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INVARIANT MANIFOLDS FOR STOCHASTIC MODELS IN FLUID DYNAMICS

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This paper is dedicated to Peter Imkeller on his Sixtieth Birthday celebration

This paper is a survey of recent results on the dynamics of Stochastic Burgers equation (SBE) and two-dimensional Stochastic Navier–Stokes Equations (SNSE) driven by affine linear noise. Both classes of stochastic partial differential equations are commonly used in modeling fluid dynamics phenomena. For both the SBE and the SNSE, we establish the *local stable manifold theorem* for hyperbolic stationary solutions, the *local invariant manifold theorem* and the *global invariant flag theorem* for ergodic stationary solutions. The analysis is based on infinite-dimensional multiplicative ergodic theory techniques developed by D. Ruelle [22] (cf. [20, 21]). The results in this paper are based on joint work of the author with T. S. Zhang and H. Zhao ([17–19]).

Keywords: Stochastic Burgers equation; stochastic Navier–Stokes equation; cocycle; stationary point; stable manifolds; invariant manifolds; invariant flag.

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1. Introduction

In this paper, we survey some results pertaining to the dynamics of the stochastic Burgers equation (SBE) on the unit interval and the 2D Stochastic Navier–Stokes equation (SNSE) on a smooth bounded planar domain. Both equations are driven by affine noise with a linear component which is white in time and an additive component which is also white in time but sufficiently smooth (colored) in space. For the stochastic Burgers equation (SBE) and the stochastic Navier–Stokes equation (SNSE), we construct Fréchet differentiable stochastic semiflows described by perfect infinite-dimensional locally compacting cocycles on the energy space. We then examine the local behavior of each cocycle in a tubular random neighborhood of a stationary solution of the underlying stochastic partial differential equation (spde). This is achieved via a nonlinear multiplicative ergodic theory in Hilbert space, developed by D. Ruelle [22] and based on Oseledec's work in the finite-dimensional setting (cf. [20, 21]). In particular, we prove the *local stable manifold* *theorem* for hyperbolic stationary solutions, the *local invariant manifold theorem* and *the global invariant foliation theorem* for the SBE and the SNSE, relative to an ergodic stationary solution. The construction of the invariant manifolds is based on the Lyapunov spectrum of the linearized cocycle along the stationary solution.

The following issues will be addressed in this paper:

- Infinite-dimensional cocycles in Hilbert space.
- Ruelle's spectral theory for compact linear cocycles in Hilbert space.
- Hyperbolicity of stationary points.
- Invariant manifolds for infinite-dimensional cocycles in Hilbert space.
- The cocycle generated by mild solutions of the stochastic Burgers and 2D Navier–Stokes equations.
- Stable/unstable, invariant manifolds and foliations for the SBE and the SNSE.

2. Stochastic Dynamics in Hilbert Space

There is a large volume of literature on stochastic dynamical systems in finite dimensions. Just to name a few references, we will refer the reader to [1, 2, 14] and the bibliographies therein. Global invariant manifolds of stochastic evolution equations with linear one-dimensional noise have been studied in [8, 9] under global Lipschitz nonlinearities and a spectral gap condition.

In this section, we describe the idea of an infinite-dimensional *cocycle* which is central to the dynamics of stochastic Burgers and 2D Navier–Stokes equations.

Denote by $\mathbb{R}$ the set of all real numbers, and by $\mathbb{R}^+ := [0, \infty)$ the set of non-negative reals. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the complete filtered Wiener space with Wiener measure P on path space $\Omega := C(\mathbb{R}, \mathbb{R}; 0)$ of all continuous paths $\omega\colon \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$. The space Ω is endowed with the topology of uniform convergence on compacta. The P-complete Borel σ-algebra on Ω is denoted by $\mathcal{F}$; and for each $t \geq 0$, $\mathcal{F}_t$ denotes the complete σ-algebra generated by all evaluations $\Omega \ni \omega \mapsto \omega(u) \in \mathbb{R}$, $u \leq t$. Let $\theta\colon \mathbb{R} \times \Omega \to \Omega$ denote the P-preserving ergodic Wiener shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega.$$

The following convention will be used throughout the paper.

Definition 2.1. (Perfection) A family of propositions $\{P(\omega)\colon \omega \in \Omega\}$ is said to *hold perfectly in* ω if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$ and $P(\omega)$ is true *for every* $\omega \in \Omega^*$.

Furthermore, we will require the following notation.

For a topological space E, we let $\mathcal{B}(E)$ be its Borel σ-algebra. If E is a Banach space, the space $L(E)$ of all bounded linear operators on E may be given the *strong topology*, which is the smallest topology with respect to which all evaluations $L(E) \ni T \mapsto T(f) \in E$, $f \in E$, are continuous. The σ-algebra generated by the strong topology on $L(E)$ is denoted by $\mathcal{B}_s(L(E))$. A stochastic process $T\colon \mathbb{R}^+ \times \Omega \to L(E)$ is said to be *strongly measurable* if it is $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}_s(L(E)))$-measurable.

Let k be a positive integer and $0 < \epsilon \leq 1$. If E and N are real Banach spaces with norms $|\cdot|$, we will denote by $L^{(k)}(E, N)$ the Banach space of all continuous k-multilinear maps $A\colon E^k \to N$ with the uniform norm $\|A\| := \sup\{|A(f_1, f_2, \ldots, f_k)|\colon f_i \in E, |f_i| \leq 1, i = 1, \ldots, k\}$. For brevity, we let $L^{(k)}(E)$ stand for $L^{(k)}(E, E)$. Suppose $G \subseteq E$ is an open set. A map $U\colon G \to N$ is said to be *of class* $C^{k,\epsilon}$ if it is C^k-Fréchet differentiable and if its kth Fréchet derivative $D^{(k)}U\colon G \to L^{(k)}(E, N)$ is ϵ-Hölder continuous on bounded sets in G. A $C^{k,\epsilon}$ map $U\colon G \to N$ is said to be *of class* $C_b^{k,\epsilon}$ if all its derivatives $D^{(j)}U, 1 \leq j \leq k$, are globally bounded on G, and $D^{(k)}U$ is ϵ-Hölder continuous on bounded subsets of G. If $G \subset E$ is open and bounded, denote by $C^{k,\epsilon}(G, N)$ the Banach space of all $C^{k,\epsilon}$ maps $U\colon G \to N$ given the norm:

$$\|U\|_{k,\epsilon} := \sup_{\substack{f \in G \\ 0 \leq j \leq k}} \|D^{(j)}U(f)\| + \sup_{\substack{f_1, f_2 \in G \\ f_1 \neq f_2}} \frac{\|D^{(k)}U(f_1) - D^{(k)}U(f_2)\|}{|f_1 - f_2|^\epsilon}.$$

A map $U\colon G \to N$ is C^∞ if it is C^k for all integers $k \geq 1$.

We may now define the concept of a $C^{k,\epsilon}$ *perfect cocycle* on a Hilbert space.

Definition 2.2. (Cocycle) Let H be a real separable Hilbert space, k a non-negative integer and $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (U, θ) on H is a $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable random field $U\colon \mathbb{R}^+ \times H \times \Omega \to H$ with the following properties:

(i) For each $\omega \in \Omega$, the map $\mathbb{R}^+ \times H \ni (t, f) \mapsto U(t, f, \omega) \in H$ is continuous; and for fixed $(t, \omega) \in \mathbb{R}^+ \times \Omega$, the map $H \ni f \mapsto U(t, f, \omega) \in H$ is $C^{k,\epsilon}$.
(ii) $U(t + s, \cdot, \omega) = U(t, \cdot, \theta(s, \omega)) \circ U(s, \cdot, \omega)$ for all $s, t \in \mathbb{R}^+$ and all $\omega \in \Omega$.
(iii) $U(0, f, \omega) = f$ for all $f \in H$ and $\omega \in \Omega$.

