1 \left[v_1(s)(y) + v_2(s)(y) \right]^2 dy \Big\} \\
\times \Big\{ \int_0^1 \left| v_1(s)(y) - v_2(s)(y) \right|^2 dy \Big\} ds \\
\leq C\|Q\|_\infty^2 \big(\rho + \|f_0\|_{L^2} \big)^2 t^{1/4} \int_0^t \frac{1}{(t-s)^{3/4}} \|v_1(s) - v_2(s)\|_{L^2}^2 \, ds \\
\leq C\|Q\|_\infty^2 \big(\rho + \|f_0\|_{L^2} \big)^2 t^{1/2} \|v_1 - v_2\|_E^2, \quad 0 \leq t \leq a. \qquad (2.27)
\end{aligned}$$

Now choose $a > 0$ sufficiently small such that

$$L := C\|Q\|_\infty \big(\rho + \|f_0\|_{L^2} \big) a^{1/4} < 1. \qquad (2.28)$$

Therefore by (2.27) and (2.28), we get

$$\|\psi(f,v_1) - \psi(f,v_2)\|_E \leq L \|v_1 - v_2\|_E \qquad (2.29)$$

for all $v_1, v_2 \in Y$, all $f \in B(f_0,\rho_0)$, where $L < 1$. Hence for each $f \in B(f_0,\rho_0)$, $\psi(f,\cdot) : Y \to Y$ is a uniform contraction on Y. By the contraction mapping theorem, $\psi(f,\cdot)$ has a unique fixed point $V(f) \in Y$; i.e.,

$$\psi\big(f, V(f)\big)(t) = V(f)(t), \quad 0 \leq t \leq a. \qquad (2.30)$$

Thus $V(f)$ is the unique local mild solution of the random Burgers pde (2.6), viz.

$$V(f)(t) = T_t(f) - \int_0^t Q(s) T_{t-s} \Big[V(f)(s) \frac{\partial V(f)(s)}{\partial \xi} \Big] ds, \quad 0 \leq t \leq a$$

for all $f \in B(f_0, \rho_0)$. Note that in (2.30), a is independent of the choice of the initial condition $f \in B(f_0, \rho_0)$ (although a is random and may still depend on the choice of $f_0 \in L^2\big([0,1], \mathbf{R}\big)$). Furthermore, the solution map $B(f_0, \rho_0) \ni f \mapsto V(f) \in C\big([0,a], L^2\big([0,1], \mathbf{R}\big)\big)$ of (2.6) is Lipschitz. This follows from the uniform contraction principle (proof of Proposition 2.3 below) and the fact that

$$\|\psi(f_1, v) - \psi(f_2, v)\|_E \leq \sup_{0 \leq t \leq a} \|T_t\|_{L(L^2)} \|f_1 - f_2\|_{L^2},$$

for all $f_1, f_2 \in B(f_0, \rho_0)$, $v \in Y$. □

The following proposition gives regularity of the local mild solution map

$$L^2\big([0,1], \mathbf{R}\big) \supset B(f_0, \rho_0) \ni f \longmapsto V(f) \in C\big([0,a], L^2\big([0,1], \mathbf{R}\big)\big)$$

of (2.6).

Proposition 2.3 (Uniform Contraction Principle). Let E, F be real Banach spaces. Suppose $B \subset F$ is an open set and $Y \subset E$ a closed ball in E. Let $\psi : B \times Y \to Y$ be a C^k map with bounded Fréchet derivatives on bounded subsets of $B \times Y$. Assume that $\psi(f, \cdot) : Y \to Y$, $f \in B$, is a uniform contraction; i.e., there exists $L \in (0,1)$ such that

$$\|\psi(f, v_1) - \psi(f, v_2)\|_E \leq L \|v_1 - v_2\|_E \tag{2.31}$$

for all $v_1, v_2 \in Y$ and all $f \in B$. Then for each $f \in B$, there is a unique $v(f) \in Y$ such that $\psi\big(f, v(f)\big) = v(f)$. Moreover, the map $B \ni f \longmapsto v(f) \in Y \subset E$ is C^k with bounded Fréchet derivatives on bounded subsets of B.

Proof. See the proof of Lemma 2.7 in ([14]). □

Theorem 2.1 (Global existence). *For each $f \in L^2\big([0,1], \mathbf{R}\big)$, Burgers spde (2.1) has a unique pathwise solution $U(f, \omega) \in C\big([0,\infty), L^2\big([0,1], \mathbf{R}\big)\big)$ such that the map*

$$L^2\big([0,1], \mathbf{R}\big) \ni f \longmapsto U(f, \omega)(t) \in L^2\big([0,1], \mathbf{R}\big)$$

is C^∞ for a.a. $\omega \in \Omega$ and all $t \geq 0$, and has bounded Fréchet derivatives on bounded sets in $L^2\big([0,1], \mathbf{R}\big)$.

Proof. As indicated previously, it is sufficient to prove the theorem for mild solutions of the random Burgers equation (2.6). Fix and suppress $\omega \in \Omega$. Also fix $f_0 \in L^2\big([0,1], \mathbf{R}\big)$. By Proposition 2.2, there exists $\rho_0, a > 0$ such that if $f \in B(f_0, \rho_0)$ and

$$\psi(f, v)(t) := T_t(f) + \frac{1}{2} \int_0^t \int_0^1 Q(s) \frac{\partial}{\partial y} p(t-s, \cdot, y) v^2(s)(y)\, dy\, ds, \quad 0 \leq t \leq a, \tag{2.32}$$

then $\psi(f, \cdot)$ has a fixed point $V(f) \in Y \subset E := C\big([0,a], L^2\big([0,1], \mathbf{R}\big)\big)$ which gives a unique local mild solution $V(f)$ of (2.6).

We will next show that the solution map

$$L^2\big([0,1],\mathbf{R}\big) \supset B(f_0,\rho_0) \ni f \longmapsto V(f) \in C\big([0,a], L^2\big([0,1],\mathbf{R}\big)\big)$$

of (2.6) is C^∞ (Fréchet) with all derivatives bounded. In view of Proposition 2.3, it is sufficient to prove that the map $\psi : B(f_0,\rho_0) \times Y \to E$ in (2.32) is C^k with bounded derivatives for all $k \geq 1$; recall that

$$Y := \big\{ v \in C\big([0,a], L^2\big([0,1],\mathbf{R}\big)\big) : \|v(t) - f_0\|_{L^2} \leq \rho \quad \forall\, t \in [0,a] \big\}\,. \tag{2.33}$$

Note first that the map

$$L^2\big([0,1],\mathbf{R}\big) \ni f \longmapsto T_{(\cdot)} f \in C\big([0,a], L^2\big([0,1],\mathbf{R}\big)\big)$$

is continuous linear (and hence C^∞). So it remains to show that the map $\phi : Y \to C\big([0,a], L^2\big([0,1],\mathbf{R}\big)\big)$, where

$$\phi(v)(t) := \frac{1}{2} \int_0^t \int_0^1 Q(s) \frac{\partial}{\partial y} p(t-s,\cdot,y) v^2(s)(y)\, dy\, ds \quad 0 \leq t \leq a,\ v \in Y, \tag{2.34}$$

is C^∞ with all derivatives bounded. To do this, consider the map

$$A : E \times E \longrightarrow E$$

defined by

$$A(v_1,v_2)(t) := \int_0^t \int_0^1 Q(s) \frac{\partial}{\partial y} p(t-s,\cdot,y) v_1(s)(y) v_2(s)(y)\, dy\, ds \tag{2.35}$$

for $0 \leq t \leq a$, $v_1, v_2 \in E$. Clearly,

$$\phi(v) = \frac{1}{2} A(v,v), \quad v \in Y. \tag{2.36}$$

We will show that A is continuous bilinear. By (2.36), this implies that ϕ is C^∞ with all derivatives bounded.

Using (2.35), we obtain

$$
\begin{aligned}
\|Av_1, v_2)(t)\|_{L^2}^2 &\le C\|Q\|_\infty^2 \int_0^1 \left(\int_0^t \frac{1}{\sqrt{t-s}} \int_0^1 e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} |v_1(s)(y)||v_2(s)(y)|\,dy\,ds\right)^2 d\xi \\
&\le C\|Q\|_\infty^2 \int_0^1 \left[\int_0^t \frac{1}{(t-s)^{3/8}} \int_0^1 \frac{e^{\frac{-(\xi-y)^2}{2c_2(t-s)}}}{\sqrt{t-s}} |v_1(s)(y)||v_2(s)(y)|\,dy \frac{1}{(t-s)^{1/8}}\,ds\right]^2 d\xi \\
&\le C\|Q\|_\infty^2 \int_0^1 \Bigg\{\int_0^t \frac{ds}{(t-s)^{3/4}} \int_0^t \frac{1}{(t-s)^{1/4}} \Bigg(\int_0^1 \frac{e^{\frac{-(\xi-y)^2}{2c_2(t-s)}}}{\sqrt{t-s}} \times \\
&\qquad\qquad \times |v_1(s)(y)||v_2(s)(y)|\,dy\Bigg)^2 ds\Bigg\}\,d\xi \\
&= C\|Q\|_\infty^2\, t^{1/4} \int_0^t \frac{1}{(t-s)^{3/4}} \left|\int_0^1 |v_1(s)(y)||v_2(s)(y)|\,dy\right| \times \\
&\qquad \times \left|\int_0^1 \int_0^1 \frac{e^{\frac{-(\xi-y)^2}{2c_2(t-s)}}}{\sqrt{t-s}}\,d\xi \cdot |v_1(s)(y)||v_2(s)(y)|\,dy\right| ds \\
&\le C\|Q\|_\infty^2\, t^{1/4} \int_0^t \frac{\|v_1(s)\|_{L^2}^2 \|v_2(s)\|_{L^2}^2}{(t-s)^{3/4}}\,ds \\
&\le C\|Q\|_\infty^2\, t^{1/2} \left(\sup_{0\le t\le a} \|v_1(s)\|_{L^2}\right)^2 \left(\sup_{0\le t\le a} \|v_2(s)\|_{L^2}\right)^2 \\
&\le C\|Q\|_\infty^2\, a^{1/2} \|v_1\|_E^2 \cdot \|v_2\|_E^2
\end{aligned}
$$

for all $t \in [0,a]$ and $v_1, v_2 \in E$. Therefore,

$$
\|A(v_1, v_2)\|_E = \sup_{0\le t\le a} \|A(v_1,v_2)(t)\|_{L^2} \le C\|Q\|_\infty\, a^{1/4} \|v_1\|_E \cdot \|v_2\|_E \tag{2.37}
$$

for all $v_1, v_2 \in E$. Hence A is continuous bilinear, ϕ and ψ are C^∞ maps with all derivatives bounded.

