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LARGE DEVIATIONS FOR STOCHASTIC SYSTEMS WITH MEMORY

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Abstract. In this paper, we develop a large deviations principle for stochastic delay equations driven by small multiplicative white noise. Both upper and lower large deviations estimates are established.

1. **Introduction.** Large deviations were studied by many authors beginning with the fundamental work of Donsker and Varadhan [4],[5],[6]. Subsequently several issues concerning large deviation principles and their applications to stochastic differential equations were studied by many authors, e.g. Freidlin and Wentzell [8], Stroock [18], Deuschel and Stroock [3], den Hollander [2], and others.

However, there is little published work on large deviations for stochastic systems with memory. The problem of large deviations for such systems was first studied by M. Scheutzow [16] within the context of additive white noise.

Stochastic systems with memory (or stochastic differential delay equations (sdde's)) serve as viable models in a variety of applications, ranging from economics and finance to signal processing (Elsanosi, Øksendal and Sulem [7], Kolmanovskii and Myshkis [10]). The origins of the qualitative theory of stochastic systems with memory goes back to work by Itô and Nisio [9], Kushner [11], Mizel and Trutzer [13], Mohammed [14], Scheutzow [17], Mao [12] and others.

In this paper we examine the question of small random perturbations of systems with memory and the associated problem of large deviations. Our analysis allows for multiplicative noise with possible dependence on the history in the diffusion coefficient. Our approach is similar to that in [1] and [18], but introduces a new induction argument in order to handle the delay.

2. **Basic Setting and Notation.** Let $W_t := \left(W_t^1, W_t^2, ..., W_t^l\right)$ denote a standard l-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, with $W_0 = 0$. Let $b = (b_1, b_2, ..., b_d) : \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d, \sigma = (\sigma_{ij})_{i=1,\cdots d, j=1,\cdots,l} : \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^l$ be Borel measurable functions. We introduce the following conditions:

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(A1) The functions b, σ satisfy a Lipschitz condition. That is, there exist constants L_1, L_2 such that for all x_1, x_2, y_1, y_2 and $t \in [0, \infty)$,

$$||\sigma(t, x_1, y_1) - \sigma(t, x_2, y_2)||_{\mathbf{R}^d \otimes \mathbf{R}^l} \leq L_1(|x_1 - x_2| + |y_1 - y_2|), \tag{1}$$

$$||b(t, x_1, y_1) - b(t, x_2, y_2)||_{\mathbf{R}^d} \leq L_2(|x_1 - x_2| + |y_1 - y_2|). \tag{2}$$

(A2) The functions $b(\cdot, x, y)$, $\sigma(\cdot, x, y)$ are continuous on $[0, \infty)$, uniformly in $x, y \in \mathbf{R}^d$, i.e.,

$$\lim_{s \to t} \sup_{x,y \in \mathbf{R}^d} |b(s, x, y) - b(t, x, y)| = 0, \tag{3}$$

$$\lim_{s \to t} \sup_{x,y \in \mathbf{R}^d} |\sigma(s, x, y) - \sigma(t, x, y)| = 0. \tag{4}$$

Let $\tau > 0$ be a fixed delay, and ψ be a given continuous function on $[-\tau, 0]$. Consider the following differential delay equation (dde):

$$\begin{aligned} dX(t) &= b(t, X(t), X(t-\tau))\, dt, \quad t \in (0, \infty) \\ X(t) &= \psi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{5}$$

and the associated perturbed sdde:

$$\begin{aligned} &dX^\varepsilon(t) = b\big(t, X^\varepsilon(t), X^\varepsilon(t-\tau)\big)dt + \varepsilon^{\frac{1}{2}} \sigma\big(t, X^\varepsilon(t), X^\varepsilon(t-\tau)\big)dW_t,\ t \in (0, \infty) \\ &X^\varepsilon(t) = \psi(t), t \in [-\tau, 0], \end{aligned} \tag{6}$$

with solution X^ε.

Throughout this paper, we will assume, without loss of generality, that the delay τ is equal to 1.

3. Statement of the Main Theorem and Proofs.

Let $C_0\big([0, m], \mathbf{R}^l\big)$ denote the space of all continuous functions $g : [0, m] \to \mathbf{R}^l$ with $g(0) = 0$. If $g \in C_0\big([0, m], \mathbf{R}^l\big)$ is absolutely continuous, set $e(g) = \int_0^m |\dot{g}(t)|^2 dt$. Otherwise, define $e(g) = \infty$. Let $F(g)$ be the solution to the dde

$$\begin{aligned} F(g)(t) &= F(g)(0) + \int_0^t b\big(s, F(g)(s), F(g)(s-1)\big)ds \\ &\quad + \int_0^t \sigma\big(s, F(g)(s), F(g)(s-1)\big)\dot{g}(s)ds, \quad 0 < t \leq m \\ F(g)(t) &= \psi(t), \qquad -1 \leq t \leq 0. \end{aligned} \tag{7}$$

Denote by $C_\psi\big([-1, m], \mathbf{R}^d\big)$ the set of all continuous functions $f : [-1, m] \to R^d$ such that $f(t) = \psi(t)$ for all $t \in [-1, 0]$.

Theorem 3.1. *Let μ_ε be the law of $X^\varepsilon(\cdot)$ on $C_\psi\big([-1, m], \mathbf{R}^d\big)$, equipped with the uniform topology. The family $\{\mu_\varepsilon, \varepsilon > 0\}$ satisfies a large deviation principle with the following good rate function*

$$I(f) := \inf \left\{ \frac{1}{2} e(g); F(g) = f \right\}, \qquad f \in C_\psi([-1, m], \mathbf{R}^d). \tag{8}$$

That is,

(i) *for any closed subset $C \subset C_\psi([-1, m], \mathbf{R}^d)$,*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(C) \leq - \inf_{f \in C} I(f), \tag{9}$$

(ii) *for any open subset* $G \subset C_\psi([-1,m],\mathbf{R}^d)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \geq -\inf_{f\in G} I(f). \tag{10}$$

The rest of the paper is devoted to the proof of this result. The proof is split into several lemmas.

