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HARTMAN-GROBMAN THEOREMS ALONG HYPERBOLIC STATIONARY TRAJECTORIES

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Abstract. We extend the Hartman-Grobman theorems for discrete random dynamical systems (RDS), proved in [7], in two directions: for continuous RDS and for hyperbolic stationary trajectories. In this last case there exists a conjugacy between travelling neighborhoods of trajectories and neighborhoods of the origin in the corresponding tangent bundle. We present applications to deterministic dynamical systems.

1. **Introduction.**

The celebrated Hartman-Grobman theorem (HGT, for short) plays a fundamental rule in the theory of dynamical systems. Essentially, among other features, it allows one to make topological classification of the dynamics in a neighborhood of hyperbolic fixed points. This classification is based on the existence of a conjugacy of the local dynamics with the linearized system at a hyperbolic fixed point. For the original papers we mention Hartman [10] and [11], and Grobman [9]. One of the first results concerning HGT in random dynamical systems (RDS, for short) is due to Wanner [19] for discrete systems. His argument was based on random difference equation, such that the construction was made ω-wise. His proof is completed by showing that the choice of random homeomorphisms is, in fact, measurable. In our intrinsic approach, Coayla-Teran and Ruffino [7], we have looked for the conjugacy in an appropriate Banach space of random homeomorphisms. The arguments in [7] correspond to an appropriate extension of the deterministic arguments, with the state space enlarged by the probability space. Hence the norms and other constants were composed with $L^1(\Omega)$ norm. In this article, we extend

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the Hartman-Grobman theorems for discrete RDS's in two directions: To cover continuous-time (hyperbolic) RDS's and hyperbolic stationary trajectories. In this latter case we show that there exists a conjugacy between traveling neighborhoods of the trajectory and neighborhoods of the origin in the corresponding tangent bundle. See figure 1 in Section 4. The proofs of the main results in this article are applications of the local discrete-time random HGT. Cf. the deterministic case in Palis and de Melo [14].

1.1. **Applications in deterministic dynamical systems.** Before we start with the technicalities of random dynamical systems we describe explicitly applications of the main theorems of this article (Theorems 4 and 6) in deterministic dynamical systems. We intend rather to show directions of further problems. The theorems below are particular cases of Corollaries 1 and 2, respectively, one just has to lift the Markovian hypothesis and consider the probability space being an ergodic invariant probability measures in the state space.

Theorem 1 (HGT, discrete chaotic systems)**.** *Let ρ be an ergodic invariant probability measure on $\mathbf{R}^d$ for an application $f : U \subset \mathbf{R}^d \to \mathbf{R}^d$. If the system is hyperbolic ρ-a.s., then, for ρ-a.e. $x \in \mathbf{R}^d$, there exists a homeomorphism $h(x) : U(x) \to W(x)$ where $U(x)$ is a neighborhood of x and $W(x)$ is a neighborhood of the origin in the tangent space $T_x\mathbf{R}^d$, such that:*

$$f(y) = h^{-1}(f(x)) \circ (D_x f(\cdot)) \circ h(x)(y),$$

for all y in the domain of the composition.

Theorem 2 (HGT, continuous deterministic systems)**.** *Let μ be an invariant ergodic probability measure on $\mathbf{R}^d$ for the flow φ_t associated to an ODE. Assume hyperbolicity in μ-a.s.. Then, for μ-a.e. $x \in \mathbf{R}^d$, there exists a homeomorphism $H(x) : U(x) \to W(x)$ where $U(x)$ is a neighborhood of x and $W(x)$ is a neighborhood of the origin in the tangent space $T_x\mathbf{R}^d$ such that:*

$$\varphi_t(y) = [H^{-1}(x_t)(\cdot) \circ D_{x_t}\varphi_t(\cdot) \circ H(x)](y),$$

for all y in the domain of the composition, where $x_t = \varphi_t(x)$ and $t \in [0, 1]$.

Among other systems, we recall that the Lorentz attractors fits perfectly for an analysis in view of Theorem 2, see e.g. Sparrow [17]. The above results suggest that one may apply the ideas of Cong [8] on topological classification of linear cocycles to characterize the dynamics in a neighborhood of stationary hyperbolic orbits; that is, with ergodic invariant measures and non-vanishing Lyapunov spectrum for the linearized cocycle. The article is organized as follows: In section 2 we present preliminaries results; mainly the HGT for hyperbolic random fixed point, Theorem 3. This result was originally proved in [7] and as we said before, this is a key result to the proof of the other theorems in this article. In Section 3 we present the first extension: we lift the fixed point hypothesis and consider hyperbolic stationary orbits on discrete systems. Finally, in Section 4 we extend the HGT to continuous RDS along stationary trajectories.

2. **Definitions and preliminary results.** In this section we present the main technical tools we shall use in the proofs of next sections. It can be skipped if the reader is already familiarized with random dynamical systems (RDS) in terms of cocycle theory, as in L. Arnold [1].