By the uniform contraction principle (Proposition 2.3), it follows that the mild solution map

$$
L^2\big([0,1],\mathbf{R}\big) \supset B(f_0,\rho_0) \ni f \longmapsto V(f) \in C\big([0,a], L^2\big([0,1],\mathbf{R}\big)\big) \tag{2.38}
$$

for (2.6) is C^∞ for some $a > 0$, and has all derivatives bounded.

We now prove existence of a global semiflow for mild solutions of (2.6). Let $\tau = \tau(\omega) > 0$ denote the supremum of all $a > 0$ such that a C^∞ solution map (2.38) for (2.6) exists on $[0,a]$ (for fixed f_0, ρ_0). We will show that $\tau = \infty$ a.s.. Suppose, if possible, that $\tau = \tau(\omega) < \infty$ for some $\omega \in \Omega$. We claim that

$$
V(f)(\tau) = \lim_{t\to\tau-} V(f)(t) = T_\tau(f) + \frac{1}{2}\int_0^\tau \int_0^1 Q(s)\frac{\partial}{\partial y} p(\tau-s,\cdot,y) V(f)^2(s)(y)\,dy\,ds \tag{2.39}
$$

for all $f \in B(f_0, \rho_0)$, where the limit is taken in $L^2\big([0,1], \mathbf{R}\big)$. Since $\lim\limits_{t\to\tau-} T_t(f) = T_\tau(f)$ by strong continuity of the heat semigroup $T_t : L^2\big([0,1], \mathbf{R}\big) \to L^2\big([0,1], \mathbf{R}\big)$, (2.39) will follow if we show that

$$\begin{aligned} \lim_{t\to\tau-} \bigg\| \int_0^\tau Q(s) \int_0^1 \frac{\partial}{\partial y} p(\tau - s, \cdot, y) V(f)^2(s)(y)\, dy\, ds \\ - \int_0^t Q(s) \int_0^1 \frac{\partial}{\partial y} p(t - s, \cdot, y) V(f)^2(s)(y)\, dy\, ds \bigg\|_{L^2} = 0. \end{aligned} \tag{2.40}$$

Denote

$$B(t)(\xi) := \int_0^t Q(s) \int_0^1 \frac{\partial}{\partial y} p(t-s, \xi, y) V(f)^2(s)(y)\, dy\, ds, \;\; 0 \le t < \tau, \;\; \xi \in [0,1]. \tag{2.41}$$

Consider

$$\begin{aligned} B(\tau)(\xi) - B(t)(\xi) = \int_t^\tau Q(s) \int_0^1 \frac{\partial}{\partial y} p(\tau - s, \xi, y) V(f)^2(s)(y)\, dy\, ds + \\ + \int_0^t Q(s) \int_0^1 \left[\frac{\partial}{\partial y} p(\tau - s, \xi, y) - \frac{\partial}{\partial y} p(t - s, \xi, y) \right] V(f)^2(s)(y)\, dy\, ds \\ = B_1(t)(\xi) + B_2(t)(\xi), \quad \xi \in [0,1], \;\; 0 \le t < \tau, \end{aligned} \tag{2.42}$$

where

$$B_1(t)(\xi) := \int_t^\tau Q(s) \int_0^1 \frac{\partial}{\partial y} p(\tau - s, \xi, y) V(f)^2(s)(y)\, dy\, ds, \tag{2.43}$$

and

$$B_2(t)(\xi) := \int_0^t \int_0^1 Q(s) \left[\frac{\partial}{\partial y} p(\tau - s, \xi, y) - \frac{\partial}{\partial y} p(t - s, \xi, y) \right] V(f)^2(s)(y)\, dy\, ds\,. \tag{2.44}$$

for $\xi \in [0,1], \;\; 0 \le t < \tau$. We will show that

$$\lim_{t\to\tau-} \|B_i(t)\|_{L^2} = 0, \quad i = 1, 2. \tag{2.45}$$

Using the estimates (2.23), (2.24) and an argument similar to that used in deriving (2.25), we obtain

$$\|B_1(t)\|_{L^2}^2 \le C \|Q\|_\infty^2 \sup_{0 \le s \le \tau} \|V(f)(s)\|_{L^2}^4 (\tau - t)^{1/2}, \quad 0 \le t < \tau. \tag{2.46}$$

Note that in (2.46), we have used the fact that

$$\sup_{0 \le s < \tau} \|V(f)(s)\|_{L^2}^4 \le \|f\|_{L^2}^4 < \infty \tag{2.47}$$

which follows from (2.10) in Proposition 2.1. Thus (2.46) implies

$$\lim_{t\to\tau-} \|B_1(t)\|_{L^2} = 0.$$

Employing similar estimates as in (2.46), the dominated convergence theorem and the fact that

$$\lim_{t\to\tau-}\left[\frac{\partial}{\partial y}p(\tau-s,\xi,y)-\frac{\partial}{\partial y}p(t-s,\xi,y)\right]=0 \tag{2.48}$$

a.e., it follows that

$$\lim_{t\to\tau-}\|B_2(t)\|_{L^2}=0. \tag{2.49}$$

This proves (2.45), (2.40) and (2.39).

By local existence, the random pde (2.6) (with Q replaced by $Q(\tau+\cdot)$) admits a local mild solution $y:[0,\epsilon]\to L^2\big([0,1],\mathbf{R}\big)$ with initial condition $V(f)(\tau)\in L^2\big([0,1],\mathbf{R}\big)$; that is

$$\begin{aligned} y(t)=T_t\big[V(f)(\tau)\big]&-\int_0^t Q(\tau+s)T_{t-s}\left[y(s)\frac{\partial y(s)}{\partial\xi}\right]ds,\\ &=T_{t+\tau}(f)-\int_0^\tau Q(s)T_{t+\tau-s}\left[V(f)(s)\frac{\partial V(f)(s)}{\partial\xi}\right]ds\\ &\quad-\int_\tau^{t+\tau}Q(s)T_{t+\tau-s}\left[y(s-\tau)\frac{\partial y(s-\tau)}{\partial\xi}\right]ds,\quad 0\le t<\epsilon\,. \end{aligned} \tag{2.50}$$

Define $\theta\in C\big([0,\tau+\epsilon],L^2\big([0,1],\mathbf{R}\big)\big)$ by

$$\theta(t):=\begin{cases}V(f)(t) & 0\le t\le\tau\\ y(t-\tau) & \tau<t\le\tau+\epsilon\,.\end{cases} \tag{2.51}$$

Therefore, (2.50) implies

$$\theta(t+\tau)=T_{t+\tau}(f)-\int_0^{t+\tau}Q(s)T_{t+\tau-s}\left[\theta(s)\frac{\partial\theta(s)}{\partial\xi}\right]ds,\quad 0\le t\le\epsilon\,. \tag{2.52}$$

For $0\le t\le\tau$, we have

$$\theta(t)=V(f)(t)=T_t(f)-\int_0^t Q(s)T_{t-s}\left[\theta(s)\frac{\partial\theta(s)}{\partial\xi}\right]ds\,. \tag{2.53}$$

Therefore, from (2.52) and (2.53), it follows that $\theta:[0,\tau+\epsilon]\to L^2\big([0,1],\mathbf{R}\big)$ is a mild solution of (2.6) on $[0,\tau+\epsilon]$ with $\theta(0)=f$. This contradicts the maximality of τ. So $\tau(\omega)=\infty$ for all $\omega\in\Omega$.

From the relation

$$U(f,\omega)(t)=V(f,\omega)(t)Q(t,\omega),\quad t\ge0,\ \omega\in\Omega, \tag{2.54}$$

we conclude that the semiflow of mild solutions:

$$L^2\big([0,1],\mathbf{R}\big)\ni f\longmapsto U(f,\omega)\in C\big([0,T],L^2\big([0,1],\mathbf{R}\big)\big)$$

of Burgers spde (2.1) is C^∞ for all $\omega\in\Omega$, all $T>0$ and has bounded Fréchet derivatives on bounded subsets of $L^2\big([0,1],\mathbf{R}\big)$. This completes the proof of Theorem 2.1. □

The next result shows that mild solutions of the stochastic Burgers equation (2.1) generate a C^∞ jointly measurable perfect cocycle

$$U : \mathbf{R}^+ \times L^2\big([0,1],\mathbf{R}\big) \times \Omega \longrightarrow L^2\big([0,1],\mathbf{R}\big)$$
$$(t,f,\omega) \longmapsto U(t,f,\omega) \equiv U(f,\omega)(t)$$

which maps bounded sets in $L^2\big([0,1],\mathbf{R}\big)$ into relatively compact sets.

Theorem 2.2 (The cocycle). *Let $U(t,f,\omega)$ be the unique global mild solution of Burgers spde (2.1) for $t \geq 0$, $f \in L^2\big([0,1],\mathbf{R}\big)$, $\omega \in \Omega$. Recall that $\theta : \mathbf{R} \times \Omega \to \Omega$ is the standard Brownian shift*

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t,s \in \mathbf{R},\ \omega \in \Omega, \tag{2.55}$$

on Wiener space $(\Omega, \mathcal{F}, P)$. Then $U : \mathbf{R}^+ \times L^2\big([0,1],\mathbf{R}\big) \times \Omega \to L^2\big([0,1],\mathbf{R}\big)$ is jointly measurable and has the following properties:

(i) *(U,θ) is a C^∞ perfect cocycle; viz.*

$$U(t_1+t_2,f,\omega) = U(t_2, U\big(t_1,f,\omega), \theta(t_1,\omega)) \tag{2.56}$$

for all $t_1,t_2 \geq 0$, $f \in L^2\big([0,1],\mathbf{R}\big)$, $\omega \in \Omega$.