It will be shown in due course that the dynamics of the stochastic Burgers equation (SBE) and the stochastic Navier–Stokes equation (SNSE) can both be captured via perfect cocycles on suitable Hilbert spaces. Such cocycles will be constructed in Secs. 3 and 4 of this paper.

For the rest of this section, we will survey (with sketchy proofs) some general results on invariant manifolds for differentiable cocycles in Hilbert space.

We begin by stating the Oseledec–Ruelle spectral theorem for the linearization of a cocycle along a *stationary point*. As may be seen from its definition below, a stationary point is a random point on the state space that is invariant under the cocycle:

Definition 2.3. (Stationary point) An $\mathcal{F}$-measurable random variable $Y\colon \Omega \to H$ is said be a *stationary point* for the cocycle (U, θ) if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{2.1}$$

for all $t \in \mathbb{R}^+$, and $\omega \in \Omega$.

Note that if Y is a stationary point of the cocycle (U,θ), then its distribution $P \circ Y^{-1}$ is an invariant measure for the one-point motion $U(\cdot, f, \cdot)$ on H.

When the cocycle (U,θ) is locally compact, we can apply the Oseledec–Ruelle spectral theorem to the strongly measurable linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t,\omega))$, $t \geq 0$, $\omega \in \Omega$ ([22], [19, Theorem 2.1.1]). This gives:

Theorem 2.1. (The Lyapunov Spectrum) *Let $(U(t,\cdot,\omega),\theta(t,\omega))$ be a $C^{k,\epsilon}$ cocycle on H such that for each $t > 0, \omega \in \Omega$, the map $U(t,\cdot,\omega)\colon H \to H$ takes bounded sets into relatively compact ones. Let $Y : \Omega \to H$ be a stationary point for the cocycle $(U(t,\cdot,\omega),\theta(t,\omega))$. Suppose that the following Oseledec integrability condition holds*

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq a} \|DU(t_2, Y(\theta(t_1,\cdot)), \theta(t_1,\cdot))\|_{L(H)} < \infty \tag{2.2}$$

for any $0 < a < \infty$. Then the following limit

$$\Lambda(\omega) := \lim_{t \to \infty} \{[DU(t, Y(\omega), \omega)]^* \circ [DU(t, Y(\omega), \omega)]\}^{1/2t} \tag{2.3}$$

exists in the uniform operator norm in $L(H)$, perfectly in ω. The Oseledec operator in (2.3) is compact, self-adjoint and non-negative with discrete spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_n} > \cdots. \tag{2.4}$$

The Lyapunov exponents $\{\lambda_n, n \geq 1\}$ correspond to values of the limit

$$\lim_{t \to \infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(g)|_H \in \{\lambda_n : n \geq 1\}$$

for any $g \in H$, perfectly in ω. Each eigenvalue e^{λ_j} has a fixed finite multiplicity m_j and a corresponding finite-dimensional eigenspace $F_j(\omega)$ with $m_j := \dim F_j(\omega)$, $j \geq 1$, $\omega \in \Omega$. Indeed, set

$$E_1(\omega) := H, \quad E_n(\omega) := \left[\bigoplus_{j=1}^{n-1} F_j(\omega)\right]^{\perp}, \quad n > 1.$$

Then for each $n \geq 1$, $\operatorname{codim} E_n(\omega) = \sum_{j=1}^{n-1} \dim F_j(\omega) < \infty$, and the following assertions are true:

$$E_n(\omega) \subset E_{n-1}(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega), \quad n > 1;$$

$$\lim_{t \to \infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(g)|_H = \lambda_n \tag{2.5}$$

for $g \in E_n(\omega)\backslash E_{n+1}(\omega)$;

$$\lim_{t\to\infty} \frac{1}{t} \log \|DU(t, Y(\omega), \omega)\|_{L(H)} = \lambda_1 \tag{2.6}$$

and

$$DU(t, Y(\omega), \omega)(E_n(\omega)) \subseteq E_n(\theta(t, \omega)) \tag{2.7}$$

for all $t \geq 0$, *perfectly in* $\omega \in \Omega$, *for all* $n \geq 1$.

Proof. We will only sketch the proof. Details may be found in [19].

Using the Oseledec integrability condition (2.2) and the Ruelle–Oseledec theorem ([19, Theorem 2.1.1]), we obtain a random family of compact self-adjoint positive operators $\Lambda(\omega) \in L(H)$, defined perfectly in ω, and satisfies

$$\Lambda(\omega) = \lim_{t\to\infty} \{[DU(t, Y(\omega), \omega)]^* \circ [DU(t, Y(\omega), \omega)]\}^{1/2t}. \tag{2.8}$$

The above almost sure limit exists in the uniform operator norm in $L(H)$, perfectly in ω. The operator $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_n} > \cdots \tag{2.9}$$

due to the ergodicity of the Brownian shift θ. Observe that the Lyapunov spectrum $\{\lambda_n : n \geq 1\}$, although fixed, may depend on the stationary point Y.

The assertions (2.6) and (2.7) of the theorem follow from the Oseledec–Ruelle spectral theorem ([19, Theorem 2.1.1]). □

Let $Y : \Omega \to H$ be a stationary point for the cocycle (U, θ) satisfying the hypotheses of Theorem 2.1. We say that Y is *hyperbolic* if it has a nonzero Lyapunov spectrum, viz. $\lambda_n \neq 0$ for all $n \geq 1$.

Theorem 2.2 below is a consequence of the nonlinear multiplicative ergodic theorem ([19, Theorem 2.2.1]). It describes the saddle-point behavior of the cocycle in the neighborhood of a hyperbolic stationary point.

For any $\rho > 0$, and any $f \in H$, we will denote by $B(f, \rho)$ the closed ball in H center f and radius ρ. Denote by $\|\cdot\|_{k,\epsilon}$ the $C^{k,\epsilon}$-norm on the Banach space $C^{k,\epsilon}(B(0, \rho), H)$.

Theorem 2.2. (The local stable manifold theorem) *Let* Y *be a hyperbolic stationary point of the cocycle* (U, θ) *satisfying the following integrability property*:

$$\int_\Omega \log^+ \sup_{0\leq t_1, t_2 \leq a} \|U(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega))\|_{k,\epsilon}\, dP(\omega) < \infty \tag{2.10}$$

for any fixed $0 < \rho, a < \infty$ *and* $\epsilon \in (0, 1]$. *Denote by* $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ *the Lyapunov spectrum of the linearized cocycle* $(DU(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ *as given in Theorem* 2.1. *Define* $i_0 := \min\{i : \lambda_i < 0\}$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ *and* $\epsilon_2 \in (0, \lambda_{i_0-1})$. *Then there exist*

(i) *a sure event* $\Omega^* \in \mathcal{F}$ *with* $\theta(t,\cdot)(\Omega^*) = \Omega^*$ *for all* $t \in \mathbb{R}$,

(ii) $\mathcal{F}$*-measurable random variables* $\rho_i, \beta_i \colon \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, *such that for each* $\omega \in \Omega^*$, *the following is true*:

There are $C^{k,\epsilon}$ *submanifolds* $\mathcal{S}(\omega), \mathcal{U}(\omega)$ *of* $B(Y(\omega), \rho_1(\omega))$ *and* $B(Y(\omega), \rho_2(\omega))$ (*respectively*) *with the following properties*:

(a) *For* $\lambda_{i_0} > -\infty$, $\mathcal{S}(\omega)$ *is the set of all* $f \in B(Y(\omega), \rho_1(\omega))$ *such that*

$$|U(n,f,\omega) - Y(\theta(n,\omega))|_H \le \beta_1(\omega) \exp\{(\lambda_{i_0} + \epsilon_1)n\}$$

for all integers $n \ge 0$. *If* $\lambda_{i_0} = -\infty$, *then* $\mathcal{S}(\omega)$ *is the set of all* $f \in B(Y(\omega), \rho_1(\omega))$ *such that*

$$|U(n,f,\omega) - Y(\theta(n,\omega))|_H \le \beta_1(\omega)\, e^{\lambda n}$$

for all integers $n \ge 0$ *and any* $\lambda \in (-\infty, 0)$. *Furthermore,*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t,f,\omega) - Y(\theta(t,\omega))|_H \le \lambda_{i_0} \tag{2.11}$$

for all $f \in \mathcal{S}(\omega)$. *The stable subspace* $\mathcal{S}^0(\omega) := E_{i_0}(\omega)$ *of the linearized cocycle* $(DU(t, Y(\omega), \cdot), \theta(t,\cdot))$ *is tangent at* $Y(\omega)$ *to the submanifold* $\mathcal{S}(\omega)$, *viz.* $T_{Y(\omega)}\mathcal{S}(\omega) = \mathcal{S}^0(\omega)$. *In particular,* $\operatorname{codim} \mathcal{S}(\omega) = \operatorname{codim} \mathcal{S}^0(\omega) = \sum_{j=1}^{i_0-1} \dim F_j(\omega)$ *is fixed and finite.*

(b)

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left[\sup \left\{ \frac{|U(t,f_1,\omega) - U(t,f_2,\omega)|_H}{|f_1 - f_2|_H} : f_1 \ne f_2,\ f_1, f_2 \in \mathcal{S}(\omega) \right\} \right] \le \lambda_{i_0}.$$

(c) (*Cocycle-invariance of the stable manifolds*):
There exists $\tau_1(\omega) \ge 0$ *such that*

$$U(t,\cdot,\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)) \tag{2.12}$$

for all $t \ge \tau_1(\omega)$. *Also*

$$DU(t, Y(\omega), \omega)(\mathcal{S}^0(\omega)) \subseteq \mathcal{S}^0(\theta(t,\omega)), \quad t \ge 0. \tag{2.13}$$

(d) $\mathcal{U}(\omega)$ *is the set of all* $f \in B(Y(\omega), \rho_2(\omega))$ *with the property that there is a discrete-time "history" process* $y(\cdot,\omega) \colon \{-n \colon n \ge 0\} \to H$ *such that* $y(0,\omega) = f$ *and for each integer* $n \ge 1$, *one has* $U(1, y(-n,\omega), \theta(-n,\omega)) = y(-(n-1),\omega)$ *and*

$$|y(-n,\omega) - Y(\theta(-n,\omega))|_H \le \beta_2(\omega) \exp\{-(\lambda_{i_0-1} - \epsilon_2)n\}.$$

If $\lambda_{i_0-1} = \infty$, $\mathcal{U}(\omega)$ *is the set of all* $f \in B(Y(\omega), \rho_2(\omega))$ *with the property that there is a discrete-time "history" process* $y(\cdot,\omega) \colon \{-n \colon n \ge 0\} \to H$ *such that* $y(0,\omega) = f$ *and for each integer* $n \ge 1$,

$$|y(-n,\omega) - Y(\theta(-n,\omega))|_H \le \beta_2(\omega) \exp\{-\lambda n\},$$

for any $\lambda \in (0, \infty)$. *Furthermore, for each* $f \in \mathcal{U}(\omega)$, *there is a unique continuous-time "history" process also denoted by* $y(\cdot, \omega)\colon (-\infty, 0] \to H$ *such that* $y(0, \omega) = f$, $U(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))|_H \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}^0(\omega)$ *of the linearized cocycle* $(DU(t, Y(\cdot), \cdot), \theta(t, \cdot))$ *is tangent at* $Y(\omega)$ *to* $\mathcal{U}(\omega)$, *viz.* $T_{Y(\omega)}\mathcal{U}(\omega) = \mathcal{U}^0(\omega)$. *In particular,* $\dim \mathcal{U}(\omega) = \sum_{j=1}^{i_0-1} \dim F_j(\omega)$ *is finite and non-random.*

(e) *Let* $y(\cdot, f_i, \omega), i = 1, 2$, *be the history processes associated with* $f_i = y(0, f_i, \omega) \in \mathcal{U}(\omega),\ i = 1, 2$. *Then*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left[\sup \left\{ \frac{|y(-t, f_1, \omega) - y(-t, f_2, \omega)|_H}{|f_1 - f_2|_H} : f_1 \neq f_2, f_i \in \mathcal{U}(\omega), i = 1, 2 \right\} \right] \leq -\lambda_{i_0-1}.$$

(f) (*Cocycle-invariance of the unstable manifolds*): *There exists* $\tau_2(\omega) \geq 0$ *such that*

$$\mathcal{U}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) \tag{2.14}$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DU(t, \cdot, \theta(-t, \omega))(\mathcal{U}^0(\theta(-t, \omega))) = \mathcal{U}^0(\omega), \quad t \geq 0;$$

and the restriction

$$DU(t, \cdot, \theta(-t, \omega))|\mathcal{U}^0(\theta(-t, \omega))\colon \mathcal{U}^0(\theta(-t, \omega)) \to \mathcal{U}^0(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

(g) *The submanifolds* $\mathcal{U}(\omega)$ *and* $\mathcal{S}(\omega)$ *are transversal at* $Y(\omega)$, *viz.*

$$H = T_{Y(\omega)}\mathcal{U}(\omega) \oplus T_{Y(\omega)}\mathcal{S}(\omega).$$

We will only give an outline of the proof of Theorem 2.2. Full details of the proof may be obtained by adapting the arguments in [16] and [19].

Proof. (An outline of the proof of Theorem 2.2)

- We develop perfect continuous-time versions of Kingman's subadditive ergodic theorem as well as the ergodic theorem ([19, Lemma 2.3.1(ii), (iii)]). The linearized cocycle $(DU(t, Y(\cdot), \cdot), \theta(t, \cdot))$ at the stationary point Y can be shown to satisfy the hypotheses of these perfect ergodic theorems. As a consequence of the perfect ergodic theorems, one obtains stable/unstable subspaces for the linearized cocycle, which will constitute tangent spaces at the stationary point Y to the local stable and unstable manifolds of the nonlinear cocycle (U, θ).

- We use hyperbolicity of the stationary point, the continuous-time integrability condition (2.10) on the cocycle and the perfect versions of the ergodic and subadditive ergodic theorems to show the existence of local stable/unstable manifolds for the discrete cocycle $(U(n,\cdot,\omega),\theta(n,\omega))$ near $Y(\omega)$ (cf. [22, Theorems 5.1 and 6.1]). These manifolds are random objects and are perfectly defined for $\omega\in\Omega$. Using interpolation between discrete times and the (continuous-time) integrability condition (2.10), it can be shown that the above manifolds for the discrete-time cocycle $(U(n,\cdot,\omega),\theta(n,\omega)), n\geq 1$, also serve as *perfectly defined* local stable/unstable manifolds for the *continuous-time* cocycle $(U(t,\cdot,\omega),\theta(t,\omega))$, $t\geq 0$, near $Y(\omega)$ (see [16, 19, 22]).
- Again, by using the integrability condition (2.10) on the nonlinear cocycle and its Fréchet derivatives, it is possible to control the excursions of the *continuous-time* cocycle $(U(t,\cdot,\cdot),\theta(t,\cdot))$, $t\geq 0$, between discrete times. In view of the perfect subadditive ergodic theorem, these estimates show that the local stable manifolds are asymptotically invariant under the nonlinear cocycle. The asymptotic invariance of the unstable manifolds is obtained via the concept of a *stochastic history process* for the cocycle. The existence of a stochastic history process is needed because the (locally compact) cocycle is *not invertible*.