2.1. **A random norm and stationary trajectories.** To set up the notation let $(\theta_t)_{t\in\mathbf{T}}$ be a group of ergodic transformations on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbf{T} = \mathbf{Z}$ or $\mathbf{R}$. A continuous (perfect) cocycle $\varphi(t, \omega)$ on $\mathbb{R}^n$ over θ is a map over the space of local diffeomorphisms $\mathrm{Diff}_{loc}(\mathbf{R}^d)$ denoted by $\varphi : \mathbf{T} \times \Omega \to \mathrm{Diff}_{loc}(\mathbf{R}^d)$ such that for all $\omega \in \Omega$:

(i) $\varphi(0, \omega) = Id$;
(ii) $\varphi(t, \omega)$ is continuous in t;
(iii) it has the cocycle property:

$$\varphi(t + s, \omega) = \varphi(t, \theta_s(\omega)) \circ \varphi(s, \omega).$$

We deal with perfect cocycles, since for every crude cocycle there exists a perfect cocycle that is indistinguishable from it; see L. Arnold and M. Scheutzow [2] or L. Arnold [1]. This cocycle generates the following random system on $\mathbb{R}^n$:

$$x_t = \varphi(t, \omega)x_0. \tag{1}$$

The concept of a cocycle over a measurable transformation on a probability space generalizes many interesting (random or not) dynamical systems, including those which are generated by random equations and stochastic differential equations, see Arnold [1]. Consider a (locally) invertible cocycle $(\varphi(t, \cdot), \theta_t), t \in \mathbf{R}$, with the origin as a fixed point. If

$$\sup_{-1\le t\le 1} \log^+ \|D_0\varphi(t, \cdot)\| \quad \text{and} \quad \sup_{-1\le t\le 1} \log^+ \|D_0\varphi^{-1}(t, \cdot)\|$$

are in $L^1(\Omega, \mathbb{P})$, then the multiplicative (Oseledec) ergodic theorem (MET) establishes a "random linear algebra" for RDS; see e.g. Arnold [1], Ruelle [15], Katok and Hasselblatt [12], among others. Moreover, this "random linear algebra" allows one to introduce a measurable random norm $\|x\|_\omega^2 = \langle x, x\rangle_\omega$, also known as Lyapunov norm (Katok and Hasselblatt [12]). For the definition of this norm and a survey of its properties, the reader may consult [1, Thm. 3.7.4] and the references therein. We only recall that the random norm is such that the exponential behavior of the linearized cocycle $D_0\varphi(t, \cdot)$ imitates the exponential behavior of a deterministic linear systems with respect to the Euclidean norm, up to an exponential ("small") correction term. More precisely: if Λ is the maximum of the modulus of the Lyapunov spectrum and $a > 0$ is smaller than the minimum of the modulus of these exponents, then,

$$e^{-(\Lambda+a)|t|} \|x\|_\omega \le \|D_0\varphi(t, \omega)x\|_{\theta_t(\omega)} \le e^{(\Lambda+a)|t|} \|x\|_\omega \quad \text{for } t \in \mathbf{T}, x \in \mathbf{R}^d.$$

for all $t \in \mathbf{R}$ and $x \in \mathbf{R}^d$. For each $\varepsilon > 0$ there exists a random variable $B_\varepsilon(.) : \Omega \to [1, +\infty)$ which provides an equivalence between the random norm and the Euclidean norm, i.e. for all $x \in \mathbf{R}^d$:

$$\frac{1}{B_\varepsilon(\omega)} \|x\| \le \|x\|_\omega \le B_\varepsilon(\omega) \|x\| \tag{2}$$

Moreover, we have that $B_\varepsilon(\theta_t(\omega)) \le B_\varepsilon(\omega)e^{\varepsilon|t|}$, see [1] or [5]. We now define the

main spaces which we are going to work with, see also [7].

a) $\mathrm{Homeo}(\Omega, \mathbf{R}^d)$ denotes the space of random homeomorphisms given by measurable random fields $h : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ such that for each $\omega \in \Omega$, $h(\omega, \cdot) : \mathbf{R}^d \to \mathbf{R}^d$ is a homeomorphism.

b) $C_b(\Omega, \mathbf{R}^d)$ denotes the space of random bounded continuous maps $u(\omega, \cdot) : \mathbf{R}^d \to \mathbf{R}^d$ such that:

$$\|u\|_{C_b(\Omega,\mathbf{R}^d)} = \mathbb{E}\left[\sup_{x\in\mathbf{R}^d} \|u(\omega,x)\|_\omega\right] < +\infty.$$

c) $C_{0,b}(\Omega, \mathbf{R}^d) \subset C_b(\Omega, \mathbf{R}^d)$ denotes the subspace of random bounded continuous maps which fix the origin, i.e. $u \in C_{0,b}(\Omega, \mathbf{R}^d)$ if $u(\omega, 0) = 0$ a.s.. We shall denote the norm in $C_b(\Omega, \mathbf{R}^d)$ restricted to this subspace by $\|\cdot\|_{C_{0,b}(\Omega,\mathbf{R}^d)}$.