For any $\varepsilon > 0$ and any $n \geq 1$, denote by $X_n^\varepsilon(\cdot)$ the solution to the sdde:

$$\begin{aligned} X_n^\varepsilon(t) &= X_n^\varepsilon(0) + \int_0^t b\big(s, X_n^\varepsilon(s), X_n^\varepsilon(s-1)\big)ds \\ &\quad + \varepsilon^{\frac{1}{2}} \int_0^t \sigma\left(\frac{[ns]}{n}, X_n^\varepsilon\left(\frac{[ns]}{n}\right), X_n^\varepsilon\left(\frac{[ns]}{n}-1\right)\right) dW_s, \quad t>0, \\ X_n^\varepsilon(t) &= \psi(t), \quad t\in[-1,0]. \end{aligned} \tag{11}$$

We need the following lemma from Stroock [18] (p. 81).

Lemma 3.2. *Let* $\alpha : [0,\infty)\times\Omega \to \mathbf{R}^d\otimes\mathbf{R}^l$ *and* $\beta : [0,\infty)\times\Omega\to\mathbf{R}^d$ *be* $(\mathcal{F}_t)_{t\geq 0}$*-progressively measurable processes. Assume that* $||\alpha(\cdot)|| \leq A$ *and* $||\beta(\cdot)|| \leq B$*, where the norm of* α *is the Hilbert-Schmidt norm and the norm of* β *is the usual norm in* $\mathbf{R}^d$*. Set* $\xi(t) := \int_0^t \alpha(s)dW_s + \int_0^t \beta(s)ds$ *for* $t\geq 0$*. Let* $T>0$ *and* $R>0$ *satisfy* $d^{\frac{1}{2}}BT < R$*. Then*

$$P\big(\sup_{0\leq t\leq T} |\xi(t)| \geq R\big) \leq 2d\exp\big(-(R-d^{\frac{1}{2}}BT)^2/2A^2dT\big). \tag{12}$$

Lemma 3.3. *In addition to (A.1) and (A.2), assume that* b,σ *are bounded. Then for any* $m\geq 1, \delta>0$*, the following is true:*

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P\big(\sup_{-1\leq t\leq m}|X^\varepsilon(t)-X_n^\varepsilon(t)|>\delta\big) = -\infty. \tag{13}$$

Proof. We prove (13) by induction on m. We first prove it for $m=1$. Set $Y_n^\varepsilon(t) := X^\varepsilon(t) - X_n^\varepsilon(t),\ t\geq 0$. Then

$$\begin{aligned} Y_n^\varepsilon(t) &= \int_0^t \big[b\big(s, X^\varepsilon(s), X^\varepsilon(s-1)\big) - b\big(s, X_n^\varepsilon(s), X_n^\varepsilon(s-1)\big)\big]\,ds \\ &\quad + \varepsilon^{\frac{1}{2}}\int_0^t \left[\sigma\Big(s, X^\varepsilon(s), X^\varepsilon(s-1)\Big) - \sigma\left(\frac{[ns]}{n}, X_n^\varepsilon\left(\frac{[ns]}{n}\right),\right.\right. \\ &\quad \left.\left. X_n^\varepsilon\left(\frac{[ns]}{n}-1\right)\right)\right] dW_s, \quad t\geq 0. \end{aligned} \tag{14}$$

For $\rho>0$, define $\tau_{n,\rho}^\varepsilon := \inf\{t\geq 0; |X_n^\varepsilon(t) - X_n^\varepsilon(\frac{[nt]}{n})| \geq \rho\}$, and set $Y_{n,\rho}^\varepsilon(t) := Y_n^\varepsilon(t\wedge\tau_{n,\rho}^\varepsilon),\ t\geq 0,\ \xi_{n,\rho}^\varepsilon := \inf\{t\geq 0, |Y_{n,\rho}^\varepsilon(t)|\geq\delta\}$. Then

$$\begin{aligned} P\big(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)|>\delta\big) &= P\big(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)|>\delta, \tau_{n,\rho}^\varepsilon\leq 1\big) \\ &\quad + P\big(\sup_{0\leq t\leq 1}|Y_n^\varepsilon(t)|>\delta, \tau_{n,\rho}^\varepsilon > 1\big) \\ &\leq P(\tau_{n,\rho}^\varepsilon\leq 1) + P(\xi_{n,\rho}^\varepsilon\leq 1). \end{aligned} \tag{15}$$

Observe that

$$P(\tau_{n,\rho}^\varepsilon \leq 1) \leq \sum_{k=1}^n P\left(\sup_{\frac{k-1}{n}\leq t\leq\frac{k}{n}} |X_n^\varepsilon(t) - X_n^\varepsilon\left(\frac{k-1}{n}\right)| \geq \rho\right). \tag{16}$$

Using Lemma 3.2, there exists a constant $c_\rho > 0$ such that

$$P(\tau^\varepsilon_{n,\rho} \le 1) \le n \exp(-nc_\rho/\varepsilon), \quad n \ge 1. \tag{17}$$

Hence,

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\tau^\varepsilon_{n,\rho} \le 1) = -\infty. \tag{18}$$

For $\lambda > 0$, define $\phi_\lambda(y) := (\rho^2 + |y|^2)^\lambda, y \in \mathbf{R}^d$. By Itô's formula,

$$M^{n,\rho}_t := \phi_\lambda(Y^\varepsilon_{n,\rho}(t)) - \int_0^{t\wedge\tau^\varepsilon_{n,\rho}} \gamma^\varepsilon_\lambda(s)ds - \rho^{2\lambda} \tag{19}$$

is a martingale with initial value zero, where,

$$\begin{aligned}
\gamma^\varepsilon_\lambda(s) \quad := \quad & 2\lambda(\rho^2 + |Y^\varepsilon_n(s)|^2)^{\lambda-1}\langle Y^\varepsilon_n(s), b(s, X^\varepsilon(s), X^\varepsilon(s-1)) \\
& -b\big(s, X^\varepsilon_n(s), X^\varepsilon_n(s-1)\big)\rangle \\
& + 2\lambda(\lambda-1)\varepsilon(\rho^2 + |Y^\varepsilon_n(s)|^2)^{\lambda-2}\big|\big(\sigma\big(s, X^\varepsilon(s), X^\varepsilon(s-1)\big) \\
& - \sigma\Big(\frac{[ns]}{n}, X^\varepsilon_n\Big(\frac{[ns]}{n}\Big), X^\varepsilon_n(\frac{[ns]}{n} - 1)\Big)\Big)^* Y^\varepsilon_n(s)\big|^2 \\
& + \lambda\varepsilon\big(\rho^2 + |Y^\varepsilon_n(s)|^2\big)^{\lambda-1}|\big|\sigma\big(s, X^\varepsilon(s), X^\varepsilon(s-1)\big) \\
& - \sigma\Big(\frac{[ns]}{n}, X^\varepsilon_n\Big(\frac{[ns]}{n}\Big), X^\varepsilon_n\Big(\frac{[ns]}{n} - 1\Big)\Big)^* |^2_{H.S}.
\end{aligned}$$