This completes the outline of the proof of Theorem 2.2. □

We conclude this section by characterizing the asymptotics of the cocycle (U,θ) relative to an ergodic stationary point $Y\colon\Omega\to H$ satisfying the requirements of Theorem 2.2. This is first done via the *local invariant manifold theorem* which characterizes the almost sure asymptotic stability of the cocycle (U,θ) in the neighborhood of the *ergodic* stationary point Y. The second result is the *global invariant flag theorem* which gives a random cocycle-invariant countable global foliation of H relative to the *ergodic* stationary point Y.

Theorem 2.3. (Local invariant manifold theorem) *Assume that (U,θ) is a $C^{k,\epsilon}$ cocycle on H with a stationary point $Y\colon\Omega\to H$ satisfying the conditions of Theorem 2.2. Assume further that Y is ergodic in the sense that the Lyapunov exponents $\lambda_i<0$ for all $i\geq 1$. Fix $\epsilon_1\in(0,-\lambda_1)$. Then there exist*

(i) *a sure event $\Omega^*\in\mathcal{F}$ with $\theta(t,\cdot)(\Omega^*)=\Omega^*$ for all $t\in\mathbb{R}$,*
(ii) *$\mathcal{F}$-measurable random variables $\rho_i,\beta_i\colon\Omega^*\to(0,1)$, $\beta_i>\rho_i\geq\rho_{i+1}>0$, $i\geq 1$, such that for each $\omega\in\Omega^*$, the following is true:*
There are $C^{k,\epsilon}$ submanifolds $\mathcal{S}_i(\omega), i\geq 1$, of $B(Y(\omega),\rho_i(\omega))$ with the following properties:

(a) *$\mathcal{S}_i(\omega)$ is the set of all $f\in B(Y(\omega),\rho_i(\omega))$ such that*
$$|U(n,f,\omega)-Y(\theta(n,\omega))|_H\leq\beta_i(\omega)\,\exp\{(\lambda_i+\epsilon_1)n\}$$
for all integers $n\geq 0$. Furthermore,
$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\log|U(t,f,\omega)-Y(\theta(t,\omega))|_H\leq\lambda_i \tag{2.15}$$

for all $f \in \mathcal{S}_i(\omega)$. *The Oseledec space* $E_i(\omega)$ *of the linearized cocycle* $(DU(t, Y(\omega), \cdot), \theta(t, \cdot))$ *is tangent at* $Y(\omega)$ *to the submanifold* $\mathcal{S}_i(\omega)$, *viz.* $T_{Y(\omega)}\mathcal{S}_i(\omega) = E_i(\omega)$. *In particular,* $\operatorname{codim} \mathcal{S}_i(\omega) = \operatorname{codim} E_i(\omega) = \sum_{j=1}^{i-1} \dim F_j(\omega)$ *(fixed and finite).*

(b)

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left[\sup \left\{ \frac{|U(t, f_1, \omega) - U(t, f_2, \omega)|_H}{|f_1 - f_2|_H} : f_1 \neq f_2,\ f_1, f_2 \in \mathcal{S}_i(\omega) \right\} \right]$$
$$\leq \lambda_i.$$

(c) (*Cocycle-invariance*):

There exists $\tau_i(\omega) \geq 0$ *such that*

$$U(t, \cdot, \omega)(\mathcal{S}_i(\omega)) \subseteq \mathcal{S}_i(\theta(t, \omega)) \tag{2.16}$$

for all $t \geq \tau_i(\omega)$. *Also*

$$DU(t, Y(\omega), \omega)(E_n(\omega)) \subseteq E_n(\theta(t, \omega)), \quad t \geq 0. \tag{2.17}$$

Proof. Let $\epsilon_1 \in (0, -\lambda_1)$. Then there exist $\Omega^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$, and $\mathcal{F}$-measurable random variables $\rho_i, \beta_i \colon \Omega^* \to (0, 1)$ such that $\beta_i(\omega) > \rho_i(\omega) > 0$, and $C^{k,\epsilon}$ local stable submanifolds $S_i(\omega) \subset B(Y(\omega), \rho_i(\omega))$ such that

$$S_i(\omega) := \{f \in B(Y(\omega), \rho_i(\omega)) \colon |U(n, f, \omega) - Y(\theta(n, \omega))|_H$$
$$\leq \beta_i(\omega) e^{(\lambda_i + \epsilon_1)n} \text{ for all } n \geq 1\}. \tag{2.18}$$

Furthermore,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t, f, \omega) - Y(\theta(t, \omega))|_H \leq \lambda_i \tag{2.19}$$

for all $f \in S_i(\omega)$.

Also, $T_{Y(\omega)}S_i(\omega) = E_i(\omega)$, the Oseledec space for the linearized cocycle $(DU (t, Y(\omega), \omega), \theta(t, \omega))$ corresponding to the Lyapunov exponent λ_i, $i \geq 1$. Following the argument in [16] and [22], the random variables $\rho_i(\omega)$, $\beta_i(\omega)$ may be selected such that

$$\rho_i(\omega) e^{(\lambda_i + \epsilon_1)t} \leq \rho_i(\theta(t, \omega)) \tag{2.20}$$

and

$$\beta_i(\omega) e^{(\lambda_i + \epsilon_1)t} \leq \beta_i(\theta(t, \omega)) \tag{2.21}$$

for all $t \geq 0$ and $\omega \in \Omega^*$.

The asymptotic cocycle-invariance property (2.16) follows from (2.20), (2.21), the cocycle property of U and the following estimate

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left[\sup_{\substack{f\in S_i(\omega)\\ f\neq Y(\omega)}} \frac{|U(t,f,\omega)-Y(\theta(t,\omega))|_H}{|f-Y(\omega)|_H} \right] \leq \lambda_i, \tag{2.22}$$

cf. [16–18, 22]. □

Now we state our second result on global invariant flags for the cocycle (U,θ) relative to an ergodic stationary point.

Theorem 2.4. (Global invariant flag theorem) *Let (U,θ) be a $C^{k,\epsilon}$ cocycle on a Hilbert space H with an ergodic stationary point $Y:\Omega\to H$ satisfying the conditions of Theorem 2.3. Denote by Ω^* the sure event given in Theorem 2.3. Define the random family of sets $\{M_i(\omega)\colon \omega\in\Omega^*,\ i\geq 1\}$ by*

$$M_i(\omega) := \left\{ f\in H\colon \overline{\lim_{t\to\infty}} \frac{1}{t}\log|U(t,f,\omega)-Y(\theta(t,\omega))|_H \leq \lambda_i \right\} \tag{2.23}$$

for $i\geq 1$, $\omega\in\Omega^*$. For fixed $i\geq 1, \omega\in\Omega^*$, define the sequence $\{S_i^n(\omega)\}_{n=1}^\infty$, inductively by:

$$S_i^1(\omega) := S_i(\omega), \tag{2.24}$$

$$S_i^n(\omega) := \begin{cases} U(n,\cdot,\omega)^{-1}[S_i(\theta(n,\omega))], & \text{if } S_i^{n-1}(\omega)\subseteq U(n,\cdot,\omega)^{-1}[S_i(\theta(n,\omega))],\\ S_i^{n-1}(\omega), & \text{otherwise,}\end{cases} \tag{2.25}$$

for all $n\geq 2$. In (2.24) and (2.25), the $S_i(\omega)$ are the local invariant $C^{k,\epsilon}$ Hilbert submanifolds of H constructed in Theorem 2.3.