The spaces $C_{0,b}(\Omega, \mathbf{R}^d)$ and $C_b(\Omega, \mathbf{R}^d)$ with the norm defined above are Banach spaces, see [7, Prop. 2.2]. Let $\varphi(t, \omega, \cdot)$ be a cocycle over a family of ergodic transformations $\theta_t : \Omega \to \Omega$ on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with $t \in \mathbf{T}$, where $\mathbf{T} = \mathbf{Z}$ or $\mathbf{R}$. We say that an $\mathcal{F}$-measurable random variable $Y : \Omega \to \mathbf{R}^d$ is a *stationary trajectory* (or *stationary point*) of $\varphi(t, \omega, \cdot)$ if

$$\varphi(t, \omega, Y(\omega)) = Y(\theta_t\omega)$$

for all $t \in \mathbf{T}$ and every $\omega \in \Omega$ ([13]). We say that a stationary trajectory Y is *hyperbolic* if the Lyapunov spectrum of the linearized cocycle $(D_{Y(\omega)}\varphi(t, \omega), \theta_t(\omega))$ (along the stationary trajectory) does not contain zero ([13]). For examples, see Section 3, where we show that support of invariant measures are stationary trajectories. Cf also [13] for related results on the existence of stable and unstable manifolds near a hyperbolic stationary trajectory. For discrete systems, frequently one finds the notation $\varphi(\omega, \cdot)$ and $\theta^k(\omega)$, with $k \in \mathbf{Z}$, more natural than $\varphi(1, \omega, \cdot)$ and $\theta_k(\omega)$, respectively. Hence, $\varphi(k, \omega, \cdot) = \varphi(\theta^{k-1}(\omega), \cdot) \circ \ldots \varphi(\omega, \cdot)$.

2.2. **Random Hartman-Grobman Theorem: Discrete, fixed-point case.** Analogous to most of the proofs of the continuous HGT, the arguments of our proofs of the random versions of this theorem along hyperbolic stationary trajectory are based on the discrete local version for a hyperbolic fixed point of random diffeomorphisms. We shall denote by $C^1_0(\Omega, \mathbf{R}^d)$ the space of random C^1-local diffeomorphisms which fix the origin. The discrete cocycle generated by the pair (f, θ), $f \in C^1_0(\Omega, \mathbf{R}^d)$, is defined by:

$$\varphi(n, \omega) = f(\theta^{n-1}) \circ \ldots \circ f(\theta(\omega)) \circ f(\omega).$$

Its linearization is still a cocycle, i.e., if we denote by $A(\omega) := D_0 f(\omega)$ the derivative of f at the origin, then,

$$D_0\varphi(n, \omega) = A(\theta^{n-1}) \circ \ldots \circ A(\theta(\omega)) \circ A(\omega).$$

For reader's convenience we rewrite the following preliminary Hartman-Grobman result for random diffeomorphisms with hyperbolic fixed points:

Theorem 3 (HGT: fixed point, local discrete case)**.** *With the notations above, let $f \in C^1_0(\Omega, \mathbf{R}^d)$ be a random local diffeomorphism such that the linearized dynamical system generated by $(A(\cdot) := D_0 f(\cdot), \theta)$ is hyperbolic. Then, for $\mathbb{P}$-a.s., there exists a positive random variable $v(\omega)$ and a homeomorphism in a $v(\omega)$-neighborhood of the origin $h \in \mathrm{Homeo}(\Omega, B(0, v(\omega)); h(B(0, v(\omega))))$ such that:*

$$f(\omega, x) = h^{-1}(\theta(\omega))A(\omega)h(\omega)(x),$$

for all x in the domain of the composition. The random conjugacy h is unique of the form $I + u$ with $u \in C_{0,b}(\Omega, \mathbf{R}^d)$.

Proof. The proof is quite technical, see [7]. To figure out the idea of the localization, we recall that there is a requirement that the non-linear component $\Psi = f - A$ is Lipschitz in a neighborhood of the origin with respect to the random norm with an apropriate Lipschitz constant L. The relation between this constant and the random radius $0 \le v(\omega) \le 1$ is such that for all x in the ball $B(0, 2v(\omega))$ we have

$$\|D_x \Psi(\omega, \cdot)\| \le \frac{L}{6\, e\, B(\omega)^2}.$$

Locally this is obviously satisfied since $D_x\Psi$ is continuous with respect to x and $D_0\Psi \equiv 0$. □

3. **Discrete case with stationary orbits.** This section presents a generalization of Theorem 3 to hyperbolic stationary orbits, in particular, to hyperbolic invariant probability measure. It corresponds to the first extension of HGT for fixed points. We shall decompose a C^1 cocycle around a stationary trajectory by writing:

$$\varphi(\omega, \cdot) = D_{Y(\omega)}\varphi(\omega, \cdot) + \Psi(\omega, \cdot),$$

where $D_{Y(\omega)}\varphi$ is the derivative of the cocycle φ at the stationary point $Y(\omega)$ and Ψ is its non-linear part.

Theorem 4 (HGT, discrete stationary orbit)**.** *Let $Y(\omega)$ be a hyperbolic stationary orbit for a discrete-time C^1-cocycle $\varphi(k, \omega, \cdot)$. Then, there exists a random homeomorphism $h(\omega) : U(\omega) \to W(\omega)$ where $U(\omega)$ is a random neighborhood of $Y(\omega)$ and $W(\omega)$ is a neighborhood of the origin in the tangent space $T_{Y(\omega)}\mathbf{R}^d$, such that:*

$$\varphi(\omega, x) = \Big[h^{-1}(\theta(\omega)) \circ \big(D_{Y(\omega)}\varphi(\omega, \cdot)\big) \circ h(\omega) \Big](x),$$

for all x in the domain of the composition.