for $s \le t \wedge \tau^\varepsilon_{n,\rho}$. Noticing that $X^\varepsilon_n(u) = \psi(u)$ and $X^\varepsilon(u) = \psi(u)$ for $u \le 0$, we see that

$$\begin{aligned}
\gamma^\varepsilon_\lambda(s) \quad \le \quad & c\lambda\phi_\lambda(Y^\varepsilon_n(s)) \\
& + \big\{4\lambda(\lambda-1)\varepsilon\big(\rho^2 + |Y^\varepsilon_n(s)|^2\big)^{\lambda-2}|Y^\varepsilon_n(s)|^2 + 2\lambda\varepsilon\big(\rho^2 + |Y^\varepsilon_n(s)|^2\big)^{\lambda-1}\big\} \\
& \times\bigg\{\Big(\sigma(s, X^\varepsilon(s), X^\varepsilon(s-1)) - \sigma\Big(\frac{[ns]}{n}, X^\varepsilon(s), X^\varepsilon(s-1)\Big)\Big)^2 \qquad (20) \\
& + |Y^\varepsilon_n(s)|^2 + \big|X^\varepsilon_n(s) - X^\varepsilon_n\Big(\frac{[ns]}{n}\Big)|^2 + \big|\psi(s-1) - \psi\Big(\frac{[ns]}{n} - 1\Big)\big|^2\bigg\}.
\end{aligned}$$

By uniform continuity, there exists an integer N so that

$$\big|\sigma(s, X^\varepsilon(s), X^\varepsilon(s-1)) - \sigma\big(\frac{[ns]}{n}, X^\varepsilon(s), X^\varepsilon(s-1)\big)\big| < \rho$$

and

$$\big|\psi(s-1) - \psi\big(\frac{[ns]}{n} - 1\big)\big| < \rho$$

for $s \le 1$ and all $n \ge N$. Thus for $n \ge N$,

$$\gamma^\varepsilon_\lambda(s) \le c(\lambda + \lambda\varepsilon + \lambda^2\varepsilon)\phi_\lambda(Y^\varepsilon_n(s)). \tag{21}$$

Choose $\lambda = \frac{1}{\varepsilon}$ and take expectations in (19) to obtain

$$E[(\rho^2 + |Y^\varepsilon_{n,\rho}(t)|^2)^{1/\varepsilon}] \le \rho^{2/\varepsilon} + \frac{C}{\varepsilon}\int_0^t E[(\rho^2 + |Y^\varepsilon_{n,\rho}(s)|^2)^{1/\varepsilon}]ds.$$

Hence,

$$E[(\rho^2 + |Y^\varepsilon_{n,\rho}(t)|^2)^{1/\varepsilon}] \le \rho^{2/\varepsilon} e^{\frac{Ct}{\varepsilon}}.$$

Since

$$(\rho^2 + \delta^2)^{\frac{1}{\varepsilon}} P(\xi^\varepsilon_{n,\rho} \le 1) \le E[(\rho^2 + |Y^\varepsilon_{n,\rho}(1)|^2)^{1/\varepsilon}],$$

we have

$$P(\xi^\varepsilon_{n,\rho} \leq 1) \leq \left(\frac{\rho^2}{\rho^2+\delta^2}\right)^{\frac{1}{\varepsilon}} e^{\frac{C}{\varepsilon}}. \tag{22}$$

Therefore,

$$\limsup_{\varepsilon\to 0} \varepsilon \log P(\xi^\varepsilon_{n,\rho} \leq 1) \leq \log\left(\frac{\rho^2}{\rho^2+\delta^2}\right) + C. \tag{23}$$

Given $M > 0$, first choose ρ sufficiently small so that $\log(\frac{\rho^2}{\rho^2+\delta^2}) + C \leq -2M$, and then use (18) to choose N so that $\limsup_{\varepsilon\to 0} \varepsilon \log P(\tau^\varepsilon_{n,\rho} \leq 1) \leq -2M$ for $n \geq N$. Combining these two facts gives

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\big(\sup_{-1\leq t\leq 1} |X^\varepsilon(t) - X^\varepsilon_n(t)| > \delta\big) \leq -M.$$

Since M is arbitrary, we have proved (13) for $m = 1$. Assume now (13) holds for some integer m. We will prove it is also true for $m+1$. Let $Y^\varepsilon_n, \tau^\varepsilon_{n,\rho}$ be defined as before. In addition, introduce two new stopping times:

$$\begin{aligned}
\tau^{1,\varepsilon}_{n,\rho} &:= \inf\{t \geq 0; |X^\varepsilon(t-1) - X^\varepsilon_n(t-1)| \geq \rho\},\\
\tau^{2,\varepsilon}_{n,\rho} &:= \inf\left\{t \geq 0; \big|X^\varepsilon_n(t-1) - X^\varepsilon_n\left(\frac{[nt]}{n} - 1\right)\big| \geq \rho\right\},
\end{aligned}$$

and define $Z^\varepsilon_{n,\rho}(t) := Y^\varepsilon_n(t \wedge \tau^{1,\varepsilon}_{n,\rho} \wedge \tau^{2,\varepsilon}_{n,\rho} \wedge \tau^\varepsilon_{n,\rho})$ and $\bar{\xi}^\varepsilon_{n,\rho} := \inf\{t \geq 0; |Z^\varepsilon_{n,\rho}(t)| \geq \delta\}$. We then have

$$\begin{aligned}
& P\big(\sup_{t\leq m+1} |Y^\varepsilon_n(t)| > \delta\big) && (24)\\
\leq\ & P(\tau^{1,\varepsilon}_{n,\rho} \wedge \tau^{2,\varepsilon}_{n,\rho} \wedge \tau^\varepsilon_{n,\rho} \leq m+1)\\
& + P\big(\sup_{t\leq m+1} |Y^\varepsilon_n(t)| > \delta, \tau^{1,\varepsilon}_{n,\rho} \wedge \tau^{2,\varepsilon}_{n,\rho} \wedge \tau^\varepsilon_{n,\rho} > m+1\big) && (25)\\
\leq\ & P(\tau^{1,\varepsilon}_{n,\rho} \leq m+1) + P(\tau^\varepsilon_{n,\rho} \wedge \tau^{2,\varepsilon}_{n,\rho} \leq m+1) + P(\bar{\xi}^\varepsilon_{n,\rho} \leq m+1).
\end{aligned}$$