Then the following is true for each $i\geq 1$ and $\omega\in\Omega^*$:

(i) *The sets $\{M_i(\omega)\colon \omega\in\Omega^*, i\geq 1\}$ are cocycle-invariant:*

$$U(t,\cdot,\omega)(M_i(\omega))\subseteq M_i(\theta(t,\omega)) \tag{2.26}$$

for all $t\geq 0$.

(ii) *$S_i^n(\omega)\subseteq S_i^{n+1}(\omega)$ for all $n\geq 1$, and*

$$M_i(\omega) = \bigcup_{n=1}^\infty S_i^n(\omega), \quad i\geq 1. \tag{2.27}$$

(iii) *$M_{i+1}(\omega)\subseteq M_i(\omega)$.*

(iv) *For any $f\in M_i(\omega)\backslash M_{i+1}(\omega)$,*

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\log|U(t,f,\omega)-Y(\theta(t,\omega))|_H \in (\lambda_{i+1},\lambda_i]. \tag{2.28}$$

Proof. Fix $\omega \in \Omega^*$, where Ω^* is defined as in Theorem 2.3.

(i) To prove the cocycle invariance property (2.26), let $f \in M_i(\omega)$ and $t_1 > 0$. Then by definition (2.23) of $M_i(\omega)$, we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t, f, \omega) - Y(\theta(t, \omega))|_H \le \lambda_i. \tag{2.29}$$

By the cocycle property of (U, θ), we have

$$\begin{aligned}
&\overline{\lim_{t\to\infty}} \frac{1}{t} \log|U(t, U(t_1, f, \omega), \theta(t_1, \omega)) - Y(\theta(t, \theta(t_1, \omega)))|_H \\
&\quad = \overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t + t_1, f, \omega) - Y(\theta(t + t_1, \omega))|_H \\
&\quad = \overline{\lim_{t\to\infty}} \frac{1}{t + t_1} \log |U(t + t_1, f, \omega) - Y(\theta(t + t_1, \omega))|_H \cdot \lim_{t\to\infty} \frac{t + t_1}{t} \\
&\quad = \overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t, f, \omega) - Y(\theta(t, \omega))|_H \\
&\quad \le \lambda_i.
\end{aligned}$$

The above inequality implies that $U(t_1, f, \omega) \in M_i(\theta(t_1, \omega))$. Hence $U(t_1, \cdot, \omega)(M_i(\omega)) \subseteq M_i(\theta(t_1, \omega))$ and so (2.26) holds for all $t \ge 0$.

(ii) To prove assertion (ii) of the theorem, observe first that (2.25) implies that $S_i^n(\omega) \subseteq S_i^{n+1}(\omega)$ for all $n \ge 1$. Next, we show the following inclusion by induction on n:

$$S_i^n(\omega) \subset M_i(\omega) \tag{2.30}$$

for all $n \ge 1$.

Let $f \in S_i^1(\omega) = S_i(\omega)$. By Theorem 2.3 and assertion (2.15), it follows that

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log |U(t, f, \omega) - Y(\theta(t, \omega))|_H \le \lambda_i \tag{2.31}$$

perfectly in ω. Therefore, $f \in M_i(\omega)$. Hence $S_i^1(\omega) = S_i(\omega) \subset M_i(\omega)$. Assume, by induction, that

$$S_i^k(\omega) \subset M_i(\omega)$$

for all $1 \le k \le n$. If $S_i^{n+1}(\omega) \nsubseteq U(n + 1, \cdot, \omega)^{-1}[S_i(\theta(n + 1, \omega))]$, then $S_i^{n+1}(\omega) = S_i^n(\omega) \subset M_i(\omega)$, by inductive hypothesis. Otherwise, $S_i^{n+1}(\omega) = U(n + 1, \cdot, \omega)^{-1}[S_i(\theta(n + 1, \omega))]$. Let $f \in S_i^{n+1}(\omega) = U(n + 1, \cdot, \omega)^{-1}[S_i(\theta(n + 1, \omega))]$. Then by the cocycle property and the definition of $S_i(\theta(n + 1, \omega))$, it follows that

$$|U(n' + n + 1, f, \omega) - Y(\theta(n' + n + 1, \omega))|_H \le \beta_i(\theta(n + 1, \omega))e^{n'\lambda_i} \tag{2.32}$$

for all $n' \ge 1$. This implies that

$$\overline{\lim_{n'\to\infty}} \frac{1}{n'} \log |U(n' + n + 1, f, \omega) - Y(\theta(n' + n + 1, \omega))|_H \le \lambda_i\,.$$

Hence

$$\overline{\lim_{n''\to\infty}} \frac{1}{n''} \log |U(n'', f, \omega) - Y(\theta(n'', \omega))|_H$$
$$= \overline{\lim_{n'\to\infty}} \frac{1}{n'+n+1} \log |U(n'+n+1, f, \omega) - Y(\theta(n'+n+1, \omega))|_H$$
$$= \overline{\lim_{n'\to\infty}} \frac{n'}{n'+n+1} \cdot \overline{\lim_{n'\to\infty}} \frac{1}{n'} \log |U(n'+n+1, f, \omega) - Y(\theta(n'+n+1, \omega))|_H$$
$$\leq \lambda_i.$$

Therefore, $f \in M_i(\omega)$, and $S_i^{n+1}(\omega) \subset M_i(\omega)$. So, by induction on n, it follows that

$$S_i^n(\omega) \subset M_i(\omega) \tag{2.33}$$

for all $n \geq 1$. Thus

$$\bigcup_{n=1}^{\infty} S_i^n(\omega) \subseteq M_i(\omega)\,. \tag{2.34}$$

In order to prove the converse inclusion

$$M_i(\omega) \subseteq \bigcup_{n=1}^{\infty} S_i^n(\omega), \tag{2.35}$$

we make the following:

Claim. *There exist an increasing (random) sequence of integers $n^k \uparrow \infty$ such that*

$$S_i^{n^k}(\omega) = U(n^k, \cdot, \omega)^{-1}[S_i\big(\theta(n^k, \omega)\big)]$$

for all $k \geq 1$.

Proof of Claim. Define $n^1 := \inf\{n > 1\colon S_i^{n-1}(\omega) \subseteq U(n, \cdot, \omega)^{-1}\ [S_i(\theta(n, \omega))]\}$. Then

$$S_i^{n^1-1}(\omega) \subseteq U(n^1, \cdot, \omega)^{-1}[S_i(\theta(n^1, \omega))]\},$$

and by definition (2.25),

$$S_i^{n_1}(\omega) = U(n_1, \cdot, \omega)^{-1}[S_i(\theta(n_1, \omega))]. \tag{2.36}$$

Furthermore, $S_i^{n-1}(\omega) \not\subseteq U(n, \cdot, \omega)^{-1}[S_i\big(\theta(n, \omega)\big)]$ for all $1 < n < n^1$, and so by (2.25),

$$S_i^n(\omega) = S_i^{n-1}(\omega) = S_i^{n-2}(\omega) = \cdots = S_i^1(\omega) = S_i(\omega)$$

for all $1 < n < n^1$. In particular,

$$S_i(\omega) = S_i^{n^1-1}(\omega) \subseteq U(n^1, \cdot, \omega)^{-1}[S_i(\theta(n^1, \omega))].$$

Therefore,

$$U(n^1, \cdot, \omega)(S_i(\omega)) \subseteq S_i(\theta(n^1, \omega)).$$

Hence

$$n^1 = \inf\{n > 1\colon U(n,\cdot,\omega)(S_i(\omega)) \subseteq S_i(\theta(n,\omega))\}. \tag{2.37}$$

Since $S_i(\omega)$ is asymptotically cocycle-invariant (Theorem 2.3(c), (2.16)), it follows from (2.37) that $1 < n^1 < \infty$. Next, define $n^2 > n^1$ by

$$n^2 := \inf\{n > n^1\colon S_i^{n-1}(\omega) \subseteq U(n,\cdot,\omega)^{-1}[S_i(\theta(n,\omega))]\}. \tag{2.38}$$