Proof. The argument is based on using a convenient centralization of the cocycle around the stationary orbit in the following sense: Define a new cocycle $(\hat\varphi, \theta)$ by

$$\hat\varphi(\omega, x) := \varphi(\omega, x + Y(\omega)) - Y(\theta(\omega)), \quad \omega \in \Omega,\ x \in \mathbf{R}^d.$$

One may easily check that $(\hat\varphi, \theta)$ is a C^1 cocycle with the origin as a hyperbolic fixed point. The linearized system is now given by $D_0\hat\varphi(\omega, \cdot) = D_{Y(\omega)}\varphi(\omega, \cdot) \circ I$, where the identity $I : T_0\mathbf{R}^d \to T_{Y(\omega)}\mathbf{R}^d$ is the derivative of the translation $x \mapsto x + Y(\omega)$. By Theorem 3, there exists a local random topological conjugacy $\hat h(\omega) : \hat U(\omega) \to \hat V(\omega)$ where $\hat U(\omega)$ and $\hat V(\omega)$ are open neighborhood of the origin in $\mathbf{R}^d$, such that

$$\hat h(\theta(\omega)) \circ \hat\varphi(\omega, \cdot) = D_0\hat\varphi(\omega, \cdot) \circ \hat h(\omega).$$

Finally, just define:

$$h(\omega)(x) = \hat h(\omega)(x - Y(\omega)).$$

The statement of the theorem then holds with $U(\omega) = Y(\omega) + \hat U(\omega)$. □

Let μ be an ergodic invariant probability measure on $\mathbf{R}^d$ for a given random dynamical systems generated by (φ, θ) as in the hypothesis of Theorem 4 (in particular for a Markovian i.i.d. systems). One method of finding stationary orbits is constructing an equivalent cocycle on an enlarged underlying probability space. Let $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be the following probability space:

$$\tilde\Omega = \Omega \times \mathbf{R}^d,\ \tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d),\ \tilde{\mathbb{P}} = \mathbb{P} \otimes \mu.$$

The family of ergodic transformation in $\tilde{\Omega}$ will be defined by:

$$\tilde{\theta}(\tilde{\omega}) = (\theta(\omega), \varphi(\omega, x))$$

with $\tilde{\omega} = (\omega, x)$. One easily sees that $\tilde{\theta}$ is $\tilde{\mathbb{P}}$-preserving and ergodic, Carverhill [6]. We define the new equivalent cocycle $\tilde{\varphi}(\tilde{\omega}, x)$ by:

$$\tilde{\varphi}(\tilde{\omega}, x) = \varphi(\omega, x).$$

Note that $\tilde{Y} : \tilde{\Omega} \to \mathbf{R}^d$ given by $\tilde{Y}(\tilde{\omega}) := x$ is a stationary orbit for the cocycle generated by $(\tilde{\varphi}, \tilde{\theta})$ over the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ (cf. [13]).

Corollary 1 (HGT for invariant probability measures)**.** *Let μ be an invariant ergodic probability measure on $\mathbf{R}^d$ for a Markov process generated by a C^1-cocycle φ. If the system is hyperbolic μ-a.s., then, for μ-a.e. $x \in \mathbf{R}^d$, there exists a random homeomorphism $h(\omega, x) : U(\omega, x) \to W(\omega, x)$ where $U(\omega, x)$ is a random local neighborhood of x and $W(\omega, x)$ is a neighborhood of the origin in $T_x\mathbf{R}^d$, such that:*

$$\varphi(\omega, y) = \Big[h^{-1}(\theta(\omega), \varphi(\omega, x)) \circ (D_x\varphi(\omega, \cdot)) \circ h(\omega, x)\Big](y),$$

for all y in the domain of the composition (which depends on (ω, x)).

Proof. As stated before, $\tilde{Y}(\tilde{\omega})$ is a stationary trajectory for the cocycle $\tilde{\varphi}(\tilde{\omega}, \cdot)$. By Theorem 4, for each $\tilde{\omega} = (x, \omega)$ there exists a neighborhood $V_{(x,\omega)}$ and a random homeomorphism $\tilde{h}(\tilde{\omega}) : V_{(x,\omega)} \to W_{(x,\omega)} \subset T_x\mathbf{R}^d$ such that

$$\tilde{h}(\tilde{\theta}(\tilde{\omega}))[\tilde{\varphi}(\tilde{\omega}, y)] = [D_x\tilde{\varphi}(\omega, \cdot) \circ \tilde{h}(\tilde{\omega})](y), \quad y \in V_{(x,\omega)}.$$