(ii) *For fixed $t > 0$ and $\omega \in \Omega$, the map $U(t,\cdot,\omega) : L^2\big([0,1],\mathbf{R}\big) \to L^2\big([0,1],\mathbf{R}\big)$ takes bounded sets into relatively compact sets in $L^2\big([0,1],\mathbf{R}\big)$.*

(iii) *For each $(t,f,\omega) \in \mathbf{R}^+ \times L^2\big([0,1],\mathbf{R}\big) \times \Omega$, the Fréchet derivative $DU(t,f,\omega) \in L\big(L^2\big([0,1],\mathbf{R}\big)\big)$ is compact linear, and the map*

$$\mathbf{R}^+ \times L^2\big([0,1],\mathbf{R}\big) \times \Omega \longrightarrow L\big(L^2\big([0,1],\mathbf{R}\big)\big)$$
$$(t,f,\omega) \longmapsto DU(t,f,\omega)$$

is strongly measurable.

(iv) *For fixed $\rho, a > 0$ and any integer $k \geq 1$,*

$$E \log^+ \sup_{\substack{0 \leq t_1,t_2 \leq a \\ \|f\|_2 \leq \rho}} \big\{ \|U(t_2,f,\theta(t_1,\cdot))\|_{L^2} + \|D^{(k)}U(t_2,f,\theta(t_1,\cdot))\|_{L^{(k)}(L^2)} \big\} < \infty \tag{2.57}$$

where $L^{(k)}(L^2)$ denotes the space of all continuous k-multilinear maps $(L^2)^k \to L^2$ given the uniform operator norm $\|\cdot\|_{L^{(k)}(L^2)}$.

Proof. Note first that Q has the cocycle property

$$Q(t_1+t_2,\omega) = Q\big(t_2,\theta(t_1,\omega)\big)Q(t_1,\omega), \quad t_1,t_2 \geq 0,\ \omega \in \Omega. \tag{2.58}$$

Secondly, (2.56) will follow from the identity

$$V(f)(t_1+t_2,\omega) = Q(t_1,\omega)^{-1} V\big[Q(t_1,\omega)V(f)(t_1,\omega)\big]\big(t_2,\theta(t_1,\omega)\big) \tag{2.59}$$

for $t_1, t_2 \geq 0$, $\omega \in \Omega$, $f \in L^2\big([0,1], \mathbf{R}\big)$. To see this, we use the relation

$$U(t,f,\omega) = Q(t,\omega)V(f)(t,\omega), \quad t \geq 0,\ \omega \in \Omega,\ f \in L^2\big([0,1], \mathbf{R}\big) \tag{2.60}$$

and uniqueness of the mild solution of (2.6). Indeed, assume that (2.59) holds. Fix $\omega \in \Omega$ and $t_1 \geq 0$ throughout this proof. Then, for $t \geq 0$, we have

$$\begin{aligned}
&U\big(t, U(t_1,f,\omega), \theta(t_1,\omega)\big) \\
&\quad = Q\big(t,\theta(t_1,\omega)\big)V\big[Q(t_1,\omega)V(f)(t_1,\omega)\big]\big(t,\theta(t_1,\omega)\big) \\
&\quad = Q(t_1+t,\omega)Q(t_1,\omega)^{-1}V\big[Q(t_1,\omega)V(f)(t_1,\omega)\big]\big(t,\theta(t_1,\omega)\big) \\
&\quad = Q(t_1+t,\omega)V(f)(t_1+t,\omega) \\
&\quad = U(t_1+t,f,\omega)\,.
\end{aligned}$$

Hence the cocycle property (2.56) holds. We now show (2.59). Define the processes

$$\left.\begin{aligned}
z(t) &:= Q(t_1,\omega)^{-1}V\big[Q(t_1,\omega)V(f)(t_1,\omega)\big]\big(t,\theta(t_1,\omega)\big) \\
\overline{z}(t) &:= V(f)(t+t_1,\omega),
\end{aligned}\right\} \tag{2.61}$$

for all $t \geq 0$. Thus,

$$\begin{aligned}
z(t) &= Q(t_1,\omega)^{-1}\big[T_t\{Q(t_1,\omega)V(f)(t_1,\omega)\} \\
&\qquad - \int_0^t Q\big(s,\theta(t_1,\omega)\big)T_{t-s}\big\{V\big(Q(t_1,\omega)V(f)(t_1,\omega)\big)\big(s,\theta(t_1,\omega)\big)\cdot \\
&\qquad \frac{\partial V}{\partial \xi}\big(Q(t_1,\omega)V(f)(t_1,\omega)\big)\big(s,\theta(t_1,\omega)\big)\big\}\, ds\big] \\
&= T_{t+t_1}(f) - \int_0^{t_1} Q(s,\omega)T_{t+t_1-s}\left[V(f)(s,\omega)\frac{\partial V(f)}{\partial \xi}(s,\omega)\right]\, ds \\
&\qquad - \int_0^t Q(s+t_1,\omega)T_{t-s}\left[z(s)\frac{\partial z(s)}{\partial \xi}\right]\, ds, \quad t \geq 0.
\end{aligned} \tag{2.62}$$

On the other hand,

$$\begin{aligned}
\overline{z}(t) &= T_{t+t_1}(f) - \int_0^{t+t_1} Q(s,\omega)T_{t+t_1-s}\left[V(f)(s,\omega)\frac{\partial V(f)}{\partial \xi}(s,\omega)\right]\, ds \\
&= T_{t+t_1}(f) - \int_0^{t_1} Q(s,\omega)T_{t+t_1-s}\left[V(f)(s,\omega)\frac{\partial V(f)}{\partial \xi}(s,\omega)\right]\, ds \\
&\qquad - \int_0^t Q(s+t_1,\omega)T_{t-s}\left[\overline{z}(s)\frac{\partial \overline{z}(s)}{\partial \xi}\right]\, ds, \quad t \geq 0.
\end{aligned} \tag{2.63}$$

Subtracting (2.63) from (2.62), taking L^2-norms and employing estimates similar to those used to derive (2.27), we obtain

$$\|z(t) - \overline{z}(t)\|_{L^2}^2 \leq \|Q\|_\infty^2 \int_0^1 \bigg(\int_0^t \int_0^1 \left|\frac{\partial}{\partial y}p(t-s,\xi,y)\right| \times$$

$$
\begin{aligned}
&\quad |z^2(s)(y) - \overline{z}^2(s)(y)|\, dy\, ds\Bigg)^2 d\xi \\
&\le C\|Q\|_\infty^2 \sup_{0\le t\le a} \left[\|z(t)\|_{L^2}^2 + \|\overline{z}(t)\|_{L^2}^2\right]\times \\
&\quad t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|z(s)-\overline{z}(s)\|_{L^2}^2\, ds \\
&= C_1 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|z(s)-\overline{z}(s)\|_{L^2}^2\, ds, \quad 0\le t\le a,
\end{aligned} \tag{2.64}
$$

where

$$
C_1 := C\|Q\|_\infty^2 \sup_{0\le t\le a}\left[\|z(t)_{L^2}^2 + \overline{z}(t)\|_{L^2}^2\right]. \tag{2.65}
$$

Iterating (2.64), we get

$$
\begin{aligned}
\|z(t)-\overline{z}(t)\|_{L^2}^2 &\le C_1^2 t^{1/4}\int_0^t\int_0^s \frac{s^{1/4}}{(t-s)^{3/4}(s-r)^{3/4}}\|z(r)-\overline{z}(r)\|_{L^2}^2\, dr\, ds \\
&= C_1^2 t^{1/4}\int_0^t \left[\int_r^t \frac{s^{1/4}}{(t-s)^{3/4}(s-r)^{3/4}}\, ds\right]\|z(r)-\overline{z}(r)\|_{L^2}^2\, dr \\
&\le C_2 t^{1/4}\int_0^t \frac{1}{(t-r)^{1/2}}\|z(r)-\overline{z}(r)\|_{L^2}^2\, dr, \quad 0\le t\le a.
\end{aligned} \tag{2.66}
$$

Again, iterating the above inequality, we obtain

$$
\begin{aligned}
\|z(t)-\overline{z}(t)\|_{L^2}^2 &\le C_2 t^{1/4}\int_0^t\int_0^s \frac{s^{1/4}}{(t-s)^{1/2}(s-r)^{1/2}}\|z(r)-\overline{z}(r)\|_{L^2}^2\, dr\, ds \\
&\le C_3\int_0^t\left[\int_r^t \frac{1}{(t-s)^{1/2}(s-r)^{1/2}}\, ds\right]\|z(r)-\overline{z}(r)\|_{L^2}^2\, dr \\
&= C_3\int_0^t\int_0^{t-r}\frac{ds}{(t-r-s)^{1/2}s^{1/2}}\|z(r)-\overline{z}(r)\|_{L^2}^2\, dr \\
&\le C_4\int_0^t \|z(r)-\overline{z}(r)\|_{L^2}^2\, dr, \quad 0\le t\le a.
\end{aligned} \tag{2.67}
$$

Now (2.67) implies that $\|z(t)-\overline{z}(t)\|_{L^2} = 0$ for all $t\ge 0$ (because a is arbitrary). Hence $z(t) = \overline{z}(t)$ for all $t\ge 0$. Therefore (2.59) holds for all t_1, ω and $t_2 = t$. Thus the cocycle property (2.56) is satisfied for all $\omega\in\Omega$, $t_1, t_2\ge 0$, $f\in L^2\big([0,1],\mathbf{R}\big)$.

To prove assertion (ii) of the theorem, it is sufficient to show that the mild solution map

$$
L^2\big([0,1],\mathbf{R}\big)\ni f\longmapsto V(f,\omega)(t)\in L^2\big([0,1],\mathbf{R}\big) \tag{2.68}
$$

takes bounded sets to relatively compact sets for fixed $t>0$, $\omega\in\Omega$. In order to do this, we establish the following claim.

Claim:

Let X be a real Banach space and $S_t : X \to X$, $t \in [0,a]$, a strongly continuous semigroup of continuous linear operators on X such that $S_t : X \to X$ is compact for each $t \in (0,a]$. If $\{x_n\}_{n=1}^\infty \subset X$ is a bounded sequence in X, then there is a subsequence $\{x_n'\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\{S_t(x_n')\}_{n=1}^\infty$ converges for each $t \in (0,a]$. (The subsequence $\{x_n'\}_{n=1}^\infty$ does not depend on the choice of $t \in (0,a]$.)

The proof of the above claim follows by a diagonalization argument. It is left to the reader.