As before, (2.25) implies that

$$S_i^{n^2-1}(\omega) = S_i^{n^1+1}(\omega) = U(n^1,\cdot,\omega)^{-1}[S_i(\theta(n^1,\omega))] \tag{2.39}$$

and

$$S_i^{n^2}(\omega) = U(n^2,\cdot,\omega)^{-1}[S_i(\theta(n^2,\omega))]. \tag{2.40}$$

Since

$$S_i^{n^2-1}(\omega) \subseteq U(n^2,\cdot,\omega)^{-1}[S_i(\theta(n^2,\omega))], \tag{2.41}$$

it follows from (2.39) that

$$U(n^1,\cdot,\omega)^{-1}[S_i(\theta(n^1,\omega))] \subseteq U(n^2,\cdot,\omega)^{-1}[S_i(\theta(n^2,\omega))].$$

Therefore,

$$U(n^2,\cdot,\omega)\{U(n^1,\cdot,\omega)^{-1}[S_i(\theta(n^1,\omega))]\} \subseteq S_i(\theta(n^2,\omega)). \tag{2.42}$$

Using the cocycle property, (2.42) implies

$$U(n^2 - n^1,\cdot,\theta(n^1,\omega))[S_i(\theta(n^1,\omega))] \subseteq S_i(\theta(n^2 - n^1,\theta(n^1,\omega))). \tag{2.43}$$

By the asymptotic cocycle-invariance of $S_i(\theta(n^1,\omega))$, it follows from (2.43) that $n^1 < n^2 < \infty$. Hence by induction, there exists an increasing sequence of integers $\{n^k\}_{k=1}^{\infty}$ such that $n^k \uparrow \infty$ as $k \to \infty$ and

$$S_i^{n^k}(\omega) = U(n^k,\cdot,\omega)^{-1}[S_i(\theta(n^k,\omega))] \tag{2.44}$$

for all integers $k \geq 1$. This completes the proof of our claim. □

We now proceed to prove the inclusion (2.35). Let $f \in M_i(\omega)$. Then by definition of $M_i(\omega)$, we have

$$\varlimsup_{t\to\infty} \frac{1}{t} \log |U(t,f,\omega) - Y(\theta(t,\omega))|_H \leq \lambda_i. \tag{2.45}$$

Fix $\epsilon_1 \in (0, -\lambda_1)$ as in Theorem 2.3. Let $0 < \epsilon < \epsilon_1$. Then, using (2.45), there exists a positive integer n_0 such that

$$\sup_{t \geq n} \frac{1}{t} \log |U(t, f, \omega) - Y(\theta(t, \omega))|_H < \lambda_i + \epsilon$$

for all $n \geq n_0$. In particular,

$$|U(n, f, \omega) - Y(\theta(n, \omega))|_H < e^{n(\lambda_i + \epsilon)}$$

for all $n \geq n_0$. Define

$$K(\omega) := \max_{1 \leq k < n_0} |U(k, f, \omega) - Y(\theta(k, \omega))|_H.$$

Therefore,

$$|U(n, f, \omega) - Y(\theta(n, \omega))|_H \leq K(\omega) e^{n(\lambda_i + \epsilon)} \tag{2.46}$$

for all $n \geq 1$.

Pick m_0 sufficiently large such that

$$K(\omega) e^{n(\epsilon - \epsilon_1)} \leq \beta_i(\theta(n, \omega)) \tag{2.47}$$

for all $n \geq m_0$. Let $n \geq m_0$, $n' \geq 0$. Using the cocycle property and (2.46), we obtain

$$\begin{aligned} &|U(n', U(n, f, \omega), \theta(n, \omega)) - Y(\theta(n', \theta(n, \omega)))|_H \\ &\qquad \leq K(\omega) e^{n(\epsilon - \epsilon_1)} \cdot e^{n'(\lambda_i + \epsilon_1)}. \end{aligned} \tag{2.48}$$

Pick $m_1 \geq m_0$ and sufficiently large such that

$$K(\omega) e^{n(\epsilon - \epsilon_1)} \leq \beta_i(\theta(n, \omega)) \tag{2.49}$$

for all $n \geq m_1$. From (2.48) and (2.49), we get

$$|U(n', U(n, f, \omega), \theta(n, \omega)) - Y(\theta(n', \theta(n, \omega)))|_H \leq \beta_i(\theta(n, \omega)) e^{n'(\lambda_i + \epsilon_1)} \tag{2.50}$$

for all $n' \geq 0$ and $n \geq m_1$. Since $|U(n, f, \omega) - Y(\theta(n, \omega))|_H \to 0$ as $n \to \infty$, then there exists $m_2 > 0$ such that $U(n, f, \omega) \in B(Y(\theta(n, \omega)), \rho_i(\theta(n, \omega)))$ for all $n \geq m_2$. Thus (2.50) implies that

$$U(n, f, \omega) \in S_i(\theta(n, \omega))$$

for all $n \geq \max(m_1, m_2)$; i.e. $f \in U(n, \cdot, \omega)^{-1}[S_i(\theta(n, \omega))]$, for all $n \geq \max(m_1, m_2)$. Now pick k sufficiently large such that $n^k \geq \max(m_1, m_2)$ and $f \in U(n^k, \cdot, \omega)^{-1}[S_i(\theta(n^k, \omega))] = S_i^{n^k}(\omega)$. This proves that $f \in \bigcup_{n=1}^{\infty} S_i^n(\omega)$; and so the inclusion (2.35) holds. The proof of (2.27) is complete; hence assertion (ii) of the theorem holds.

Assertions (iii) and (iv) of the theorem follow directly from the definition (2.23) of the flag $M_i(\omega)$, $i \geq 1$. □

Remark 2.1. It is not clear if the $M_i(\omega)$ in Theorem 2.4 are $C^{k,\epsilon}$ immersed submanifolds in H. This would require transversality of the cocycle $U(n, \cdot, \omega)$ and the local stable manifold $S_i(\theta(n, \omega))$.

3. Stochastic Burgers Equation

In this section we study the dynamics of the following one-dimensional stochastic Burgers equation (SBE) with affine linear white noise:

$$\left.\begin{aligned} du(t) &= \nu\Delta u\,dt - u\frac{\partial u}{\partial \xi}\,dt + \gamma u(t)\,dt + \sum_{k=1}^{\infty}\sigma_k u(t)\,dW_k(t) + \sigma_0(\xi)\,dW_0(t), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0, \quad \xi \in [0,1], \\ u(t,0) &= u(t,1) = 0, \quad \text{for all } t > 0, \\ u(0,\xi) &= f(\xi), \quad \xi \in [0,1]. \end{aligned}\right\} \tag{3.1}$$

In the above SBE, the noise coefficients σ_k, $k \geq 1$, are constants such that $\sum_{k=1}^{\infty}\sigma_k^2 < \infty$; the W_k, $k \geq 0$, are independent standard Brownian motions defined on the complete Wiener space $(\Omega, \mathcal{F}, P)$; σ_0 is a smooth function on $[0,1]$; $\gamma u(t)\,dt$ is a deterministic linear drift term with a fixed parameter γ; the positive constant ν is the viscosity coefficient; and f is the initial function in the energy space $L^2([0,1],\mathbb{R})$. Note that the external stochastic forcing in the SBE (3.1) is provided by the linear drift term $\gamma u(t)\,dt$, the linear white noise term $\sum_{k=1}^{\infty}\sigma_k u(t)\,dW_k(t)$ and the additive spacetime noise term $\sigma_0(\xi)\,dW_0(t)$.

Burgers spde with noise has been studied extensively by many authors, mainly due to its significance in modeling turbulence in physics and engineering. The reader may refer to works by [5, 7, 10, 15, 23, 26, 27] and the references therein.