□

Example: HGT for random perturbation of hyperbolic systems. Let $F :$

$U \subset \mathbf{R}^n \to U$ be a hyperbolic local diffeomorphism. We shall denote a random perturbation in a certain parameter of F by $F(\omega, x)$, that is, F is now a random variables in the space of local diffeomorphisms, where ω is in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For many interesting case, the random system generated by H will be a cocycle with respect to an ergodic transformation $\theta : \Omega \to \Omega$, that is:

$$F^{(n)}(\omega, x) = F(\theta^{n-1}\omega, \cdot) \circ F(\theta^{n-2}\omega, \cdot) \circ \ldots \circ F(\theta\omega, \cdot) \circ F(\omega, x).$$

Assume that the system $F(\omega, x)$ is still hyperbolic and that ρ is an (ergodic) invariant probability measure on $\mathbf{R}^n$ for this perturbed system. By the corollary above, there exists a random homeomorphism $h(\omega, x) : V(\omega, x) \to W(\omega, x)$ defined in the probability space $(U \times \Omega, \mathcal{B} \otimes \mathcal{F}, \mathbf{P} \otimes \rho)$, with the ergodic transformation $(\omega, x) \mapsto (\theta\omega, F(\omega, x))$ such that, if y is in the corresponding neighborhood $V(\omega, x)$, then :

$$F(\omega, y) = h^{-1}\left(\theta\omega, F(\omega, x)\right) \circ dF_x(\omega) \circ h\left(\omega, x\right)(y).$$

4. **Continuous Case.** To establish a Hartman-Grobman theorem for hyperbolic stationary trajectories, we shall first establish the result for a continuous-time C^1-cocycle φ which has the origin as a hyperbolic fixed point. We shall assume that the cocycle φ is a continuous process in the group of C^1 diffeomorphisms. In fact most of the interesting cocycles in finite dimensional state space, independently of how they are generated (stochastic differential equation, random equation, etc) satisfy this assumption: See Arnold [1] and references therein. Initially, consider discrete-time systems generated by the time-one map $\varphi(1, \omega, \cdot)$. If the origin is a hyperbolic fixed point for the continuous system, it is also a hyperbolic fixed point for this discretization. Let $V(\omega) := B(0, v(\omega))$ be the neighborhood of conjugacy for this discrete cocycle, as stated in Theorem 3. With this notation we have:

Lemma 1. *Let $\varphi(t, \omega, \cdot)$ be a C^1-cocycle on $\mathbf{R}^d$ such that the origin is a hyperbolic fixed point. There exists a random variable $u : \Omega \to \mathbf{R}^+$ such that $u(\omega) \leq v(\omega)$ for all $\omega \in \Omega$, and*

$$\varphi(t, \omega, x) \in V(\theta_t(\omega)),$$

for all $x \in U(\omega) := B(0, u(\omega))$ and $t \in [0, 2]$. Obviously $u(\omega) \leq v(\omega)$.

Proof. First, note that, by hyperbolicity, the derivative $D_0\varphi(t, \omega, \cdot)$ is a non-singular matrix, hence, there exists a ball centered at zero with radius $R(t, \omega) > 0$ where $\varphi(t, \omega, \cdot)$ is invertible. Furthermore, the process $R(t, \omega)$ may be selected to be jointly measurable and sample-continuous in t. Let $R(\omega)$ be the (measurable) random radius defined by

$$R(\omega) = \inf\{R(t, \omega); t \in [0, 2]\}.$$

By continuity of $R(t, \omega)$ in t, it follows that $R(\omega)$ is strictly positive. We claim that $\varphi(1, \theta_s(\omega), x)$ is continuous in $s \in [0, 2]$ for all x in $B(0, R(\omega))$. In fact, in this case:

$$\varphi(1, \theta_s(\omega), x) = \phi(1 + s, \omega, \cdot) \circ \phi(s, \omega, x)^{-1},$$

and the expression on the right hand side is continuous in s. The continuity of $\varphi(1, \theta_s(\omega), x)$ in s implies that the radius of conjugacy in the discrete (time-one) case $v(\theta_s(\omega))$, as defined in the proof of Theorem 3, is also continuous in $s \in [0, 2]$. Hence,

$$r := \inf\{v(\theta_s(\omega)), s \in [0, 2]\} > 0,$$

for all $\omega \in \Omega$. By continuity of $\varphi(t, \omega, x)$ in t, it follows that for each ω there exists a strictly positive radius $u(\omega)$ such that $\|\varphi(t, \omega, x)\| < r(\omega)$ for all x with $\|x\| < u(\omega)$. Moreover, one can choose $u(\omega)$ such that it is measurable in ω. □

In view of the above lemma, the local HGT may be proved using the same ideas as in the global version, [7, Thm 5.1]. This is in fact a "random" adaptation of the ideas in Palis and de Melo [14] or S. Sternberg [18, Lemma 4]. We shall write, as before

$$\varphi(t, \omega, \cdot) = \Phi(t, \omega, \cdot) + \Psi(t, \omega, \cdot),$$

where Φ is the linearization at the origin $\Phi(t, \omega, \cdot) = D_0\varphi(t, \omega, \cdot)$.