As in the proof of (18),

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\tau^\varepsilon_{n,\rho} \wedge \tau^{2,\varepsilon}_{n,\rho} \leq m+1) = -\infty. \tag{26}$$

By the induction hypothesis,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P(\tau^{1,\varepsilon}_{n,\rho} \leq m+1)\\
&\leq \lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\big(\sup_{-1\leq t\leq m} |X^\varepsilon(t) - X^\varepsilon_n(t)| > \rho\big) = -\infty.
\end{aligned} \tag{27}$$

Again by Itô's formula,

$$\bar{M}^{n,\rho}_t := \phi_\lambda(Z^\varepsilon_{n,\rho}(t)) - \int_0^{t\wedge\tau^\varepsilon_{n,\rho}\wedge\tau^{1,\varepsilon}_{n,\rho}\wedge\tau^{2,\varepsilon}_{n,\rho}} \gamma^\varepsilon_\lambda(s)ds - \rho^{2\lambda} \tag{28}$$

is a martingale with $\bar{M}_0^{n,\rho} = 0$, where

$$
\begin{aligned}
\gamma_\lambda^\varepsilon(s) \quad &\leq \quad 2\lambda(\rho^2 + |Y_n^\varepsilon(s)|^2)^{\lambda-1}|Y_n^\varepsilon(s)|(|Y_n^\varepsilon(s)| + |X^\varepsilon(s-1) - X_n^\varepsilon(s-1)|) \\
&\quad + \big\{4\lambda(\lambda-1)\varepsilon(\rho^2 + |Y_n^\varepsilon(s)|^2)^{\lambda-2}|Y_n^\varepsilon(s)|^2 + 2\lambda\varepsilon(\rho^2 + |Y_n^\varepsilon(s)|^2)^{\lambda-1}\big\} \\
&\quad \times \Bigg\{ \Big(\sigma(s, X^\varepsilon(s), X^\varepsilon(s-1)) - \sigma\Big(\frac{[ns]}{n}, X^\varepsilon(s), X^\varepsilon(s-1)\Big)^2 \\
&\quad + |Y_n^\varepsilon(s)|^2 + \Big|X_n^\varepsilon(s) - X_n^\varepsilon\Big(\frac{[ns]}{n}\Big)\Big|^2 \\
&\quad + \Big|X_n^\varepsilon(s-1) - X_n^\varepsilon\Big(\frac{[ns]}{n} - 1\Big)\Big|^2 + |X^\varepsilon(s-1) - X_n^\varepsilon(s-1)|^2 \Bigg\} \\
&\leq \quad 2\lambda(\rho^2 + Y_n^\varepsilon(s)^2)^{\lambda-1}|Y_n^\varepsilon(s)|(|Y_n^\varepsilon(s)| + \rho) \\
&\quad + \big\{4\lambda(\lambda-1)\varepsilon(\rho^2 + |Y_n^\varepsilon(s)|^2)^{\lambda-2}|Y_n^\varepsilon(s)|^2 + 2\lambda\varepsilon(\rho^2 + |Y_n^\varepsilon(s)|^2)^{\lambda-1}\big\} \\
&\quad \times (|Y_n^\varepsilon(s)|^2 + 4\rho^2) \\
&\leq \quad c(\lambda + \lambda\varepsilon + \lambda^2\varepsilon)\phi(Y_n^\varepsilon(s))
\end{aligned}
\tag{29}
$$

for $s \leq \tau_{n,\rho}^\varepsilon \wedge \tau_{n,\rho}^{1,\varepsilon} \wedge \tau_{n,\rho}^{2,\varepsilon} \wedge (m+1)$, and sufficiently large n.

Using (25), (26) and following the proof of the case for $m = 1$, we see that (13) is also true for $m + 1$. This completes the proof of the lemma. □

For $n \geq 1$, define the map $F_n(\cdot) : C_0([0, m], \mathbf{R}^l) \to C_\psi([-1, m], \mathbf{R}^d)$ by

$$
\begin{aligned}
F_n(\omega)(t) :=& \psi(t), \qquad -1 \leq t \leq 0 \\
F_n(\omega)(t) :=& F_n(\omega)\Big(\frac{k}{n}\Big) + \int_{\frac{k}{n}}^t b(s, F_n(\omega)(s), F_n(\omega)(s-1))ds \\
& + \sigma\Big(\frac{k}{n}, F_n(\omega)\Big(\frac{k}{n}\Big), F_n(\omega)\Big(\frac{k}{n} - 1\Big)\Big)\Big(\omega(t) - \omega\Big(\frac{k}{n}\Big)\Big)
\end{aligned}
\tag{30}
$$

for $\frac{k}{n} \leq t \leq \frac{k+1}{n}$. It is easy to see that $F_n : C_0([0, m], \mathbf{R}^l) \to C_\psi([-1, m], \mathbf{R}^d)$ is continuous.