We next show that the solution map

$$L^2\big([0,1],\mathbf{R}\big) \ni f \longmapsto V(f,\omega)(t) \in L^2\big([0,1],\mathbf{R}\big)$$

is compact for each $t > 0$, $\omega \in \Omega$. To do so, let $\{f_n\}_{n=1}^\infty$ be any bounded sequence in $L^2\big([0,1],\mathbf{R}\big)$. Then by compactness and strong continuity of the heat semigroup $T_t : L^2\big([0,1],\mathbf{R}\big) \to L^2\big([0,1],\mathbf{R}\big)$, $t > 0$, the above claim gives a subsequence $\{\tilde{f}_n\}_{n=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $\{T_t(\tilde{f}_n)\}_{n=1}^\infty$ is convergent for each $t \in (0,a]$. Now, using estimates similar to (2.27), we get

$$\begin{aligned} \|V(\tilde{f}_n,\omega)(t) - &V(\tilde{f}_m,\omega)(t)\|_{L^2}^2 \\ &\leq 2\|T_t(\tilde{f}_n) - T_t(\tilde{f}_m)\|_{L^2}^2 + \\ &\qquad + Ct^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|V(\tilde{f}_n)(s) - V(\tilde{f}_m)(s)\|_{L^2}^2\, ds \end{aligned} \tag{2.69}$$

for all $0 < t \leq a$, $\omega \in \Omega$. Set

$$\phi(t) := \limsup_{m,n\to\infty} \|V(\tilde{f}_n)(t) - V(\tilde{f}_m)(t)\|_{L^2}, \quad 0 < t \leq a.$$

Taking $\limsup\limits_{m,n\to\infty}$ on both sides of (2.69) and using the fact that

$$\limsup_{m,n\to\infty} \|T_t(\tilde{f}_n) - T_t(\tilde{f}_m)\|_{L^2}^2 = 0,$$

we get

$$\phi(t) \leq Ct^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\phi(s)\, ds, \quad 0 < t \leq a. \tag{2.70}$$

Iterating (2.70) twice as in the proof of (2.67), it follows that $\phi(t) = 0$ for all $t \in (0,a]$. Therefore, for each $t \in (0,a]$, $\{V(\tilde{f}_n)(t)\}_{n=1}^\infty$ is a Cauchy sequence in $L^2\big([0,1],\mathbf{R}\big)$ and hence it converges. This proves compactness of the mild solution map (2.68) of (2.6), and completes the proof of assertion (ii) of the theorem.

To prove assertion (iv) of the theorem, use the definition (2.21) of ψ, and linearize the fixed-point relation

$$V(f)(t) = \psi\big(f, V(f)\big)(t), \quad 0 \leq t \leq a \tag{2.71}$$

to obtain

$$
\begin{aligned}
DV(f)(t)(g) &= D_1\psi\big(f, V(f)\big)(t)(g) + D_2\psi\big(f, V(f)\big)(t)(g) \\
&= T_t(g) + \int_0^t Q(s)\int_0^1 \frac{\partial}{\partial y}p(t-s,\xi,y)\big[DV(f)(s)(g)(y)V(f)(s)(y)\big]\,dy\,ds
\end{aligned}
\tag{2.72}
$$

for all $t \in [0,a]$, $f, g \in L^2\big([0,1],\mathbf{R}\big)$. Let $\rho > 0$ and suppose $g, f \in L^2\big([0,1],\mathbf{R}\big)$ are such that $\|f\|_{L^2} \le \rho$ and $\|g\|_{L^2} \le 1$. Take L^2-norms of (2.72) and use C as a generic deterministic constant that could change from line to line. This gives

$$
\begin{aligned}
\|DV(f)(t)(g)\|_{L^2}^2 &\le 2\|T_t\|_{L(L^2)}^2\|g\|_{L^2}^2 + \\
&\quad + C\|Q\|_\infty^2 \int_0^1 \Big(\int_0^t \big|\frac{\partial}{\partial y}p(t-s,\xi,y)\big| \times \\
&\qquad \times |DV(f)(s)(g)(y)V(f)(s)(y)|\,dy\,ds\Big)^2 d\xi \\
&\le 2 + C\|Q\|_\infty^2 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|V(f)(s)\|_{L^2}^2\|DV(f)(s)(g)\|_{L^2}^2\,ds \\
&\le 2 + C\|Q\|_\infty^2\|f\|_{L^2}^2 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|DV(f)(s)\|_{L(L^2)}^2\|g\|_{L^2}^2\,ds \\
&\le 2 + C\|Q\|_\infty^2\rho^2 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|DV(f)(s)\|_{L(L^2)}^2\,ds \\
&\le 2 + C\|Q\|_\infty^2 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|DV(f)(s)\|_{L(L^2)}^2\,ds, \quad 0 \le t \le a.
\end{aligned}
$$

Hence,

$$
\|DV(f)(t)\|_{L(L^2)}^2 \le 2 + C\|Q\|_\infty^2 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\|DV(f)(s)\|_{L(L^2)}^2\,ds, \tag{2.73}
$$

for $0 \le t \le a$ and $f \in L^2\big([0,1],\mathbf{R}\big)$ with $\|f\| \le \rho$. Define

$$
\eta(t) := \sup_{\|f\|\le\rho}\|DV(f)(s)\|_{L(L^2)}^2\,, \quad 0 \le t \le a\,. \tag{2.74}
$$

Then (2.73) and (2.74) give

$$
\eta(t) \le 2 + C\|Q\|_\infty^2 t^{1/4}\int_0^t \frac{1}{(t-s)^{3/4}}\,\eta(s)\,ds, \quad 0 \le t \le a\,. \tag{2.75}
$$

Iterating (2.75) yields

$$
\eta(t) \le 2 + C\|Q\|_\infty^2 + C\|Q\|_\infty^4\int_0^t \frac{1}{(t-s)^{1/2}}\,\eta(s)\,ds, \quad 0 \le t \le a\,. \tag{2.76}
$$

Again iterate (2.76) to obtain

$$\eta(t) \leq 2 + C\|Q\|_\infty^6 + C\|Q\|_\infty^8 \int_0^t \eta(s)\,ds, \quad 0 \leq t \leq a\,. \tag{2.77}$$

By Gronwall's lemma, (2.77) implies

$$\eta(t) \leq \left[2 + C\|Q\|_\infty^6\right] e^{C\|Q\|_\infty^8 t}, \quad 0 \leq t \leq a\,.$$

Therefore,

$$\log^+ \sup_{\substack{0\leq s\leq a\\ \|f\|\leq\rho}} \|DV(f)(s)\|_{L(L^2)} \leq \log\left[2 + C\|Q\|_\infty^6\right] + C\|Q\|_\infty^8\,. \tag{2.78}$$

By the proof of the contraction mapping theorem (via successive approximation) and using the joint measurability of

$$\begin{aligned} \psi : \Omega \times B(f_0,\rho_0) \times Y &\longrightarrow E \\ (\omega, f, v) &\longmapsto \psi(f,v,\omega)(t) \\ &\qquad = T_t(f) + \frac{1}{2}\int_0^t \int_0^1 Q(s)\frac{\partial}{\partial y} p(t-s,\cdot,y) v^2(s)(y)\,dy\,ds, \end{aligned}$$

it follows that the maps

$$\begin{aligned} [0,a] \times L^2\big([0,1],\mathbf{R}\big) \times \Omega &\longrightarrow L^2\big([0,1],\mathbf{R}\big) \\ (t,f,\omega) &\longmapsto V(f,\omega)(t) \end{aligned}$$

and

$$\begin{aligned} [0,a] \times L^2\big([0,1],\mathbf{R}\big) \times \Omega &\longrightarrow L^2\big([0,1],\mathbf{R}\big) \\ (t,f,\omega) &\longmapsto DV(f,\omega)(t)(g) \end{aligned}$$

are jointly measurable (for each $g \in L^2\big([0,1],\mathbf{R}\big)$). This proves the strong measurability assertion in (iii) of the theorem. The proof of the first part of assertion (iii) follows from the Fréchet smoothness of U and assertion (ii) of the theorem.

Using the martingale property of Q and the relation

$$Q(t) = \exp\left\{\gamma t + \sum_{k=1}^\infty \left(\sigma_k W_k(t) - \frac{1}{2}\sigma_k^2 t\right)\right\}, \quad t \geq 0,$$

it is easy to see that

$$E\|Q\|_\infty^p < \infty \tag{2.79}$$

for all $p \geq 1$. Taking expectations on both sides of (2.78), we get

$$E \log^+ \sup_{\substack{0\leq s\leq a\\ \|f\|_2\leq\rho}} \|DV(f)(t)\|_{L(L^2)} < \infty\,. \tag{2.80}$$

Now by (2.60), we have

$$DU(t, f, \omega) = Q(t, \omega) DV(f, \omega)(t). \tag{2.81}$$

Assertion (iv) of Theorem 2.2, for $k = 1$, now follows from (2.80) and (2.81). To complete the proof of the theorem, we indicate the proof of the estimate (2.57) for $k \geq 2$. From the proof of Theorem 2.1, recall that $E := C([0, a], L^2([0, 1], \mathbf{R})$ and $\psi : L^2([0, 1], \mathbf{R}) \times E \to E$ is given by

$$\psi(f, v)(t) := T_t(f) - \frac{1}{2} A(v, v)(t), \quad f \in E, \ 0 \leq t \leq a, \tag{2.82}$$

where $A : E \times E \to E$ is the continuous bilinear map

$$A(v_1, v_2)(t) := \int_0^t \int_0^1 Q(s) \frac{\partial}{\partial y} p(t - s, \cdot, y) v_1(s)(y) v_2(s)(y) \, dy \, ds, \quad v_1, v_2 \in E, \ 0 \leq t \leq a. \tag{2.83}$$

Therefore, (2.71) becomes

$$V(f)(t) := T_t(f) - \frac{1}{2} A(V(f), V(f))(t), \quad f \in L^2([0, 1], \mathbf{R}), \ 0 \leq t \leq a. \tag{2.84}$$

Fix $t \in [0, a]$ and take Fréchet derivatives on both sides of the above equation to obtain

$$DV(f)(t) := T_t - \frac{1}{2} A(\cdot, V(f))(t) \circ DV(f)(t), \quad f \in L^2([0, 1], \mathbf{R}). \tag{2.85}$$

Since V is C^∞, then using the fact that A is continuous symmetric bilinear, we can differentiate the above equation once more to obtain

$$D^{(2)} V(f)(\cdot, \cdot)(t) = A(DV(f)(\cdot), DV(f)(\cdot))(t) - A(\cdot, V(f)) \circ D^{(2)} V(f)(\cdot, \cdot)(t) \tag{2.86}$$

for all $f \in L^2([0, 1], \mathbf{R})$ and $t \in [0, a]$.

