Stochastic Dynamical Systems in Infinite Dimensions[†]

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Abstract

We study the local behavior of infinite-dimensional stochastic semiflows near hyperbolic equilibria. The semiflows are generated by stochastic differential systems with finite memory, stochastic evolution equations and semilinear stochastic partial differential equations.

1 Introduction

In this article, we summarize some results on the existence and qualitative behavior of stochastic dynamical systems in infinite dimensions. The three main examples covered are stochastic systems with finite memory (stochastic functional differential equations-sfde's), semilinear stochastic evolution equations (see's) and stochastic partial differential equations (spde's). Due to limitations of space, our summary is by no means intended to be exhaustive: The emphasis will be mainly on the local behavior of infinite-dimensional stochastic dynamical systems near hyperbolic equilibria (or stationary solutions).

The main highlights of the article are:

- Infinite-dimensional cocycles
- Ruelle's spectral theory for compact linear cocycles in Hilbert space
- Stationary points (equilibria). Hyperbolicity
- Existence of stable/unstable manifolds near equilibria
- Cocycles generated by regular sfde's. Singular sfde's
- Cocycles generated by semilinear see's and spde's
- Solutions of anticipating semilinear sfde's and see's.

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[†]This article is dedicated to Heinrich von Weizsäcker on his sixtieth birthday celebration.

2 What is a stochastic dynamical system?

We begin by formulating the idea of a *stochastic semiflow* or an infinite-dimensional *cocycle* which is central to the analysis in this work.

First, we establish some notation.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $\bar{\mathcal{F}}$ the P-completion of $\mathcal{F}$, and let $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions ([32]).

If E is a topological space, we denote by $\mathcal{B}(E)$ its Borel σ-algebra. If E is a Banach space, we may give the space $L(E)$ of all bounded linear operators on E the *strong topology*, viz. the smallest topology with respect to which all evaluations $L(E) \ni T \mapsto T(x) \in E$, $x \in E$, are continuous. Denote by $\mathcal{B}_s(L(E))$ the σ-algebra generated by the strong topology on $L(E)$. Let $\mathbf{R}$ denote the set of all reals, and $\mathbf{R}^+ := [0, \infty)$. We say that a process $T : \mathbf{R}^+ \times \Omega \to L(E)$ is *strongly measurable* if it is $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}_s(L(E)))$-measurable.

Let k be a positive integer and $0 < \epsilon \leq 1$. If E and N are real Banach spaces with norms $|\cdot|$, we will denote by $L^{(k)}(E, N)$ the Banach space of all continuous k-multilinear maps $A : E^k \to N$ with the uniform norm $\|A\| := \sup\{|A(v_1, v_2, \cdots, v_k)| : v_i \in E, |v_i| \leq 1, i = 1, \cdots, k\}$. We let $L^{(k)}(E)$ stand for $L^{(k)}(E, E)$. Suppose $U \subseteq E$ is an open set. A map $f : U \to N$ is said to be *of class* $C^{k,\epsilon}$ if it is C^k and if $D^{(k)} f : U \to L^{(k)}(E, N)$ is ϵ-Hölder continuous on bounded sets in U. A $C^{k,\epsilon}$ map $f : U \to N$ is said to be *of class* $C_b^{k,\epsilon}$ if all its derivatives $D^{(j)} f, 1 \leq j \leq k$, are globally bounded on U, and $D^{(k)} f$ is ϵ-Hölder continuous on U. If $U \subset E$ is open and bounded, denote by $C^{k,\epsilon}(U, N)$ the Banach space of all $C^{k,\epsilon}$ maps $f : U \to N$ given the norm:

$$\|f\|_{k,\epsilon} := \sup_{\substack{x \in U \\ 0 \leq j \leq k}} \|D^j f(x)\| + \sup_{\substack{x_1, x_2 \in U \\ x_1 \neq x_2}} \frac{|D^k f(x_1) - D^k f(x_2)|}{|x_1 - x_2|^\epsilon}.$$

We now define a *cocycle* on Hilbert space.

Definition 2.1 (Cocycle). Let $\theta : \mathbf{R} \times \Omega \to \Omega$ be a $(\mathcal{B}(\mathbf{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable group of P-preserving transformations on the probability space $(\Omega, \mathcal{F}, P)$, H a real separable Hilbert space, k a non-negative integer and $\epsilon \in (0, 1]$. A $C^{k,\epsilon}$ *perfect cocycle* (U, θ) on H is a $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable random field $U : \mathbf{R}^+ \times H \times \Omega \to H$ with the following properties:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times H \ni (t, x) \mapsto U(t, x, \omega) \in H$ is continuous; and for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $H \ni x \mapsto U(t, x, \omega) \in H$ is $C^{k,\epsilon}$.

(ii) $U(t + s, \cdot, \omega) = U(t, \cdot, \theta(s, \omega)) \circ U(s, \cdot, \omega)$ for all $s, t \in \mathbf{R}^+$ and all $\omega \in \Omega$.