Theorem 5 (HGT for continuous cocycles, fixed point)**.** *Consider a C^1-cocycle $(\varphi(t, \omega), \theta_t)$ on $\mathbf{R}^d$, with the origin as a hyperbolic fixed point. Then there exists a random conjugacy between φ and its linearization $\Phi(t, \omega) = D_0\varphi(t, \omega)$. More precisely: there exists a random neighborhood $U(\omega) = B(0, u(\omega))$ of 0 and a measurable random homeomorphism $H(\omega) : U(\omega) \to H(\omega)(U(\omega))$ such that*

$$\varphi(t, \omega, x) = [H^{-1}(\theta_t(\omega))\Phi(t, \omega)H(\omega)](x)$$

for all x in $U(\omega)$ and $t \in [-1, 1]$.

Proof. Let h be the random conjugacy of Theorem 4 for the discrete system generated by $\varphi(1, \omega, x)$. Let $u(\omega)$ be the strictly positive random radius defined in Lemma 4.1. For $x \in U(\omega) := B(0, u(\omega))$, define:

$$H(\omega)(x) := \int_0^1 \Phi(-s, \theta_s(\omega))h(\theta_s(\omega))\varphi(s, \omega, x)ds. \tag{3}$$

Lemma 1 guarantees that the composition makes sense and that the integrand is continuous in s. Hence $H(\omega)$ is well-defined. Note also that, by hyperbolicity, $\Phi(s, \omega)$ is an invertible cocycle and $\Phi(-s, \theta_s(\omega)) = \Phi(s, \omega)^{-1}$. Let $t \in [-1, 1]$. Then

$$\Phi(t, \omega) \circ H(\omega) = \int_0^1 \Phi(t - s, \theta_s(\omega)) \circ h(\theta_s(\omega)) \circ \varphi(s - t, \theta_t(\omega), \cdot)ds \ \circ \varphi(t, \omega).$$

Using the change of variable $r = s - t$, we have:

$$\begin{aligned} \Phi(t, \omega) \circ H(\omega) &= \int_{-t}^{1-t} \Phi(-r, \theta_{r+t}(\omega)) \circ h(\theta_{r+t}(\omega)) \circ \varphi(r, \theta_t(\omega), .)dr \ \circ \varphi(t, \omega) \\ &= \left[\int_{-t}^{0} \Phi(-r, \theta_{r+t}(\omega)) \circ h(\theta_{r+t}(\omega)) \circ \varphi(r, \theta_t(\omega), \cdot)dr \right. \\ &\quad \left. + \int_0^{1-t} \Phi(-r, \theta_{r+t}(\omega)) \circ h(\theta_{r+t}(\omega)) \circ \varphi(r, \theta_t(\omega), \cdot)dr \right] \ \circ \varphi(t, \omega). \end{aligned}$$

The first integral inside the bracket equals:

$$\begin{aligned} \int_{-t}^{0} \Phi(-r - 1, \theta_{r+t+1}(\omega)) \circ [\Phi(1, \theta_{t+r}(\omega)) \circ h(\theta_{r+t}(\omega)) \circ \varphi(-1, \theta_{r+t+1}(\omega), \cdot)] \\ \circ \varphi(r + 1, \theta_t(\omega), \cdot)dr. \end{aligned} \tag{4}$$

But Theorem 3 states that

$$\Phi(1, \theta_{t+r}(\omega)) \circ h(\theta_{r+t}(\omega)) \circ \varphi(-1, \theta_{r+t+1}(\omega)) = h(\theta(\theta_{r+t}(\omega))).$$

Note that the domain makes sense because the formula above is applied to a point

$$[\varphi(r + 1, \theta_t(\omega)) \circ \varphi(t, \omega)](x) = \varphi(t + r + 1, \omega, x) \in B(0, u(\theta_{t+r+1}(\omega))),$$

with $t + r + 1 < 2$. Hence, expression (4) equals:

$$\int_{-t}^{0} \Phi(-r - 1, \theta_{r+t+1}(\omega)) \circ h(\theta(\theta_{r+t}(\omega))) \circ \varphi(r + 1, \theta_t(\omega), \cdot)dr,$$

which, changing variables again with $s = r + 1$, becomes:

$$\int_{1-t}^{1} \Phi(-s, \theta_{s+t}(\omega)) \circ h(\theta_{s+t}(\omega))) \circ \varphi(s, \theta_t(\omega), \cdot)ds.$$

Hence,

$$\begin{aligned}\Phi(t,\omega)\circ H(\omega) &= \left[\int_{1-t}^{1}\Phi(-s,\theta_{s+t}(\omega))\circ h(\theta_{s+t}(\omega))\circ\varphi(s,\theta_t(\omega),\cdot)ds\right.\\ &\quad \left.+\int_0^{1-t}\Phi(-r,\theta_{r+t}(\omega))\circ h(\theta_{r+t}(\omega))\circ\varphi(r,\theta_t(\omega),\cdot)dr\right]\circ\varphi(t,\omega)\\ &= \left[\int_0^1\Phi(-s,\theta_{s+t}(\omega))\circ h(\theta_{s+t}(\omega)))\circ\varphi(s,\theta_t(\omega),\cdot)ds\right]\circ\varphi(t,\omega)\\ &= H(\theta_t(\omega))\circ\varphi(t,\omega).\end{aligned}$$