In the remaining estimates we will denote by C a generic deterministic positive constant that may change from line to line.

Taking $L^{(2)}(L^2)$-norms on both sides of (2.86) and using an argument similar to the proof of (2.73), we get

$$\begin{aligned} \|D^{(2)} V(f)(t)\|^2 \leq C \|Q\|_\infty^2 \sup_{0 \leq t \leq a} \|DV(f)(\cdot)(t)\|_{L(L^2)} \\ + C \big[\|Q\|_\infty^2 t^{1/4} \int_0^t \frac{1}{(t - s)^{3/4}} \|V(f)(s)\|_{L^2}^2 \cdot \|D^{(2)} V(f)(s)\|^2 \, ds, \end{aligned} \tag{2.87}$$

for all $f \in L^2([0, 1], \mathbf{R})$ and $t \in [0, a]$. Let $\eta(t)$, $t \in [0, a]$, be defined as in (2.74). Define

$$\beta(t) := \sup_{\|f\| \leq \rho} \|D^{(2)} V(f)(t)\|_{L^{(2)}(L^2)}^2, \quad 0 \leq t \leq a\,. \tag{2.88}$$

From (2.77), we know that

$$\eta(t) \leq \big[2 + C \|Q\|_\infty^6\big] e^{C \|Q\|_\infty^8 t}, \quad 0 \leq t \leq a\,. \tag{2.89}$$

Hence (2.87) implies

$$\beta(t) \leq C\|Q\|_\infty^8 e^{C\|Q\|_\infty^8} + C\|Q\|_\infty^2 t^{1/4} \int_0^t \frac{\beta(s)}{(t-s)^{3/4}}\, ds, \quad 0 \leq t \leq a\,. \tag{2.90}$$

Iterating the above inequality gives

$$\beta(t) \leq C\|Q\|_\infty^{10} e^{C\|Q\|_\infty^8} + C\|Q\|_\infty^4 \int_0^t \frac{\beta(s)}{(t-s)^{1/2}}\, ds, \quad 0 \leq t \leq a, \tag{2.91}$$

and iterating once more, we obtain

$$\beta(t) \leq C\|Q\|_\infty^{14} e^{C\|Q\|_\infty^8} + C\|Q\|_\infty^8 \int_0^t \beta(s)\, ds, \quad 0 \leq t \leq a, \tag{2.92}$$

Then Gronwall's lemma implies

$$\beta(t) \leq C\|Q\|_\infty^{14} e^{C\|Q\|_\infty^8 t}, \quad 0 \leq t \leq a\,. \tag{2.93}$$

Since Q has finite moments of all orders, the above inequality implies

$$E \log^+ \sup_{\substack{0\leq t\leq a\\ \|f\|\leq\rho}} \|D^{(2)}V(f)(s)\|_{L^{(2)}(L^2)}^2 < \infty. \tag{2.94}$$

To complete the proof of (2.57), one may take higher-order Fréchet derivatives of (2.86) and then repeat the above argument to obtain

$$E \log^+ \sup_{\substack{0\leq t\leq a\\ \|f\|\leq\rho}} \|D^{(k)}V(f)(s)\|_{L^{(k)}(L^2)}^2 < \infty, \quad k \geq 1, \tag{2.95}$$

by induction on k. This completes the proof of assertion (iv) of the theorem. □

3 The Dynamics-Affine noise

The results and methods introduced in the last section extend to the case of additive space-time noise that is smooth in space and white in time (viz. Burgers spde (1.1)). One motivation for dealing with this scenario is that the presence of the additive noise term allows for the existence of non-trivial stationary points for the cocycle.

In this section, we will only outline the construction of the cocycle for Burgers spde (1.1) and leave the rest of the details to the reader.

Recall Burgers spde (1.1) with affine (additive + linear) white noise:

$$\left.\begin{aligned} &du(t) = \nu\Delta u\, dt - u\frac{\partial u}{\partial \xi}\, dt + \gamma u(t)\, dt + \sum_{k=1}^\infty \sigma_k u(t)\, dW_k(t) + \sigma_0(\xi) dW_0(t), \quad t>0, \quad \xi\in[0,1],\\ &u(t,0) = u(t,1) = 0 \quad \text{for all } t>0,\\ &u(0,\xi) = f(\xi),\ \xi\in[0,1]. \end{aligned}\right\} \tag{1.1}$$

As in the previous section, our objective is to show that the random field of mild solutions of (1.1) generates a Fréchet smooth locally compacting perfect cocycle $U : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \Omega \to L^2([0,1],\mathbf{R})$. The construction also yields Oseledec-type integrability estimates on the cocycle and its Fréchet derivatives (cf. Theorem 2.2).

Recall that Q satisfies

$$E\|Q\|_\infty < \infty, \tag{3.1}$$

where

$$\|Q\|_\infty \equiv \|Q(\cdot,\omega)\|_\infty := \sup_{0\le t\le T} Q(t,\omega), \quad \omega \in \Omega, \tag{3.2}$$

for any finite positive T. Moreover,

$$\left.\begin{aligned} dQ^{-1}(t) &= (\sum_{k=1}^\infty \sigma_k^2 - \gamma)Q^{-1}(t)\,dt - \sum_{k=1}^\infty \sigma_k Q^{-1}(t)\,dW_k(t), \quad t \ge 0, \\ Q(0) &= 1. \end{aligned}\right\} \tag{3.3}$$

Set

$$V(t,\xi) =: u(t,\xi)Q^{-1}(t), \quad t \ge 0,\ \xi \in [0,1], \tag{3.4}$$

where u is the solution of Burgers spde (1.1). By Itô's formula (the product rule), we have

$$\begin{aligned} dV(t) &= \nu Q^{-1}(t)\Delta u\,dt - Q^{-1}(t)u\frac{\partial u}{\partial \xi}\,dt + Q^{-1}(t)\sigma_0(\xi)dW_0(t) \\ &= \nu\Delta V(t)dt - Q(t)V\frac{\partial V}{\partial \xi}\,dt + Q^{-1}(t)\sigma_0(\xi)\,dW_0(t),\ \ t > 0. \end{aligned} \tag{3.5}$$

Let $Z(t,\xi)$ be the solution of the spde:

$$\left.\begin{aligned} dZ(t) &= \nu\Delta Z(t)dt + Q^{-1}(t)\sigma_0(\xi)dW_0(t),\ t > 0, \xi \in [0,1], \\ Z(0,\xi) &= 0,\ \xi \in [0,1] \\ Z(t,0) &= 0, \quad t > 0, \\ Z(t,1) &= 0, \quad t > 0. \end{aligned}\right] \tag{3.6}$$

Then Z is given by

$$Z(t,\xi) = \int_0^t Q^{-1}(s)T_{t-s}\sigma_0(\xi)dW_0(s),\ t > 0, \xi \in [0,1]. \tag{3.7}$$

Define $V_0(t,\xi) := V(t,\xi) - Z(t,\xi)$. Then V_0 solves the random pde:

$$\left.\begin{aligned} \frac{\partial V_0}{\partial t} &= \nu\Delta V_0(t) - Q(t)(V_0(t,\xi) + Z(t,\xi))\frac{\partial (V_0(t,\xi)+Z(t,\xi))}{\partial \xi} \\ &= \nu\Delta V_0(t) - Q(t)V_0(t,\xi)\frac{\partial V_0(t,\xi)}{\partial \xi} - Q(t)V_0(t,\xi)\frac{\partial Z(t,\xi)}{\partial \xi} \\ &\quad - Q(t)Z(t,\xi)\frac{\partial V_0(t,\xi)}{\partial \xi} - Q(t)Z(t,\xi)\frac{\partial Z(t,\xi)}{\partial \xi},\ t > 0, \xi \in [0,1] \\ V_0(0,\xi) &= f(\xi),\ \xi \in [0,1], \\ V_0(t,0) &= 0, \quad t > 0, \\ V_0(t,1) &= 0, \quad t > 0. \end{aligned}\right] \tag{3.8}$$

Reversing the above formal procedure, it is not hard to see that if V_0 is a mild solution of (3.8), then

$$u(t,\xi) := Q(t)[V_0(t,\xi) + Z(t,\xi)] \tag{3.9}$$

is a mild solution of Burgers spde (1.1). Thus to get a perfect cocycle for the mild solution $u(t), t \geq 0$ of (1.1), it is sufficient to analyze the family of mild solutions to the random pde (3.8) perfectly in $\omega \in \Omega$. To this end, the following a priori estimate is needed.

Proposition 3.1. For $f \in L^2\big([0,1], \mathbf{R}\big)$, let $V_0(t,f,\omega)$ be a mild solution of the initial boundary value problem (3.8) for $0 < t < T$ and some $T > 0$. Then for each $\omega \in \Omega$ and $t \in [0,T]$,

$$\begin{aligned} \|V_0(t,f,\omega)\|^2_{L^2([0,1],\mathbf{R})} \quad &+ \quad \nu \int_0^t \left\| \frac{\partial V_0(s,f,\omega)}{\partial \xi} \right\|^2_{L^2([0,1],\mathbf{R})} ds \\ &\leq \quad C_T(\omega) \left[\|f\|^2_{L^2([0,1],\mathbf{R})} + \int_0^t Q(s) \left\| Z(s,\cdot) \frac{\partial Z(s,\cdot)}{\partial \xi} \right\|^2_{L^2} ds \right] \end{aligned} \tag{3.10}$$

for all $t \in [0,T)$, and all $\omega \in \Omega$, where $C_T(\omega)$ is a constant depending only on ω and T.