Our analysis of the above SBE is focused on the following objectives:

- To describe the dynamics of the SBE (3.1) via a perfect locally compacting cocycle (semiflow) generated by mild solutions of the equation. This cocycle turns out to be C^∞ and satisfies the Oseledec–Ruelle integrability condition of Sec. 2.
- To characterize the almost sure long-time asymptotics for the cocycle of (3.1) using the Lyapunov spectrum of its linearization along a stationary solution of the SBE. The Lyapunov spectrum is countable and non-random.
- To prove that the SBE satisfies the *local stable manifold theorem* near a hyperbolic stationary solution, *the local invariant manifold theorem* and *the global invariant flag theorem* relative to an ergodic stationary solution.

The above issues will be addressed via direct appplication of the general results developed in the previous section (Theorems 2.1–2.4).

We use the energy space $H := L^2([0,1],\mathbb{R})$ with the Hilbert norm

$$\|f\|_{L^2} := \int_0^1 f(t)\,dt, \quad f \in L^2([0,1],\mathbb{R}),$$

as state space for the dynamics of the SBE (3.1). It is well known that for each $f \in H$, a unique mild solution $u(\cdot, f, \cdot)\colon \mathbb{R}^+ \times \Omega \to H$ of the SBE (3.1) exists. See [7] and the references therein.

Our first result shows that the mild solutions of the SBE (3.1) generate a perfect Fréchet smooth cocycle (u, θ) on the energy space $H = L^2([0,1], \mathbb{R})$ which satisfies the Oseledec integrability condition (3.4) below.

Theorem 3.1. (The cocycle) *Let $u(t, f, \omega)$ be the unique global mild solution of the SBE (3.1) for $t \geq 0$, $f \in L^2([0,1], \mathbb{R})$, $\omega \in \Omega$. Recall that $\theta \colon \mathbb{R} \times \Omega \to \Omega$ is the standard Brownian shift*

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \ \omega \in \Omega, \tag{3.2}$$

on Wiener space $(\Omega, \mathcal{F}, P)$. Then $u \colon \mathbb{R}^+ \times L^2([0,1], \mathbb{R}) \times \Omega \to L^2([0,1], \mathbb{R})$ is jointly measurable and has the following properties:

(i) *(u, θ) is a C^∞ perfect cocycle; viz.*

$$u(t_2, u(t_1, f, \omega), \theta(t_1, \omega)) = u(t_1 + t_2, f, \omega) \tag{3.3}$$

for all $t_1, t_2 \geq 0$, $f \in L^2([0,1], \mathbb{R})$, $\omega \in \Omega$.

(ii) *For fixed $t > 0$ and $\omega \in \Omega$, the map $u(t, \cdot, \omega) \colon L^2([0,1], \mathbb{R}) \to L^2([0,1], \mathbb{R})$ takes bounded sets into relatively compact sets in $L^2([0,1], \mathbb{R})$.*

(iii) *For each $(t, f, \omega) \in \mathbb{R}^+ \times L^2([0,1], \mathbb{R}) \times \Omega$, the Fréchet derivative $DU(t, f, \omega) \in L(L^2([0,1], \mathbb{R}))$ is compact linear, and the map*

$$\begin{aligned} \mathbb{R}^+ \times L^2([0,1], \mathbb{R}) \times \Omega &\longrightarrow L(L^2([0,1], \mathbb{R})) \\ (t, f, \omega) &\longmapsto Du(t, f, \omega) \end{aligned}$$

is strongly measurable.

(iv) *For fixed $\rho, a > 0$ and any integer $k \geq 1$,*

$$E \log^+ \sup_{\substack{0 \leq t_1, t_2 \leq a \\ \|f\|_{L^2} \leq \rho}} \{\|u(t_2, f, \theta(t_1, \cdot))\|_{L^2} + \|D^{(k)} u(t_2, f, \theta(t_1, \cdot))\|_{L^{(k)}(L^2)}\} < \infty \tag{3.4}$$

where $L^{(k)}(L^2)$ denotes the space of all continuous k-multilinear maps $(L^2)^k \to L^2$ given the uniform operator norm $\| \cdot \|_{L^{(k)}(L^2)}$.

Proof. We will only outline the construction of the cocycle $u \colon \mathbb{R}^+ \times L^2([0,1], \mathbb{R}) \times \Omega \to L^2([0,1], \mathbb{R})$. The main idea is a reduction of the SBE (3.1) to a random pde of Burgers type which can be solved pathwise by a lengthy uniform contraction mapping argument. The construction also yields Oseledec-type integrability estimates on the cocycle and its Fréchet derivatives (cf. Theorem 2.2). The reader may check the details ([17]).

Set

$$Q(t) := \exp\left\{ \sum_{k=1}^{\infty} \sigma_k W_k(t) - \frac{t}{2} \sum_{k=1}^{\infty} \sigma_k^2 + \gamma t \right\}, \quad t \geq 0. \tag{3.5}$$

Define the random field Z by

$$Z(t,\xi) := \int_0^t Q^{-1}(s) T_{t-s}\sigma_0(\xi) dW_0(s), \quad t > 0, \quad \xi \in [0,1]. \tag{3.6}$$

Let V_0 be a mild solution of the random pde:

$$\left.\begin{aligned} \frac{\partial V_0}{\partial t} &= \nu \Delta V_0(t) - Q(t)V_0(t,\xi)\frac{\partial V_0(t,\xi)}{\partial \xi} - Q(t)V_0(t,\xi)\frac{\partial Z(t,\xi)}{\partial \xi} \\ &\quad - Q(t)Z(t,\xi)\frac{\partial V_0(t,\xi)}{\partial \xi} - Q(t)Z(t,\xi)\frac{\partial Z(t,\xi)}{\partial \xi}, \quad t > 0, \quad \xi \in [0,1] \\ V_0(0,\xi) &= f(\xi), \quad \xi \in [0,1], \\ V_0(t,0) &= 0, \quad t > 0, \\ V_0(t,1) &= 0, \quad t > 0. \end{aligned}\right] \tag{3.7}$$

One can check that

$$u(t,\xi) := Q(t)[V_0(t,\xi) + Z(t,\xi)], \quad t \geq 0, \ \xi \in [0,1],$$

is a mild solution of the SBE (3.1). Thus to get a perfect cocycle for $u(t), t \geq 0$, it is sufficient to solve the random pde (3.7) perfectly in $\omega \in \Omega$. A unique local mild solution of (3.7) is obtained using a contraction mapping argument. The local solution may be shown to be a global one by virtue of the following *a priori* estimate:

For $f \in L^2([0,1],\mathbb{R})$, let $V_0(t,f,\omega)$ be a mild solution of the initial boundary value problem (3.7) for $0 < t < T$ and some $T > 0$. Then for each $\omega \in \Omega$ and $t \in [0,T]$,

$$\begin{aligned} &\|V_0(t,f,\omega)\|_{L^2}^2 + \nu \int_0^t \left\| \frac{\partial V_0(s,f,\omega)}{\partial \xi} \right\|_{L^2}^2 ds \\ &\qquad \leq C_T(\omega) \left[\|f\|_{L^2}^2 + \int_0^t Q(s) \left\| Z(s,\cdot)\frac{\partial Z(s,\cdot)}{\partial \xi} \right\|_{L^2}^2 ds \right] \end{aligned} \tag{3.8}$$

for all $t \in [0,T)$, and all $\omega \in \Omega$, where $C_T(\omega)$ is a constant depending only on ω and T. □

If $u(t,Y,\cdot)$ is a stationary solution of the SBE (3.1) with a random initial condition $Y : \Omega \to L^2([0,1],\mathbb{R})$, then it follows from Theorem 2.1 that the linearized cocycle $(DU(\cdot,Y,\cdot),\theta(t,\cdot))$ has a fixed discrete Lyapunov spectrum $\{\lambda_i : i \geq 1\}$. Furthermore, the following theorems hold verbatim for the SBE (3.1) with $U = u, H = L^2([0,1],\mathbb{R})$: Theorem 2.1 (the Lyapunov spectrum), Theorem 2.2 (the local stable manifold theorem), Theorem 2.3 (the local invariant manifold theorem) and Theorem 2.4 (the global invariant flag theorem).