Lemma 3.4. $\lim_{n\to\infty} \sup_{\{g; e(g) \leq \alpha\}} \sup_{-1 \leq t \leq m} |F_n(g)(t) - F(g)(t)| = 0.$

Proof. Note that for g with $e(g) \leq \alpha$,

$$
\begin{aligned}
F_n(g)(t) =& F_n(g)(0) + \int_0^t b\big(s, F_n(g)(s), F_n(g)(s-1)\big)ds \\
& + \int_0^t \sigma\Big(\frac{[ns]}{n}, F_n(g)\Big(\frac{[ns]}{n}\Big), F_n(g)\Big(\frac{[ns]}{n} - 1\Big)\Big)\dot{g}(s)ds.
\end{aligned}
\tag{31}
$$

Thus,

$$
\begin{aligned}
& F_n(g)(t) - F(g)(t) \\
=& \int_0^t \big[b(s, F_n(g)(s), F_n(g)(s-1)) - b\big(s, F(g)(s), F(g)(s-1)\big)\big]ds \\
& + \int_0^t \Big[\sigma\Big(\frac{[ns]}{n}, F_n(g)\Big(\frac{[ns]}{n}\Big), F_n(g)\Big(\frac{[ns]}{n} - 1\Big)\Big) \\
& - \sigma\big(s, F(g)(s), F(g)(s-1)\big)\Big]\dot{g}(s)ds.
\end{aligned}
\tag{32}
$$

By the linear growth condition on b and σ, we have

$$\begin{aligned}|F_n(g)(t)| &\le |\psi(0)| + C\int_0^t \big(1 + 2\sup_{-1\le u\le s}|F_n(g)(u)|\big)ds \\ &\quad + C\int_0^t \big(1 + 2\sup_{-1\le u\le s}|F_n(g)(u)|\big)|\dot g|(s)ds.\end{aligned}$$

Using Grownwall's inequality, this implies that

$$\sup_{n\ge 1}\sup_{-1\le u\le m}|F_n(g)|(u) \le C\exp\big(2m + 2e(g)\big). \tag{33}$$

In particular,

$$M_\alpha = \sup_{g;e(g)\le\alpha}\sup_{n\ge 1}\sup_{-1\le u\le m}|F_n(g)(u)|)| \le C\exp(2m+2\alpha) < \infty. \tag{34}$$

Again by the linear growth condition and (30), we have

$$\begin{aligned}&\Big|F_n(g)(t) - F_n(g)\Big(\frac{[ns]}{n}\Big)\Big| \\ \le\; & \int_{\frac{[nt]}{n}}^t |b\big(s, F_n(g)(s), F_n(g)(s-1)\big)|ds + \int_{\frac{[nt]}{n}}^t \Big|\sigma\Big(\frac{[ns]}{n}, F_n(g)\Big(\frac{[ns]}{n}\Big), \\ & F_n(g)\Big(\frac{[ns]}{n} - 1\Big)\Big)\Big| |\dot g|(s)ds \qquad (35) \\ \le\; & C_\alpha M_\alpha\Big(\frac{1}{n}\Big)^{\frac12} \to 0\end{aligned}$$

uniformly over the set $\{g; e(g)\le\alpha\}$. Thus,

$$\begin{aligned}&|F_n(g)(t) - F(g)(t)| \\ \le\; & C\int_0^t |F_n(g)(s) - F(g)(s)|ds + C\int_0^t |F_n(g)(s-1) - F(g)(s-1)|ds \\ & + \int_0^t \sup_{x,y}\Big|\sigma\Big(\frac{[ns]}{n}, x, y\Big) - \sigma(s,x,y)\Big||\dot g|(s)ds \\ & + C\int_0^t \big[|F_n(g)(s-1) - F(g)(s-1)| + |F_n(g)(s) - F(g)(s)|\big]|\dot g|(s)ds \\ & + \int_0^t \Big[\Big|F_n(g)(s) - F_n(g)\Big(\frac{[ns]}{n}\Big)\Big| \\ & + \Big|F_n(g)(s-1) - F_n(g)\Big(\frac{[ns]}{n} - 1\Big)\Big|\Big]|\dot g|(s)ds \qquad (36) \\ \le\; & C_\alpha\Big[\Big(\frac{1}{n}\Big)^{\frac12} + \sup_s\sup_{x,y}\Big|\sigma\Big(\frac{[ns]}{n}, x, y\Big) - \sigma(s,x,y)\Big|^2\Big] \\ & + 2\int_0^t \sup_{-1\le u\le s}||F_n(g)(u) - F(g)(u)|ds \\ & + 2\int_0^t \sup_{-1\le u\le s}||F_n(g)(u) - F(g)(u)||\dot g|(s)ds. \qquad (37)\end{aligned}$$

This gives,

$$\sup_{-1\le u\le m}|F_n(g)(u)-F(g)(u)| \le \tilde{C}C_\alpha\left[\left(\frac{1}{n}\right)^{\frac{1}{2}}+\big|\sup_s\sup_{x,y}\Big|\sigma\Big(\frac{[ns]}{n},x,y\Big)-\sigma(s,x,y)\Big|^2\right]. \tag{38}$$

Hence,

$$\lim_{n\to\infty}\sup_{\{g;e(g)\le\alpha\}}\sup_{-1\le t\le m}|F_n(g)(t)-F(g)(t)|=0.$$

This proves the lemma. □

Proof of Theorem 3.1 when b,σ are bounded. Notice that $X_n^\varepsilon(s)=F_n(\varepsilon^{\frac{1}{2}}W)(s)$, where W is the Brownian motion. The theorem follows from Lemma 3.3, Lemma 3.4 and the generalized contraction principle (Theorem 4.2.23 [1]) in large deviations theory.. □

Next, we remove the boundedness assumptions on b and σ. We begin with

Proposition 3.5. *Assume that*

$$|\sigma(t,x,y)|\le C(1+|x|+|y|), \tag{39}$$

$$|b(t,x,y)|\le C(1+|x|+|y|), \tag{40}$$

for all $x,y\in\mathbf{R}^d$. Then for each integer $m\ge 1$,

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P\big(\sup_{-1\le t\le m}|X^\varepsilon(t)|>R\big)=-\infty \tag{41}$$

where $X^\varepsilon(\cdot)$ is the solution to equation (6).