Proof. As in the proof of Proposition 2.1, it is sufficient to assume that $f \in C_0^\infty\big([0,1], \mathbf{R}\big)$ and V_0 is a classical solution of (3.8). We fix and suppress $\omega \in \Omega$ throughout. Applying the chain rule we obtain

$$\begin{aligned} \|V_0(t)\|^2_{L^2} \quad = \quad & \|f\|^2_{L^2} - 2\nu \int_0^t \int_0^1 \left(\frac{\partial V_0(s)}{\partial \xi} \right)^2 d\xi\, ds \\ & -2 \int_0^t \int_0^1 Q(s) V_0^2(s,\xi) \frac{\partial V_0(s,\xi)}{\partial \xi} d\xi\, ds - 2 \int_0^t \int_0^1 Q(s) V_0^2(s,\xi) \frac{\partial Z(s,\xi)}{\partial \xi} d\xi\, ds \\ & -2 \int_0^t \int_0^1 Q(s) Z(s,\xi) V_0(s,\xi) \frac{\partial V_0(s,\xi)}{\partial \xi} d\xi\, ds \\ & -2 \int_0^t \int_0^1 Q(s) Z(s,\xi) V_0(s,\xi) \frac{\partial Z(s,\xi)}{\partial \xi} d\xi\, ds \end{aligned}$$

$$\begin{aligned} \leq \quad & \|f\|^2_{L^2} - 2\nu \int_0^t \int_0^1 \left(\frac{\partial V_0(s)}{\partial \xi} \right)^2 d\xi\, ds \\ & -\frac{2}{3} \int_0^t \int_0^1 Q(s) \frac{\partial V_0^3(s,\xi)}{\partial \xi} d\xi\, ds + 2 \int_0^t Q(s) \left\| \frac{\partial Z(s,\cdot)}{\partial \xi} \right\|_{L^\infty} \|V_0(s,\cdot)\|^2_{L^2} ds \\ & +\nu \int_0^t \int_0^1 \left(\frac{\partial V_0(s)}{\partial \xi} \right)^2 d\xi\, ds + C_\nu \int_0^t Q^2(s) \|Z(s,\cdot)\|^2_{L^\infty} \|V_0(s,\cdot)\|^2_{L^2} ds \\ & + \int_0^t Q(s) \|V_0(s,\cdot)\|^2_{L^2} ds + \int_0^t Q(s) \left\| Z(s,\cdot) \frac{\partial Z(s,\cdot)}{\partial \xi} \right\|^2_{L^2} ds \end{aligned} \tag{3.11}$$

for all $t \in [0,T)$. Note that

$$\int_0^1 \frac{\partial V_0^3(s,\xi)}{\partial \xi} d\xi = V_0^3(s,1) - V_0^3(s,0) = 0.$$

Using Young's and Gronwall's inequalities it follows from (3.11) that

$$
\begin{aligned}
&\|V_0(t,f,\omega)\|^2_{L^2([0,1],\mathbf{R})} + \nu \int_0^t \left\| \frac{\partial V_0(s,f,\omega)}{\partial \xi} \right\|^2_{L^2([0,1],\mathbf{R})} ds \\
\leq \; & \left[\|f\|^2_{L^2([0,1],\mathbf{R})} + \int_0^t Q(s) \left\| Z(s,\cdot) \frac{\partial Z(s,\cdot)}{\partial \xi} \right\|^2_{L^2} ds \right] \\
& \times \exp \left\{ C \int_0^t \{ Q(s) \left\| \frac{\partial Z(s,\cdot)}{\partial \xi} \right\|_{L^\infty} + Q^2(s) \|Z(s,\cdot)\|^2_{L^\infty} + Q(s) \} ds \right\} \quad (3.12)
\end{aligned}
$$

for all $t \in [0,T)$. □

Emphasizing the dependence on the initial function f, we denote by $U(t,f,\omega)$ the mild solution $u(t,\xi)$ of Burgers spde (1.1). To check that the random field $U(t,f,\omega)$ gives rise to a cocycle on $L^2([0,1],\mathbf{R})$, we will verify the perfect cocycle identity

$$U(t+s,f,\omega) = U(t,U(s,f,\omega),\theta(s,\omega)), \quad t,s \geq 0, \omega \in \Omega. \tag{3.13}$$

Note that

$$U(t+s,f,\omega) = Q(t+s)[V_0(t+s,f,\omega) + Z(t+s,\omega)] \tag{3.14}$$

$$U(t,U(s,f,\omega),\theta(s,\omega)) = Q^{-1}(s,\omega)Q(t+s)[V_0(t,U(s,f,\omega),\theta(s,\omega) + Z(t,\theta(s,\omega))] \tag{3.15}$$

Thus to prove (3.13), we need to show that

$$V_0(t+s,f,\omega) = Q^{-1}(s,\omega)V_0(t,U(s,f,\omega),\theta(s,\omega)) + Q^{-1}(s,\omega)Z(t,\theta(s,\omega)) - Z(t+s,\omega) \tag{3.16}$$

It is easy to show that

$$Z(t+s,f,\omega) = Q^{-1}(s,\omega)Z(t,\theta(s,\omega)) + T_t(Z(s,\omega)), \quad t,s \geq 0, \omega \in \Omega. \tag{3.17}$$

Then (3.16) reduces to

$$V_0(t+s,f,\omega) = Q^{-1}(s,\omega)V_0(t,U(s,f,\omega),\theta(s,\omega)) - T_t(Z(s,\omega)) \tag{3.18}$$

Set

$$L(t) := Q^{-1}(s,\omega)V_0(t,U(s,f,\omega),\theta(s,\omega)) - T_t(Z(s,\omega)),$$

and

$$M(t) := V_0(t+s,f,\omega).$$

It is possible to show that $L(t)$ and $M(t)$ satisfy the same random integral equation. So, by uniqueness, $L(t) = M(t)$ for all $t \geq 0$, and (3.18) follows.

Following the arguments in Section 2, the reader may show that all the assertions of Theorems 2.1 and 2.2 hold for Burgers spde (1.1).

4 Stability-Affine Noise

In this section we characterize the behavior of solutions of the Burgers spde (1.1) near a general *equilibrium* or a *stationary point/solution*.

We first describe the concepts of a general stationary point and its hyperbolicity for the Burgers spde (1.1).

Definition 4.1 (Stationary point/equilibrium). An $\mathcal{F}$-measurable random variable $Y : \Omega \to L^2([0,1],\mathbf{R})$ is said be a **stationary random point** or **equilibrium** for the cocycle (U,θ) of (1.1) if

$$U(t, Y(\omega), \omega) = Y(\theta(t,\omega)) \tag{4.1}$$

for all $t \in \mathbf{R}^+$, and $\omega \in \Omega$.

Remark.

(i) For Burgers spde with additive (not linear) spatially smooth noise, it is known that a stationary random point exists (Liu and Zhao [12], Theorem 3.4). Under periodic boundary conditions and sufficiently spatially smooth (C^3) additive noise, stationary random points are established in work by Sinai ([18]), E, Khanin, Mazel and Sinai ([9]).

(ii) It is easy to see that the distribution $P \circ Y^{-1}$ of a stationary random point $Y : \Omega \to L^2([0,1],\mathbf{R})$ is an invariant measure for the underlying Markov process of solutions of Burgers spde. The existence of invariant measures for Burgers spde has been analyzed by a number of authors (See [6] and the references therein). Conversely, under suitable enlargement of the underlying probability space, one can show that an invariant measure for the one-point motion induces a stationary random point for the stochastic semiflow. To see this we proceed as follows: Let μ be an invariant probability measure on $L^2([0,1],\mathbf{R})$ for the Markov process generated by mild solutions of Burgers spde. Denote by $\mathcal{B}(L^2)$ the Borel σ-algebra of $L^2([0,1],\mathbf{R})$. Define the enlarged probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by setting

$$\tilde{\Omega} := \Omega \times L^2([0,1],\mathbf{R}), \;\; \tilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{B}(L^2), \;\; \tilde{P} := P \otimes \mu.$$

Sample points $\tilde{\omega} \in \tilde{\Omega}$ are given by $\tilde{\omega} := (\omega, f)$, $\omega \in \Omega$, $f \in L^2([0,1],\mathbf{R})$; and a $\tilde{P}$-preserving semigroup $\tilde{\theta} : \mathbf{R}^+ \times \tilde{\Omega} \to \tilde{\Omega}$ is obtained by setting $\tilde{\theta}(t,\tilde{\omega}) := (\theta(t,\omega), U(t,f,\omega))$, $t \geq 0, \omega \in \Omega$, $f \in L^2([0,1],\mathbf{R})$. The reader may easily check the latter statement using the invariance of the measure μ under the one-point motion and of the probability measure P under the Wiener shift θ. Furthermore, define the extended cocycle $\tilde{U} : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \tilde{\Omega} \to L^2([0,1],\mathbf{R})$ by $\tilde{U}(t,f,\tilde{\omega}) := U(t,f,\omega)$, $t \geq 0, f \in L^2([0,1],\mathbf{R})$, $\tilde{\omega} = (\omega, f) \in \tilde{\Omega}$. It is easy to check that $(\tilde{U}, \tilde{\theta})$ is a perfect cocycle on $L^2([0,1],\mathbf{R})$. Finally, we define the $\tilde{\mathcal{F}}$-measurable random variable $\tilde{Y} : \tilde{\Omega} \to$

$L^2([0,1],\mathbf{R})$ by $\tilde{Y}(\tilde{\omega}) := f$, $\tilde{\omega} := (\omega, f) \in \tilde{\Omega}$. It follows immediately from the definition of $\tilde{U}$ that $\tilde{U}(t, \tilde{Y}(\tilde{\omega}), \tilde{\omega}) = \tilde{Y}(\tilde{\theta}(t, \tilde{\omega}))$ for all $t \in \mathbf{R}^+$, and $\tilde{\omega} \in \tilde{\Omega}$. Hence $\tilde{Y}$ is a stationary random point for the cocycle $(\tilde{U}, \tilde{\theta})$ in the sense of Definition 4.1. Note further that an Oseledec integrability property similar to (iv) of Theorem 2.2 also holds for the extended cocycle $(\tilde{U}, \tilde{\theta})$.

Let $Y : \Omega \to L^2([0,1],\mathbf{R})$ be a stationary random point for the cocycle (U,θ) of (1.1) with $E\log^+ \|Y\|_{L^2} < \infty$. It is easy to see that $(DU(t, Y(\omega), \omega), \theta(t,\omega))$ is a compact linear cocycle. So by the integrability condition (2.57) (for (1.1)) and the Ruelle-Oseledec theorem, it has a discrete fixed Lyapunov spectrum

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}.$$

The stationary point Y is said to be *hyperbolic* if $\lambda_i \neq 0$ for all $i \geq 1$.