(iii) $U(0, x, \omega) = x$ for all $x \in H, \omega \in \Omega$.

Using Definition 2.1, it is easy to check that a cocycle (U, θ) corresponds to a one-parameter semigroup on $H \times \Omega$.

Throughout this article we will assume that each P-preserving transformation $\theta(t, \cdot) : \Omega \to \Omega$ is ergodic.

3 Spectral theory of linear cocycles: Hyperbolicity

The question of hyperbolicity is central to many studies of finite and infinite-dimensional (stochastic) dynamical systems. This question focuses on the characterization of almost-sure "saddle-like behavior" of the nonlinear stochastic dynamical system when linearized at a given statistical equilibrium. Statistical equilibria are viewed as random points in the infinite-dimensional state space called *stationary points* of the non-linear cocycle. For the underlying stochastic differential equation, the stationary points correspond to *stationary solutions*.

The main results in this section are the *spectral theorem* for a compact *linear* infinite-dimensional cocycle (Theorem 3.1) and the *saddle-point property* in the hyperbolic case (Theorem 3.2). A discrete version of the spectral theorem was established in the fundamental work of D. Ruelle ([34]), using multiplicative ergodic theory techniques. A continuous version of the spectral theorem is developed in [19] within the context of linear stochastic systems with finite memory. See also work by the author and M. Scheutzow on regular stochastic systems with finite memory ([22]), and joint work with T.S. Zhang and H. Zhao ([27]). The spectral theorem gives a deterministic discrete *Lyapunov spectrum* or *set of exponential growth rates* for the linear cocycle. The proof of the spectral theorem uses infinite-dimensional discrete multiplicative ergodic theory techniques and interpolation arguments in order to control the excursions of the cocycle between discrete times. A linear cocycle is *hyperbolic* if its Lyapunov spectrum does not contain zero.

For a nonlinear cocycle, a stationary point is defined to be *hyperbolic* if the linearized cocycle (at the stationary point) is hyperbolic. Under such a hyperbolicity condition, one may obtain a *local stable manifold theorem* for the non-linear cocycle near the stationary point (Theorem 4.1).

Throughout the article we will use the following convention:

Definition 3.1 (Perfection). A family of propositions $\{P(\omega) : \omega \in \Omega\}$ is said to *hold perfectly in* ω if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$ and $P(\omega)$ is true *for every* $\omega \in \Omega^*$.

We now define a *stationary point* for a cocycle (U, θ) in Hilbert space H.

Definition 3.2 (Stationary Point). An $\mathcal{F}$-measurable random variable $Y : \Omega \to H$ is said be a *stationary random point* for the cocycle (U, θ) if it satisfies the following identity:

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \tag{3.1}$$

for all $t \in \mathbf{R}^+$, perfectly in $\omega \in \Omega$.

The reader may note that the above definition is an infinite-dimensional analogue of a corresponding concept of cocycle-invariance that was used by the author in joint work with M. Scheutzow to give a proof of the stable manifold theorem for stochastic ordinary differential equations (sode's) (Definition 3.1, [21]). Definition 3.2 above essentially gives a useful realization of the idea of an invariant measure for a stochastic dynamical system generated by an sode, a stochastic functional differential equation (sfde), a stochastic evolution equation (see) or an spde. Such a realization allows us to analyze the local *almost sure* stability properties of the stochastic semiflow in the neighborhood of the stationary point. The existence (and uniqueness/ergodicity) of a stationary random point for various classes of spde's and see's has been studied by many researchers; see for example [7] and the references therein.

The following spectral theorem gives a fixed discrete set of Lyapunov exponents for a compact linear cocycle (T, θ) on H. The discreteness of the Lyapunov spectrum is a consequence of the compactness of the cocycle, while the ergodicity of the shift θ guarantees that the spectrum is deterministic. This fact allows us to define hyperbolicity of the linear cocycle (T, θ) and hence that of the stationary point Y of a nonlinear cocycle (U, θ).

Theorem 3.1 (Oseledec-Ruelle). *Let H be a real separable Hilbert space. Suppose (T, θ) is an $L(H)$-valued strongly measurable cocycle such that there exists $t_0 > 0$ with $T(t, \omega)$ compact for all $t \geq t_0$. Assume that $T : \mathbf{R}^+ \times \Omega \to L(H)$ strongly measurable and*

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\|_{L(H)} + E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(H)} < \infty.$$

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbf{R}^+$, and for each $\omega \in \Omega_0$, the limit

$$\Lambda(\omega) := \lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)}$$

exists in the uniform operator norm. Each linear operator $\Lambda(\omega)$ is compact, non-negative and self-adjoint with a discrete spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the Lyapunov exponents λ_i's are distinct and non-random. Each eigenvalue $e^{\lambda_i} > 0$ has a fixed finite non-random multiplicity m_i and a corresponding eigen-space $F_i(\omega)$, with $m_i := \dim F_i(\omega)$. Set $i = \infty$ when $\lambda_i = -\infty$. Define

$$E_1(\omega) := H, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \ i > 1, \ E_\infty := ker \Lambda(\omega).$$