Proof. We use induction on m. We first prove (40) for $m=1$. For $\lambda>0$, set $\phi_\lambda(y):=(1+|y|^2)^\lambda, y\in\mathbf{R}^d$. By Itô's formula, the process

$$M_t^\lambda:=\phi_\lambda(X^\varepsilon(t))-\int_0^t\gamma_\lambda^\varepsilon(s)ds-(1+|x|^2)^\lambda,\quad t\ge 0, \tag{42}$$

is a martingale with initial value zero, where

$$\begin{aligned}\gamma_\lambda^\varepsilon(s)&=2\lambda\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-1}\big\langle X^\varepsilon(s),b\big(s,X^\varepsilon(s),X^\varepsilon(s-1)\big)\big\rangle\\&\quad+2\lambda(\lambda-1)\varepsilon\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-2}|\big(\sigma(s,X^\varepsilon(s),X^\varepsilon(s-1)\big)^*X^\varepsilon(s)|^2\\&\quad+\lambda\varepsilon\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-1}||\sigma\big(s,X^\varepsilon(s),X^\varepsilon(s-1)\big)^*|_{H.S}^2,\end{aligned}$$

for $s\le 1$. Since $X^\varepsilon(u)=\psi(u)$ for $u\le 0$, it follows that

$$\begin{aligned}\gamma_\lambda^\varepsilon(s)&\le 2\lambda\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-1}|X^\varepsilon(s)|\big[1+|X^\varepsilon(s)|+\sup_{-1\le t\le 0}|\psi(t)|\big]\\&\quad+2\lambda(\lambda-1)\varepsilon\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-2}|X^\varepsilon(s)|\big[1+|X^\varepsilon(s)|\\&\quad+\sup_{-1\le t\le 0}|\psi(t)|\big]^2+\lambda\varepsilon\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-1}\big[1+|X^\varepsilon(s)|\\&\quad+\sup_{-1\le t\le 0}|\psi(t)|\big]^2\le C_\psi\big(\lambda+\lambda(\lambda+1)\varepsilon\big)\phi_\lambda\big(X^\varepsilon(s)\big).\end{aligned} \tag{43}$$

Let $\xi_R^\varepsilon := \inf\{t \geq 0, |X^\varepsilon(t)| > R\}$. Choosing $\lambda = \frac{1}{\varepsilon}$, it follows from (42) that

$$\begin{aligned} &E\big[(1+|X^\varepsilon\big(t\wedge\xi_R^\varepsilon)|^2\big)^{\frac{1}{\varepsilon}}\big] \\ \leq&(1+|x|^2)^{\frac{1}{\varepsilon}} + \frac{C}{\varepsilon}\int_0^t E\big[(1+|X^\varepsilon(s\wedge\xi_R^\varepsilon)|^2)^{\frac{1}{\varepsilon}}\big]ds. \end{aligned} \tag{44}$$

Hence,

$$E\big[(1+|X^\varepsilon(t\wedge\xi_R^\varepsilon)|^2)^{\frac{1}{\varepsilon}}\big] \leq (1+|x|^2)^{\frac{1}{\varepsilon}} e^{\frac{C}{\varepsilon}}. \tag{45}$$

This implies

$$\begin{aligned} &P(\sup_{-1\leq t\leq 1}|X^\varepsilon(t)| > R) \leq P\big(\xi_R^\varepsilon \leq 1\big) \\ \leq&(1+R^2)^{\frac{1}{2}}(1+|x|^2)^{\frac{1}{\varepsilon}} e^{\frac{C}{\varepsilon}}. \end{aligned} \tag{46}$$

Hence,

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P\big(\sup_{-1\leq t\leq 1}|X^\varepsilon(t)| > R\big) = -\infty. \tag{47}$$

Assume now that (40) holds for some m. We will prove that it is also true for $m+1$. For $R_1 > 0$, set $\xi_1^\varepsilon := \inf\{t \geq 0, |X^\varepsilon(t-1)| \geq R_1\}$ and $X_1^\varepsilon(t) := X^\varepsilon(t\wedge\xi_1^\varepsilon)$. Define $\xi_R^\varepsilon := \inf\{t \geq 0, |X_1^\varepsilon(t)| \geq R\}$. Then,

$$P(\sup_{-1\leq t\leq m+1}|X^\varepsilon(t)| > R) \tag{48}$$

$$\leq P(\xi_1^\varepsilon \leq m+1) + P(\xi_R^\varepsilon \leq m+1) \tag{49}$$

$$= P(\sup_{-1\leq t\leq m}|X^\varepsilon(t)| > R_1) + P(\xi_R^\varepsilon \leq m+1). \tag{50}$$

As before, by Itô's formula,

$$\bar{M}_t^\lambda := \phi_\lambda(X_1^\varepsilon(t)) - \int_0^{t\wedge\xi_1^\varepsilon}\gamma_\lambda^\varepsilon(s)ds - (1+|x|^2)^\lambda, \quad t \geq 0, \tag{51}$$

is a martingale with initial value zero, where

$$\begin{aligned} \gamma_\lambda^\varepsilon(s) \leq& 2\lambda\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-1}|X^\varepsilon(s)|\big[1+|X^\varepsilon(s)|+R_1\big] \\ &+ 4\lambda(\lambda-1)\varepsilon(1+|X^\varepsilon(s)|^2)^{\lambda-2}|X^\varepsilon(s)|^2[1+|X^\varepsilon(s)|+R_1]^2 \\ &+ 2\lambda\varepsilon\big(1+|X^\varepsilon(s)|^2\big)^{\lambda-1}\big[1+|X^\varepsilon(s)|+R_1\big]^2 \\ \leq& C_{R_1}\big(\lambda+\lambda(\lambda+1)\varepsilon\big)\phi_\lambda\big(X^\varepsilon(s)\big). \end{aligned} \tag{52}$$

for $s \leq 1\wedge\xi_1^\varepsilon$.