In order to analyze the dynamics of the Burgers spde (1.1) near a general equilibrium or stationary point $Y : \Omega \to L^2([0,1],\mathbf{R})$, we linearize the smooth cocycle $U : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \Omega \to L^2([0,1],\mathbf{R})$ at the stationary point Y. This gives a linear cocycle of Fréchet derivatives $DU(t, Y(\omega), \omega) \in L(L^2([0,1],\mathbf{R}))$ satisfying the following random equations

$$DU(t,Y) = Q(t,\cdot)DV_0(t,Y), \quad t \geq 0, \tag{4.2}$$

and

$$\begin{aligned} DV_0(t,Y)(g) &= g - \int_0^t \nu \Delta DV_0(s,Y)(g) ds - \int_0^t Q(s) DV_0(s,Y)(g) \cdot \frac{\partial V_0(s,Y)}{\partial \xi} ds \\ &\quad - \int_0^t Q(s) V_0(s,f) \cdot \frac{\partial DV_0(s,Y)(g)}{\partial \xi} ds \\ &\quad - \int_0^t Q(s) DV_0(s,Y)(g) \cdot \frac{\partial Z(s)}{\partial \xi} ds \\ &\quad - \int_0^t Q(s) Z(s) \cdot \frac{\partial V_0(s,Y)}{\partial \xi} ds, \quad t \geq 0, \; g \in L^2([0,1],\mathbf{R}), \end{aligned} \tag{4.3}$$

where V_0 and Z are defined as in Section 3.

We next apply the Oseledec-Ruelle spectral theorem to the compact linear cocycle $\big(DU(t, Y(\omega), \omega), \theta(t,\omega)\big)$, $t \geq 0$, $\omega \in \Omega$ ([16]; Theorem 2.1.1, [15]). This gives

Theorem 4.1 (The Lyapunov spectrum: general equilibrium). *Let $\big(U(t,\cdot,\omega), \theta(t,\omega)\big)$ be the C^∞ cocycle on $L^2([0,1],\mathbf{R})$ generated by Burgers spde (1.1). Suppose that $Y : \Omega \to L^2([0,1],\mathbf{R})$ is a stationary random point for the cocycle (U,θ) of the Burgers spde (1.1) with $E\log^+ \|Y\|_{L^2} < \infty$. Then the following limit*

$$\Lambda(\omega) := \lim_{t\to\infty} \big\{ \big[DU(t, Y(\omega), \omega)\big]^* \circ \big[DU(t, Y(\omega), \omega)\big] \big\}^{1/2t} \tag{4.4}$$

exists in the uniform operator norm in $L(L^2([0,1],\mathbf{R}))$*, perfectly in* ω*. The Oseledec operator* $\Lambda(\omega)$ *in* (4.4) *is compact, self-adjoint and non-negative with discrete non-random spectrum*

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_n} > \cdots \tag{4.5}$$

The Lyapunov exponents $\{\lambda_n\}_{n=1}^{\infty}$ *correspond to values of the limit*

$$\lim_{t\to\infty} \frac{1}{t} \log \|DU(t,Y(\omega),\omega)(g)\|_{L^2} \in \{\lambda_n\}_{n=1}^{\infty}$$

for any $g \in L^2([0,1],\mathbf{R})$*, perfectly in* ω*. Each eigenvalue* e^{λ_j} *has a fixed finite multiplicity* m_j *with a corresponding finite-dimensional eigenspace* $F_j(\omega)$ *such that* $m_j := dim\, F_j(\omega), j \geq 1, \omega \in \Omega$*. If we set*

$$E_1(\omega) := L^2([0,1],\mathbf{R}), \quad E_n(\omega) := \left[\bigoplus_{j=1}^{n-1} F_j(\omega)\right]^{\perp}, \quad n > 1,$$

then for each $n \geq 1$*,* $codim\, E_n(\omega) = \sum_{j=1}^{n-1} m_j < \infty$*, and the following assertions are true:*

$$E_n(\omega) \subset E_{n-1}(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = L^2([0,1],\mathbf{R}), \quad n > 1;$$

$$\lim_{t\to\infty} \frac{1}{t} \log \|DU(t,Y(\omega),\omega)(g)\|_{L^2} = \lambda_n \tag{4.6}$$

for $g \in E_n(\omega)\backslash E_{n+1}(\omega)$ *;*

$$\lim_{t\to\infty} \frac{1}{t} \log \|DU(t,Y(\omega),\omega)\|_{L(L^2)} = \lambda_1; \tag{4.7}$$

and

$$DU(t,Y(\omega),\omega)(E_n(\omega)) \subseteq E_n(\theta(t,\omega)) \tag{4.8}$$

for all $t \geq 0$*, perfectly in* $\omega \in \Omega$*, for all* $n \geq 1$*.*

Proof. The Oseledec integrability condition

$$E \log^+ \sup_{0\leq t_1,t_2\leq a} \|DU(t_2, Y(\theta(t_1,\cdot)), \theta(t_1,\cdot))\|_{L(L^2)} < \infty \tag{4.9}$$

for any $0 < a < \infty$, follows from (2.57) in Theorem 2.2. Using the above integrability condition and the Ruelle-Oseledec theorem (Theorem 2.1.1, [15]), there is a random family of compact self-adjoint positive operators $\Lambda(\omega) \in L(L^2)$, defined perfectly in ω, and satisfies

$$\Lambda(\omega) := \lim_{t\to\infty} \left\{ \left[DU(t,Y(\omega),\omega)\right]^* \circ \left[DU(t,Y(\omega),\omega)\right] \right\}^{1/2t}. \tag{4.10}$$

The above almost sure limit exists in the uniform operator norm in $L(L^2)$, perfectly in ω. The operator $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_n} > \cdots \tag{4.11}$$

due to the ergodicity of the Brownian shift θ.

The remaining assertions (4.6)-(4.8) of the theorem follow from the Oseledec-Ruelle spectral theorem (Theorem 2.1.1, [15]). ∎

Theorem 4.2 and 4.3 below are consequences of the nonlinear multiplicative ergodic theorem ([15], Theorem 2.2.1). Theorem 4.2 (*the local stable manifold theorem*) describes the saddle-point behavior of the random flow of the Burgers spde (1.1) in the neighborhood of any hyperbolic equilibrium. Theorem 4.3 (*the local invariant manifold theorem*) gives local invariant manifolds near an ergodic equilibrium. Details of the proofs of both theorems are left to the reader ([15]).

Theorem 4.2 (The local stable manifold theorem). *Assume that* $Y:\Omega\to L^2([0,1],\mathbf{R})$ *is a hyperbolic stationary random point for the cocycle* (U,θ) *of the Burgers spde (1.1) with* $E\log^+\|Y\|_{L^2}<\infty$. *Denote by* $\{\cdots<\lambda_{i+1}<\lambda_i<\cdots<\lambda_2<\lambda_1\}$ *the Lyapunov spectrum of the linearized cocycle* $(DU(t,Y(\omega),\omega),\theta(t,\omega),t\geq 0)$ *as given in Theorem 4.1. Define* $i_0:=min\{i:\lambda_i<0\}$.

Fix $\epsilon_1\in(0,-\lambda_{i_0})$ *and* $\epsilon_2\in(0,\lambda_{i_0-1})$. *Then there exist*

(i) *a sure event* $\Omega^*\in\mathcal{F}$ *with* $\theta(t,\cdot)(\Omega^*)=\Omega^*$ *for all* $t\in\mathbf{R}$,

(ii) $\mathcal{F}$*-measurable random variables* $\rho_i,\beta_i:\Omega^*\to(0,1)$, $\beta_i>\rho_i>0$, $i=1,2$, *such that for each* $\omega\in\Omega^*$, *the following is true:*

There are C^∞ *submanifolds* $\mathcal{S}(\omega)$, $\mathcal{U}(\omega)$ *of* $B(Y(\omega),\rho_1(\omega))$ *and* $B(Y(\omega),\rho_2(\omega))$ *(resp.) with the following properties:*

(a) *For* $\lambda_{i_0}>-\infty$, $\mathcal{S}(\omega)$ *is the set of all* $f\in B(Y(\omega),\rho_1(\omega))$ *such that*

$$\|U(n,f,\omega)-Y(\theta(n,\omega))\|_{L^2}\leq\beta_1(\omega)\exp\{(\lambda_{i_0}+\epsilon_1)n\}$$

for all integers $n\geq 0$. *If* $\lambda_{i_0}=-\infty$, *then* $\mathcal{S}(\omega)$ *is the set of all* $f\in B(Y(\omega),\rho_1(\omega))$ *such that*

$$\|U(n,f,\omega)-Y(\theta(n,\omega))\|_{L^2}\leq\beta_1(\omega)\,e^{\lambda n}$$

for all integers $n\geq 0$ *and any* $\lambda\in(-\infty,0)$. *Furthermore,*

$$\limsup_{t\to\infty}\frac{1}{t}\log\|U(t,f,\omega)-Y(\theta(t,\omega))\|_{L^2}\leq\lambda_{i_0}\tag{4.12}$$

for all $f\in\mathcal{S}(\omega)$. *The stable subspace* $\mathcal{S}^0(\omega)$ *of the linearized cocycle* $(DU(t,Y(\omega),\cdot),\theta(t,\cdot))$ *is tangent at* $Y(\omega)$ *to the submanifold* $\mathcal{S}(\omega)$, *viz.* $T_{Y(\omega)}\mathcal{S}(\omega)=\mathcal{S}^0(\omega)$. *In particular, codim* $\mathcal{S}(\omega)=$ *codim* $\mathcal{S}^0(\omega)=\sum_{j=1}^{i_0-1}dim\,F_j(\omega)$ *is fixed and finite.*

(b) $\displaystyle\limsup_{t\to\infty}\frac{1}{t}\log\left[\sup\left\{\frac{\|U(t,f_1,\omega)-U(t,f_2,\omega)\|_{L^2}}{\|f_1-f_2\|_{L^2}}:f_1\neq f_2,\ f_1,f_2\in\mathcal{S}(\omega)\right\}\right]\leq\lambda_{i_0}.$

(c) *(Cocycle-invariance of the stable manifolds):*
There exists $\tau_1(\omega)\geq 0$ *such that*

$$U(t,\cdot,\omega)(\mathcal{S}(\omega))\subseteq\mathcal{S}(\theta(t,\omega))\tag{4.13}$$

for all $t \geq \tau_1(\omega)$. Also

$$DU(t, Y(\omega), \omega)(\mathcal{S}^0(\omega)) \subseteq \mathcal{S}^0(\theta(t, \omega)), \quad t \geq 0. \tag{4.14}$$