We still have to prove that $H(\omega)$ is indeed a homeomorphism. We shall conclude this fact by showing that $H = I + u$ with u in the space $C_{0,b}(\Omega, \mathbf{R}^d)$. Therefore, by the uniqueness stated in Theorem 3, $H(\omega) = h(\omega)$ a.s.. For readers convenience, we leave this part of the prove to be done in the next lemma. □

Lemma 2. *The random conjugacy H has the form $H = I + u$ with u in the space $C_{0,b}(\Omega, \mathbf{R}^d)$.*

Proof. We write

$$H(\omega) = I + \int_0^1 \Phi(-s,\theta_s(\omega))(\Psi(s,\omega) + u(\theta_s(\omega))\varphi(s,\omega))ds$$

and note that the component

$$\int_0^1 \Phi(-s,\theta_s(\omega))u(\theta_s(\omega))\circ\varphi(s,\omega)ds$$

belongs to $C_{0,b}(\Omega, \mathbf{R}^d)$, since $u \in C_{0,b}(\Omega, \mathbf{R})$ and Φ is linear. Moreover, given an $\varepsilon > 0$, there exists a random variable $\delta(\omega) \leq u(\omega)$ such that, if $x \in B(0, \delta(\omega))$, then for $s \in [0, 1]$:

$$\|\Psi(s,\omega,x)\|_{\theta_s(\omega)} \leq \varepsilon.$$

Hence the other component

$$\int_0^1 \Phi(-s,\theta_s(\omega))\circ\Psi(s,\omega)$$

also belongs to $C_{0,b}(\Omega, \mathbf{R}^d)$. □

Theorem 6 (HGT for continuous stationary trajectories)**.** *Let $Y(\omega)$ be a hyperbolic stationary trajectory for a C^1 cocycle $\varphi(t,\omega,\cdot)$. Then, there exists a random homeomorphism $H(\omega) : U(\omega) \to W(\omega)$ where $U(\omega)$ is a neighborhood of $Y(\omega)$ and $W(\omega)$ is a neighborhood of origin in the tangent space $T_{Y(\omega)}\mathbf{R}^d$ such that:*

$$\varphi(t,\omega) = H^{-1}(\theta_t(\omega))\circ\ D_{Y(\omega)}\varphi(t,\omega)\ \circ H(\omega),$$

for all $t \in [-1, 1]$.

Proof. Again, using the same centralization argument as in Theorem 4, define a new cocycle $\hat{\varphi}$ by:

$$\hat{\varphi}(t,\omega,x) := \varphi(t,\omega,x+Y(\omega)) - Y(\theta_t(\omega)).$$

The linearized flow is now given by $D_0\hat{\varphi}(t,\omega) = D_{Y(\omega)}\varphi(t,\omega)\circ I$, where the identity $I : T_0\mathbf{R}^d \to T_{Y(\omega)}\mathbf{R}^d$ is the derivative of the translation $x \mapsto x + Y(\omega)$ at $x = 0$.

By Theorem 5, there exists a random conjugacy $\hat{H}(\omega) : U(\omega) \to V(\omega)$ where $U(\omega)$ and $V(\omega)$ are open neighborhood of the origin in $\mathbf{R}^d$, such that

$$\hat{\varphi}(t, \omega, \cdot) = \hat{H}^{-1}(\theta_t(\omega)) \circ \ D_0\hat{\varphi}(t, \omega, \cdot) \ \circ \hat{H}(\omega).$$

Finally, as in the discrete case, just define:

$$H(\omega)(x) = \hat{H}(\omega)(x - Y(\omega)).$$

□

Note that, if φ is a solution flow of an Itô stochastic differential equation, the cocycle $\hat{\varphi}$ is non-adapted (hence is not a solution of any Itô stochastic equation). Due to this fact, there is a good advantage on having a HGT for general cocycle instead of the particular case of solutions of stochastic differential equations. Suppose the cocycle $(\varphi(t, \cdot), \theta_t)$ generates a Markov processes (e.g. flows of stochastic differential equations). Suppose the Markov process has an ergodic invariant probability measure μ on $\mathbf{R}^d$. This induces a stationary trajectory for an equivalent cocycle on a suitably enlarged probability space: Consider the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, as defined prior to Corollary 1. The family of ergodic $\tilde{\mathbb{P}}$-preserving transformations is now given by:

$$\tilde{\theta}_t(\tilde{\omega}) := (\theta_t(\omega), \varphi(t, \omega, x))$$

Define a new cocycle $\tilde{\varphi}(t, \tilde{\omega}, \cdot)$ by:

$$\tilde{\varphi}(t, \tilde{\omega}, x) = \varphi(t, \omega, x).$$

Note that $\tilde{Y} : \tilde{\Omega} \to \mathbf{R}^d$ given by $\tilde{Y}(\tilde{\omega}) = x$ is a stationary trajectory for the cocycle $(\tilde{\varphi}(t, \cdot), \tilde{\theta})$ over the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ ([13]).