Using (51) and the proof of (46), we get

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\varepsilon\log P(\xi_R^\varepsilon \leq m+1) \\ &\qquad\leq -\log(1+R^2) + \log(1+|x|^2) + C_{R_1}(m+1). \end{aligned} \tag{53}$$

Thus it follows from (49) that

$$\begin{aligned} &\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{t\leq m+1}|X^\varepsilon(t)| \geq R) \\ \leq&(\limsup_{\varepsilon\to 0}\varepsilon\log P\big(\sup_{t\leq m}|X^\varepsilon(t)| \geq R_1)\big) \\ &\vee\big(-\log(1+R^2) + \log(1+|x|^2) + C_{R_1}(m+1)\big). \end{aligned} \tag{54}$$

Hence,

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{t\le m+1}|X^\varepsilon(t)|\ge R)$$
$$\le \limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{t\le m}|X^\varepsilon(t)|\ge R_1).$$

Using the induction hypothesis and letting $R_1\to\infty$ we obtain (40) for $m+1$. This completes the proof of the proposition. □

For $R>0$, define $m_R:=\sup\{|b(t,x,y)|,|\sigma(t,x,y)|; t\in[0,m],|x|\le R,|y|\le R\}$ and $b_i^R:=(-m_R-1)\vee b_i\wedge(m_R+1)$, $\sigma_{i,j}^R:=(-m_R-1)\vee\sigma_{i,j}\wedge(m_R+1)$, $1\le i,j\le d$. Put $b_R:=(b_1^R,b_2^R,...,b_d^R)$ and $\sigma_R:=(\sigma_{i,j}^R)_{1\le i,j\le d}$. Then

$$b_R(t,x,y)=b(t,x,y),\quad \sigma_R(t,x,y)=\sigma(t,x,y),$$

for $t\in[0,m]$, $|x|\le R$, $|y|\le R$. Furthermore, b_R and σ_R satisfy the Lipschitz condition (A.1) with the same Lipschitz constant.

Let $X_R^\varepsilon(\cdot)$ be the solution to the sdde

$$\begin{aligned} X_R^\varepsilon(t)&=X_R^\varepsilon(0)+\int_0^t b_R(s,X_R^\varepsilon(s),X_R^\varepsilon(s-1))ds\\ &\quad+\varepsilon^{\frac12}\int_0^t\sigma_R(s,X_R^\varepsilon(s),X_R^\varepsilon(s-1))dW_s,\ t>0,\\ X_R^\varepsilon(t)&=\psi(t),\quad t\in[-1,0]. \end{aligned}\tag{55}$$

□

Proposition 3.6. *Fix $m\ge 1$. Then*

$$\lim_{R\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{-1\le t\le m}|X^\varepsilon(t)-X_R^\varepsilon(t)|>\delta)=-\infty.\tag{56}$$

Proof. Again we will use induction. We omit the proof for the case $m=1$ since it is similar to that of Lemma 3.3. Let us assume that (55) holds for some m. We will prove that it also holds for $m+1$. Set $Y_R^\varepsilon(t):=X^\varepsilon(t)-X_R^\varepsilon(t)$. For $R_1>0$, define $\xi_{R_1}^\varepsilon:=\inf\{t\ge 0;|X^\varepsilon(t)|\ge R_1\}$. For any $R\ge R_1$ we have

$$\begin{aligned} &Y_R^\varepsilon(t\wedge\xi_{R_1}^\varepsilon)\\ &=\int_0^{t\wedge\xi_{R_1}^\varepsilon}\big[b_R(s,X^\varepsilon(s),X^\varepsilon(s-1)-b_R(s,X_R^\varepsilon(s),X_R^\varepsilon(s-1)\big]ds\\ &+\varepsilon^{\frac12}\int_0^{t\wedge\xi_{R_1}^\varepsilon}\big[\sigma_R\big(s,X^\varepsilon(s),X^\varepsilon(s-1)\big)\\ &-\sigma_R\big(s,X_R^\varepsilon(s),X_R^\varepsilon(s-1)\big)\big]dW_s,\quad t\ge 0. \end{aligned}\tag{57}$$

For $\rho>0$, let $\phi_\lambda(y):=(\rho^2+|y|^2)^\lambda$ and $\tau_{R,\rho}^\varepsilon:=\inf\{t\ge 0;|X^\varepsilon(t-1)-X_R^\varepsilon(t-1)|\ge\rho\}$. Set $Y_{R,\rho}^\varepsilon(t):=Y_R^\varepsilon(t\wedge\xi_{R_1}^\varepsilon\wedge\tau_{R,\rho}^\varepsilon)$ and $\xi_{R,\rho}^\varepsilon:=\inf\{t\ge 0;|Y_{R,\rho}^\varepsilon(t)|\ge\delta\}$. Then

$$P\Big(\sup_{-1\le t\le m+1}|Y_R^\varepsilon(t)|>\delta\Big)\tag{58}$$

$$\begin{aligned} \le\ & P(\xi_{R_1}^\varepsilon\le m+1)\\ &+P(\tau_{R,\rho}^\varepsilon\le m+1)+P(\xi_{R,\rho}^\varepsilon\le m+1) \end{aligned}\tag{59}$$

$$\begin{aligned} \le\ & P\Big(\sup_{-1\le t\le m+1}|X^\varepsilon(t)|>R_1\Big)\\ &+P(\sup_{-1\le t\le m}|X^\varepsilon(t)-X_R^\varepsilon(t)|>\rho)+P(\xi_{R,\rho}^\varepsilon\le m+1). \end{aligned}$$

By the induction hypothesis,

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{-1\le t\le m} |X^\varepsilon(t) - X^\varepsilon_R(t)| > \rho \Big) = -\infty. \tag{60}$$

By Itô's formula,

$$\phi_\lambda(Y^\varepsilon_{R,\rho}(t)) - \int_0^{t\wedge\xi^\varepsilon_{R_1}\wedge\tau^\varepsilon_{R,\rho}} \gamma^\varepsilon_\lambda(s)ds - \rho^{2\lambda} = M^{R,\rho}_t \tag{61}$$

is a martingale with initial value zero, where, as in the proof of Lemma 3.3, for $s \le t\wedge\xi^\varepsilon_{R_1}\wedge\tau^\varepsilon_{R,\rho}$,

$$\gamma^\varepsilon_\lambda(s) \le C(\lambda + \lambda(\lambda+1)\varepsilon)\phi_\lambda(Y^\varepsilon_R(s)). \tag{62}$$

As before, this implies that

$$\limsup_{\varepsilon\to 0} \varepsilon\log P(\xi^\varepsilon_{R,\rho} \le m+1) \le \log\left(\frac{\rho^2}{\rho^2+\delta^2}\right) + C. \tag{63}$$