(d) *$\mathcal{U}(\omega)$ is the set of all $f \in B(Y(\omega), \rho_2(\omega))$ with the property that there is a discrete-time "history" process $y(\cdot, \omega) : \{-n : n \geq 0\} \to L^2([0,1], \mathbf{R})$ such that $y(0, \omega) = f$ and for each integer $n \geq 1$, one has $U(1, y(-n, \omega), \theta(-n, \omega)) = y(-(n-1), \omega)$ and*

$$\|y(-n, \omega) - Y(\theta(-n, \omega))\|_{L^2} \leq \beta_2(\omega) \exp\{-(\lambda_{i_0-1} - \epsilon_2)n\}.$$

If $\lambda_{i_0-1} = \infty$, $\mathcal{U}(\omega)$ is the set of all $f \in B(Y(\omega), \rho_2(\omega))$ with the property that there is a discrete-time "history" process $y(\cdot, \omega) : \{-n : n \geq 0\} \to L^2([0,1], \mathbf{R})$ such that $y(0, \omega) = f$ and for each integer $n \geq 1$,

$$\|y(-n, \omega) - Y(\theta(-n, \omega))\|_{L^2} \leq \beta_2(\omega) \exp\{-\lambda n\},$$

for any $\lambda \in (0, \infty)$. Furthermore, for each $f \in \mathcal{U}(\omega)$, there is a unique continuous-time "history" process also denoted by $y(\cdot, \omega) : (-\infty, 0] \to L^2([0,1], \mathbf{R})$ such that $y(0, \omega) = f$, $U(t, y(s, \omega), \theta(s, \omega)) = y(t+s, \omega)$ for all $s \leq 0, 0 \leq t \leq -s$, and

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\|_{L^2} \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}^0(\omega)$ of the linearized cocycle $(DU(t, Y(\cdot), \cdot), \theta(t, \cdot))$ is tangent at $Y(\omega)$ to $\mathcal{U}(\omega)$, viz. $T_{Y(\omega)}\mathcal{U}(\omega) = \mathcal{U}^0(\omega)$. In particular, $\dim \mathcal{U}(\omega) = \sum_{j=1}^{i_0-1} \dim F_j(\omega)$ is finite and non-random.

(e) *Let $y(\cdot, f_i, \omega), i = 1, 2$, be the history processes associated with $f_i = y(0, f_i, \omega) \in \mathcal{U}(\omega)$, $i = 1, 2$. Then*

$$\limsup_{t \to \infty} \frac{1}{t} \log\Big[\sup\Big\{\frac{\|y(-t, f_1, \omega) - y(-t, f_2, \omega)\|_{L^2}}{\|f_1 - f_2\|_{L^2}} : f_1 \neq f_2,\ f_i \in \mathcal{U}(\omega), i = 1, 2\Big\}\Big] \leq -\lambda_{i_0-1}.$$

(f) *(Cocycle-invariance of the unstable manifolds):*
There exists $\tau_2(\omega) \geq 0$ such that

$$\mathcal{U}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) \tag{4.15}$$

for all $t \geq \tau_2(\omega)$. Also

$$DU(t, \cdot, \theta(-t, \omega))(\mathcal{U}^0(\theta(-t, \omega))) = \mathcal{U}^0(\omega), \quad t \geq 0;$$

and the restriction

$$DU(t, \cdot, \theta(-t, \omega))|\mathcal{U}^0(\theta(-t, \omega)) : \mathcal{U}^0(\theta(-t, \omega)) \to \mathcal{U}^0(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

(g) *The submanifolds* $\mathcal{U}(\omega)$ *and* $\mathcal{S}(\omega)$ *are transversal, viz.*

$$L^2([0,1],\mathbf{R}) = T_{Y(\omega)}\mathcal{U}(\omega) \oplus T_{Y(\omega)}\mathcal{S}(\omega).$$

Theorem 4.3 (Local invariant manifold theorem). *Assume that* $Y : \Omega \to L^2([0,1],\mathbf{R})$ *is a stationary random point for the cocycle* (U,θ) *of the Burgers spde (1.1) with* $E\log^+ \|Y\|_{L^2} < \infty$. *Consider the Lyapunov spectrum* $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ *of the linearized cocycle* $(DU(t,Y(\omega),\omega), \theta(t,\omega), t \geq 0)$ *given in Theorem 4.1. Assume further that* Y *is ergodic in the sense that the Lyapunov exponents* $\lambda_i < 0$ *for all* $i \geq 1$. *Fix* $\epsilon_1 \in (0, -\lambda_1)$. *Then there exist*

(i) *a sure event* $\Omega^* \in \mathcal{F}$ *with* $\theta(t,\cdot)(\Omega^*) = \Omega^*$ *for all* $t \in \mathbf{R}$,

(ii) $\mathcal{F}$*-measurable random variables* $\rho_i, \beta_i \colon \Omega^* \to (0,1)$, $\beta_i > \rho_i \geq \rho_{i+1} > 0$, $i \geq 1$, *such that for each* $\omega \in \Omega^*$, *the following is true:*

There are C^∞ *submanifolds* $\mathcal{S}_i(\omega), i \geq 1$, *of* $B(Y(\omega), \rho_i(\omega))$ *with the following properties:*

(a) $\mathcal{S}_i(\omega)$ *is the set of all* $f \in B(Y(\omega), \rho_i(\omega))$ *such that*

$$|U(n,f,\omega) - Y(\theta(n,\omega))|_H \leq \beta_i(\omega) \exp\{(\lambda_i + \epsilon_1)n\}$$

for all integers $n \geq 0$. *Furthermore,*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t,f,\omega) - Y(\theta(t,\omega))|_H \leq \lambda_i \tag{4.16}$$

for all $f \in \mathcal{S}_i(\omega)$. *The Oseledec space* $E_i(\omega)$ *of the linearized cocycle* $(DU(t,Y(\omega),\cdot), \theta(t,\cdot))$ *is tangent at* $Y(\omega)$ *to the submanifold* $\mathcal{S}_i(\omega)$, *viz.* $T_{Y(\omega)}\mathcal{S}_i(\omega) = E_i(\omega)$. *In particular,* $\operatorname{codim} \mathcal{S}_i(\omega) = \operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} \dim F_j(\omega)$ *(fixed and finite).*

(b)

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log\left[\sup\left\{\frac{|U(t,f_1,\omega) - U(t,f_2,\omega)|_H}{|f_1 - f_2|_H} : f_1 \neq f_2,\ f_1, f_2 \in \mathcal{S}_i(\omega)\right\}\right] \leq \lambda_i.$$

(c) *(Cocycle-invariance):*
There exists $\tau_i(\omega) \geq 0$ *such that*

$$U(t,\cdot,\omega)(\mathcal{S}_i(\omega)) \subseteq \mathcal{S}_i(\theta(t,\omega)) \tag{4.17}$$

for all $t \geq \tau_i(\omega)$. *Also*

$$DU(t,Y(\omega),\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega)), \quad t \geq 0. \tag{4.18}$$

Our final result is the *global invariant flag theorem* which gives a random cocycle-invariant countable global foliation of the energy space $L^2([0,1,\mathbf{R})$ relative to the *ergodic* stationary point Y of Burgers spde (1.1). The reader may adapt the proof of Theorem 4.6 in [16].

Theorem 4.4 (Global invariant flag theorem). *Assume the conditions and notations of Theorem 4.3. Define the family of random sets* $\{M_i(\omega)\colon \omega \in \Omega^*,\ i \geq 1\}$ *by*

$$M_i(\omega) := \left\{ f \in H\colon \varlimsup_{t\to\infty} \frac{1}{t} \log |U(t,f,\omega) - Y(\theta(t,\omega))|_H \leq \lambda_i \right\} \tag{4.19}$$

for $i \geq 1$, $\omega \in \Omega^*$. *For fixed* $i \geq 1, \omega \in \Omega^*$, *define the sequence* $\{S_i^n(\omega)\}_{n=1}^{\infty}$, *inductively by:*

$$S_i^1(\omega) := S_i(\omega), \tag{4.20}$$

$$S_i^n(\omega) := \begin{cases} U(n,\cdot,\omega)^{-1}\big[S_i\big(\theta(n,\omega)\big)\big], & \text{if } S_i^{n-1}(\omega) \subseteq U(n,\cdot,\omega)^{-1}\big[S_i\big(\theta(n,\omega)\big)\big], \\ S_i^{n-1}(\omega), & \text{otherwise,} \end{cases} \tag{4.21}$$

for all $n \geq 2$. *In* (4.20) *and* (4.21), *the* $S_i(\omega)$ *are the local invariant* C^∞ *Hilbert submanifolds of* $L^2([0,1],\mathbf{R})$ *constructed in Theorem 4.3.*

Then the following is true for each $i \geq 1$ *and* $\omega \in \Omega^*$:

(i) The sets $\{M_i(\omega)\colon \omega \in \Omega^*,\ i \geq 1\}$ *are cocycle-invariant:*

$$U(t,\cdot,\omega)\big(M_i(\omega)\big) \subseteq M_i\big(\theta(t,\omega)\big) \tag{4.22}$$

for all $t \geq 0$.

(ii) $S_i^n(\omega) \subseteq S_i^{n+1}(\omega)$ *for all* $n \geq 1$, *and*

$$M_i(\omega) = \bigcup_{n=1}^{\infty} S_i^n(\omega), \qquad i \geq 1. \tag{4.23}$$

(iii) $M_{i+1}(\omega) \subseteq M_i(\omega)$.

(iv) For any $f \in M_i(\omega) \backslash M_{i+1}(\omega)$,

$$\varlimsup_{t\to\infty} \frac{1}{t} \log |U(t,f,\omega) - Y(\theta(t,\omega))|_H \in (\lambda_{i+1}, \lambda_i]. \tag{4.24}$$

Remark. It is not clear if the $M_i(\omega)$ in Theorem 4.4 are C^∞ immersed submanifolds in $L^2([0,1],\mathbf{R})$. This would require transversality of the cocycle $U(n,\cdot,\omega)$ and the local stable manifold $S_i\big(\theta(n,\omega)\big)$.

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