In the absence of additive noise ($\sigma_0 = 0$), one can explicitly compute the Lyapunov spectrum of the zero equilibrium ($Y \equiv 0$) for the SBE (3.1). Indeed by linearizing (the mild form of) the SBE at $Y \equiv 0$, one obtains $\lambda_i = -\nu\pi^2 i^2 + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2$ for each $i \geq 1$.

4. Stochastic 2*D* Navier–Stokes Equation

Existence and uniqueness of solutions, asymptotic compactness, ergodicity and large deviations for $2D$ SNSE's have been studied by several authors; e.g. [3, 4, 7, 11–13, 24].

In this section, we consider the two-dimensional stochastic Navier–Stokes equation (SNSE) on a smooth bounded domain D, driven by affine linear multiplicative white noise:

$$\left.\begin{aligned} du - \nu\triangle u\, dt + (u\cdot\nabla)u\, dt + \nabla p\, dt &= \gamma u\, dt + \sigma_0\, dW_0(t,x) + \sum_{k=1}^{\infty}\sigma_k u(t)\, dW_k(t),\\ (\nabla\cdot u)(t,x) &= 0, \quad x\in D,\ t>0,\\ u(t,x) &= 0, \quad x\in\partial D,\ t>0,\\ u(0,x) &= f(x), \quad x\in D. \end{aligned}\right\} \tag{4.1}$$

In the above SNSE, D is a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂D; $u(t,x)\in\mathbb{R}^2$ is the velocity field at time t and position $x\in D$; $\triangle$ is the Dirichlet Laplacian on D; $p(t,x)$ is the pressure field; $0<\nu$ is the viscosity coefficient; $W_0(t,x)$ is additive space-time noise, white in t, and smooth in x; σ_0 is a constant; $\sigma_k \in L(\mathbb{R}^2)$, $k\geq 1$, are commuting, symmetric (2×2)-matrices with $\sum_{k=1}^{\infty}|\sigma_k|^2 < \infty$, $|\sigma_k|^2 := \mathrm{tr}(\sigma_k^2)$; $W_k, k\geq 1$, are independent one-dimensional standard Brownian motions, defined on the complete filtered Wiener space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$; W_k, $k\geq 1$, are independent of W_0; $\gamma u\, dt$ is a deterministic linear drift term with a fixed parameter γ; $f\colon D\to\mathbb{R}^2$ is the initial velocity.

We next formulate the above SNSE as a stochastic evolution equation on the space of divergence-free vector fields. Consider the Hilbert space

$$V := \{v\in H_0^1(D,\mathbb{R}^2)\colon \nabla\cdot v = 0 \text{ a.e. in } D\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v(x)|^2\, dx\right)^{\frac{1}{2}}$$

and inner product $\langle\langle\cdot,\cdot\rangle\rangle$. Denote by H the closure of V in the L^2-norm

$$|v|_H := \left(\int_D |v(x)|^2\, dx\right)^{\frac{1}{2}}, \quad v\in V.$$

Let $\langle\cdot,\cdot\rangle$ be the inner product on H, and $P_H\colon L^2(D,\mathbb{R}^2)\to H$ denote the Helmholtz–Hodge projection. The (Stokes) operator A in H is given by

$$Au := -\nu P_H\triangle u, \quad u\in H^2(D,\mathbb{R}^2)\cap V.$$

Define the bilinear operator B by $B(u, v) := P_H((u \cdot \nabla)v)$ whenever u, v are such that $(u \cdot \nabla v)$ belongs to L^2. As short-hand, use $B(u) := B(u, u)$. Apply the projection P_H to each term of the SNSE (4.1) and get the abstract form:

$$\left.\begin{aligned} du(t) + Au(t)\,dt + B(u(t))\,dt = \gamma u(t)\,dt + \sigma_0\,dW_0(t) + \sum_{k=1}^{\infty} \sigma_k^H u(t)\,dW_k(t) \\ u(0) = f \in H \end{aligned}\right\} \tag{4.2}$$

in $L^2(0, T; V')$; V' is the dual of V; $\sigma_k^H f := P_H(\sigma_k \circ f)$, $f \in H$. It is known that the SNSE (4.1) has a unique global mild solution $u(\cdot, f, \cdot) \in L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times [0, T]; V)$ for each $f \in H$ ([3]). Furthermore, mild solutions of the SNSE (4.1) generate a Fréchet $C^{1,1}$ locally compacting cocycle (viz. stochastic semiflow) $u\colon \mathbb{R}^+ \times H \times \Omega \to H$ on the Hilbert space H. To see this one uses a variational technique which transforms the SNSE (4.2) into a *random* NSE. Indeed, we can define $u\colon \mathbb{R}^+ \times H \times \Omega \to H$ by setting

$$u(t, f, \omega) := Q(t, \omega)[v(t, f, \omega) + Z(t, \omega)],$$

for $t \geq 0,\ \omega \in \Omega, f \in H$, where $Q\colon [0, \infty) \times \Omega \to L(\mathbb{R}^2)$ satisfies

$$\begin{aligned} Q(t) &= I + \gamma \int_0^t Q(s)\,ds + \sum_{k=1}^{\infty} \int_0^t \sigma_k^H Q(s)\,dW_k(s), \quad t \geq 0, \\ Z(t) &: = \sigma_0 \int_0^t Q(s)^{-1} T_{t-s}\,dW_0(s), \quad t \geq 0, \end{aligned} \tag{4.3}$$

$T_t := \exp(-tA)\colon H \to H$, $t \geq 0$, is the semigroup generated by $-A$, and $v(t) \equiv v(t, f)$ satisfy the random NSE:

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt - Q(t)B(Q(t)(v(t) + Z(t)), v(t) + Z(t))\,dt, \quad t \geq 0, \\ v(0) &= f \in H\,. \end{aligned}\right\} \tag{4.4}$$

Existence of a unique global mild solution to the above random NSE (4.4) follows by Galerkin approximations, *a priori* estimates and compactness of the embedding $V \to H$ (cf. [25]). Lipschitz and Fréchet differentiability ($C^{1,1}$) properties for v and hence for u are obtained using lengthy *a priori* estimates on v and its Gateaux derivatives. Uniqueness of the solution to the random NSE (4.4) implies the identity

$$\begin{aligned} &Q(t_1, \omega)[v(t_1 + t_2, f, \omega) + T_{t_2} Z(t_1, \omega)] \\ &\qquad = v(t_2, Q(t_1, \omega)[v(t_1, f, \omega) + Z(t_1, \omega)], \theta(t_1, \omega)) \end{aligned} \tag{4.5}$$

for $t_1, t_2 \geq 0, \omega \in \Omega, f \in H$. The cocycle property for u follows from the above identity (4.5). If $Y : \Omega \to H$ is a stationary point of the SNSE such that $E \log^+ |Y| < \infty$, then the Oseledec integrability condition (2.10) holds for $U = u, k = 1, \epsilon = 1$. This follows from the *a priori* estimates on v. One may then conclude that the linearized cocycle at Y has a fixed discrete Lyapunov spectrum and Theorems 2.1–2.4 hold verbatim for the SNSE (4.1) with $U = u, k = 1, \epsilon = 1$. These results follow by extending the arguments in [18].

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