Hence, it follows from (57), (58) and (60) that

$$\begin{aligned}
&\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \varepsilon\log P(\sup_{-1\le t\le m+1} |Y^\varepsilon_R(t)| > \delta)\\
&\le \big(\limsup_{\varepsilon\to 0}\varepsilon\log P(\sup_{-1\le t\le m+1}|X^\varepsilon(t)| > R_1)\big)\\
&\vee\Big\{\log\left(\frac{\rho^2}{\rho^2+\delta^2}\right) + C\Big\}.
\end{aligned} \tag{64}$$

By Proposition 3.5, letting first $\rho\to 0$ and then, $R_1\to\infty$, we obtain (55) for $m+1$. The proof of Proposition 3.6 is complete. □

For g with $e(g) < \infty$, let $F_R(g)$ be the solution to the dde

$$\begin{aligned}
F_R(g)(t) =& F_R(g)(0) + \int_0^t b_R\big(s, F_R(g)(s), F_R(g)(s-1)\big)ds\\
&+ \int_0^t \sigma_R\big(s, F_R(g)(s), F_R(g)(s-1)\big)\dot g(s)ds\\
F_R(g)(t) =& \psi(t), \qquad -1\le t\le 0.
\end{aligned} \tag{65}$$

Define

$$I_R(f) := \inf\left\{\frac{1}{2}e(g); F_R(g) = f\right\} \tag{66}$$

for each $f \in C_\psi\big([-1,m]\to \mathbf{R}^d\big)$. If $\sup_{-1\le t\le m} |F(g)(t)| \le R$, then $F(g) = F_R(g)$. Therefore,

$$I(f) = I_R(f), \qquad \text{for all } f \text{ with } \sup_{-1\le t\le m}|f(t)| \le R. \tag{67}$$

Lemma 3.7. *$I(\cdot)$ is a good rate function on $C_\psi([-1,m],\mathbf{R}^d)$; that is, for any $\alpha\ge 0$, the level set $\{f; I(f)\le\alpha\}$ is compact.*

Proof. As in Lemma 3.4, we can show that $\lim_{R\to\infty}\sup_{e(g)\le\alpha}|F_R(g) - F(g)| = 0$. In particular, this implies that $F(\cdot)$ is continuous on each level set $\{g; e(g)\le\alpha\}$. Since $e(\cdot)$ is a good rate function, this is sufficient to conclude that $I(\cdot)$ is also a good rate function. □

Proof of Theorem 3.1 in the unbounded case. For $R > 0$ and a closed subset $C \subset C_\psi([-1, m], \mathbf{R}^l)$, set $C_R = C \cap \{f; ||f||_\infty \le R\}$. C_R^δ denotes the δ-neighborhood of C_R. Denote by μ_ε^R the law of X_R^ε. Then we have

$$\begin{aligned}\mu_\varepsilon(C) &\le \mu_\varepsilon(C_{R_1}) + P(\sup_{-1\le t\le m} |X^\varepsilon(t)| > R_1)\\ &\le \mu_\varepsilon^R(C_{R_1}^\delta) + P(\sup_{-1\le t\le m} |X^\varepsilon(t) - X_R^\varepsilon(t)| > \delta)\\ &\quad + P(\sup_{-1\le t\le m} |X^\varepsilon(t)| > R_1).\end{aligned} \tag{68}$$

Using the large deviation principle for $\{\mu_\varepsilon^R, \varepsilon > 0\}$, we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon(C)\\ &\le \big(-\inf_{f\in C_{R_1}^\delta} I_R(f)\big) \vee \big(\limsup_{\varepsilon\to0} \varepsilon \log P(\sup_{-1\le t\le m} |X^\varepsilon(t)| > R_1)\big)\\ &\vee (\limsup_{\varepsilon\to0} \varepsilon \log P(\sup_{-1\le t\le m} |X^\varepsilon(t) - X_R^\varepsilon(t)| > \delta).\end{aligned} \tag{69}$$

Sending R to infinity gives

$$\begin{aligned}&\limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon(C)\\ &\le \big(-\inf_{f\in C_{R_1}^\delta} I(f)\big) \vee \big(\limsup_{\varepsilon\to0} \varepsilon \log P(\sup_{-1\le t\le m} |X^\varepsilon(t)| > R_1)\big).\end{aligned} \tag{70}$$

Letting first $\delta \to 0$, and then $R_1 \to \infty$, we obtain

$$\limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon(C) \le -\inf_{f\in C} I(f)$$

which is the upper bound (9) in Theorem 3.1.

Let G be an open subset of $C_\psi([-1, m] \to \mathbf{R}^d)$. Fix any $\phi_0 \in G$ and choose $\delta > 0$ such that $B(\phi_0, \delta) = \{f; ||f - \phi_0||_\infty \le \delta\} \subset G$. Then

$$\begin{aligned}-I_R(\phi_0) &\le \limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon^R\left(B(\phi_0, \frac{\delta}{2})\right)\\ &\le \big(\limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon(G)\big)\\ &\quad \vee \left((\limsup_{\varepsilon\to0} \varepsilon \log P\left(\sup_{-1\le t\le m} |X^\varepsilon(t) - X_R^\varepsilon(t)| > \frac{\delta}{2}\right)\right).\end{aligned} \tag{71}$$

Note that $I_R(\phi_0) = I(\phi_0)$ for all $R \ge ||\phi_0||_\infty$. So, letting $R \to \infty$ in the above inequality, we get

$$-I(\phi_0) \le \limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon(G). \tag{72}$$

Since ϕ_0 is arbitrary, it follows that

$$-\inf_{f\in G} I(f) \le \limsup_{\varepsilon\to0} \varepsilon \log \mu_\varepsilon(G)$$

which is the lower bound (10) in Theorem 3.1. The proof of Theorem 3.1 is now complete. □

Remark. The results in this paper can be easily extended to the case, where different delays τ_1, τ_2 are allowed in (6):

$$\begin{aligned} dX^\varepsilon(t) &= b\big(t, X^\varepsilon(t), X^\varepsilon(t-\tau_1)\big)dt + \varepsilon^{\frac{1}{2}}\sigma\big(t, X^\varepsilon(t), X^\varepsilon(t-\tau_2)\big)dW_t, \quad t \in (0,\infty), \\ X^\varepsilon(t) &= \psi(t), \quad t \in [-(\tau_1 \vee \tau_2), 0]. \end{aligned} \tag{73}$$

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