Corollary 2 (HGT on invariant probability measures)**.** *Let μ be an invariant ergodic probability measure on $\mathbf{R}^d$ for a Markov process associated with the cocycle φ (e.g. a solution of a stochastic differential equation). Assume hyperbolicity μ-a.s.. Then, for μ-a.e. $x \in \mathbf{R}^d$, there exists a random homeomorphism $H(\omega, x) : U(\omega, x) \to W(\omega, x)$ where $U(\omega, x)$ is a random neighborhood of x and $W(\omega, x)$ is a neighborhood of the origin in the tangent space $T_x\mathbf{R}^d$ such that:*

$$\varphi(t, \omega, y) = [H^{-1}(\theta_t(\omega)), x_t(\omega)) \circ D_{x_t(\omega)}\varphi(t, \omega) \circ H(\omega, x)](y),$$

for all y in the domain of the composition, where $x_t(\omega) := \varphi(t, \omega, x)$ and $t \in [0, 1]$.

Proof. The proof is essentially the same as in the discrete case. Take $\tilde{Y}(\omega)$ the stationary trajectory for the cocycle $\tilde{\varphi}(t, \tilde{\omega}, \cdot)$. By Theorem 6, for each $\tilde{\omega} = (\omega, x)$ there exists a neighborhood $V_{(\omega,x)}$ and a random homeomorphism $\tilde{H}(\tilde{\omega}) : V_{(\omega,x)} \to W_{(\omega,x)} \subset T_x\mathbf{R}^d$ such that

$$\tilde{H}(\tilde{\theta}_t(\tilde{\omega})) \circ \tilde{\varphi}(t, \tilde{\omega}, x) = [T_{\tilde{Y}(\tilde{\omega})}\tilde{\varphi}(t, \omega) \circ \tilde{H}(\tilde{\omega})](x).$$

Now state the natural definition $H(\omega, x) = \tilde{H}(\tilde{\omega})$. □

Figure 1 illustrates the result of Theorem 6. The linearized trajectory $v_t = D\varphi(v_0)$ equals the image $H(\omega, x_t(\omega))(y_t)$ of some $y_t = \varphi(t, \omega, y_0)$ with y_t in the neighborhood $V(\theta_t(\omega), x_t(\omega))$ for all $t \in [-1, 1]$. It illustrates the fact that the linearized cocycle $D\varphi$ together with the random homeomorphism H characterize completely the non-linear cocycle φ in a neighborhood of a hyperbolic stationary trajectory. For stochastic dynamical systems on a Riemannian manifold, this centralization argument does not work straightforward once the group structure of $\mathbf{R}^d$

is intrinsic in the argument. Nevertheless, by local coordinates, the conjugacy in $\mathbf{R}^d$ is transported to the manifold, namely:

Corollary 3. *Let φ be a C^1-cocycle on a Riemannian manifold M. If $Y(\omega)$ is a hyperbolic stationary point, then, there exist a random local homeomorphisms $H(\omega) : U(\omega) \subset M \to W(\omega)$ where $W(\omega)$ is a neighborhood of origin in $T_{Y(\omega)}M$ such that:*

$$\varphi(t,\omega) = H^{-1}(\theta_t(\omega)) \circ \ D_{Y(\omega)}\varphi(t,\omega,\cdot) \ \circ H(\omega),$$

on the domain of the composition (restricted to local coordinate neighborhoods).

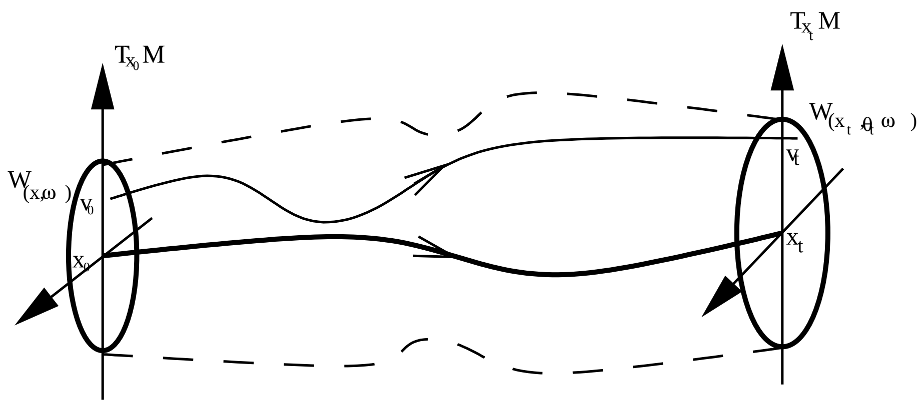

Figure 1: Tubular neighborhood of conjugacy in the tangent bundle.

We recall the example of Baxendale [4] of a hyperbolic Brownian motion on the flat torus T^2 (moreover, with non-zero rotation number, see Ruffino [16]). The ergodic invariant probability measure is the Lebesgue measure λ. The corollary above states that for $\mathbb{P}\otimes\lambda$-a.e. (ω, x_0), given the trajectory $x_t(\omega) = \varphi_t(\omega, x_0)$ of this Brownian motion, there is a 'traveling' neighborhood of $x_t(\omega)$ such that the dynamics around $x_t(\omega)$ in the torus is conjugate to the linearized dynamics in the tangent bundle.

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