Then

$$E_\infty \subset \cdots \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = H,$$

$$\lim_{t \to \infty} \frac{1}{t} \log |T(t, \omega) x| = \begin{cases} \lambda_i & if \quad x \in E_i(\omega) \backslash E_{i+1}(\omega), \\ -\infty & if \quad x \in E_\infty(\omega), \end{cases}$$

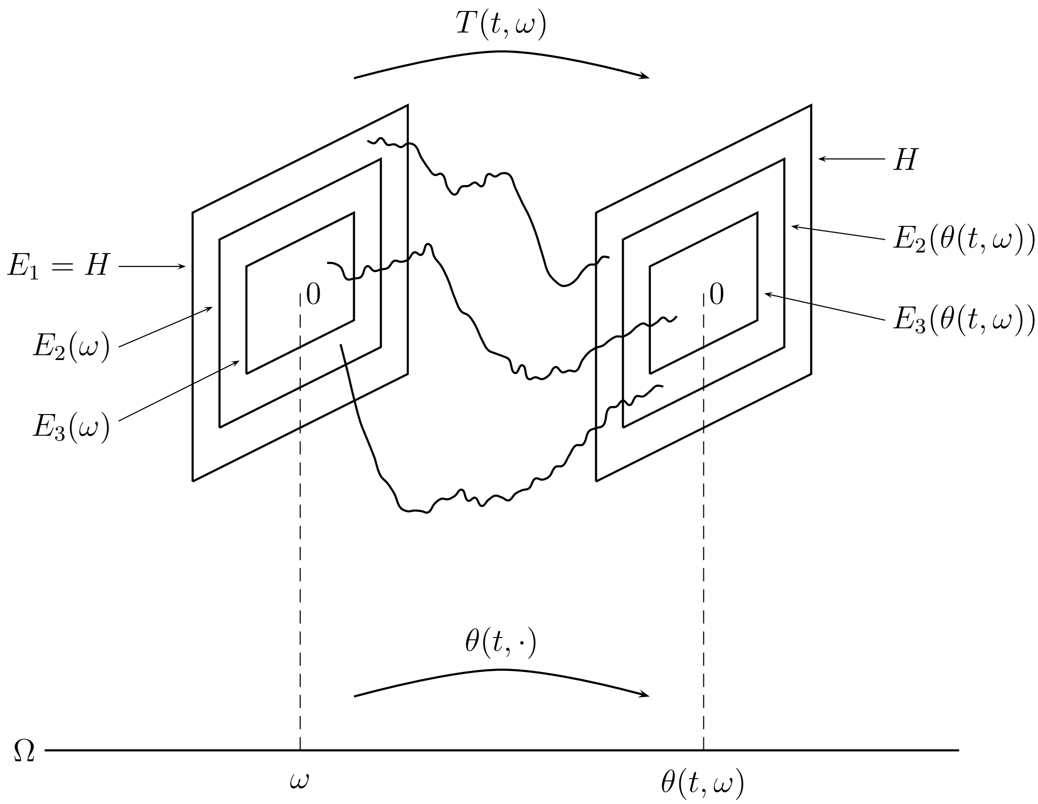
and

$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0, \ i \geq 1$.

The following figure illustrates the Oseledec-Ruelle theorem.

The Spectral Theorem



Proof of Theorem 3.1.

The proof is based on a discrete version of Oseledec's multiplicative ergodic theorem and the perfect ergodic theorem ([33], pp. 303-304; cf. [31], [19], Lemma 5). Details of the extension to continuous time are given in [19] within the context of linear stochastic functional differential equations with finite memory. The arguments in [19] extend directly to general linear cocycles in a separable Hilbert space. Cf. [10]. □

Definition 3.3. Let (T, θ) be a linear cocycle on a Hilbert space H satisfying all the conditions of Theorem 3.1. The cocycle (T, θ) is said to be *hyperbolic* if its *Lyapunov spectrum* $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ does not vanish, in the sense that $\lambda_i \neq 0$ for all $i \geq 1$.

The following result is a "random saddle point property" for hyperbolic linear cocycles. A proof is given in ([19], Theorem 4, Corollary 2; [24], Theorem 5.3) within the context of stochastic differential systems with finite memory; but the arguments therein extend immediately to linear cocycles in a separable Hilbert space.

Theorem 3.2 (The saddle point property). *Let (T, θ) be a hyperbolic linear cocycle on a Hilbert space H. Assume that*

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq 1} \|T(t_2, \theta(t_1, \cdot))\|_{L(H)} < \infty;$$

and denote by $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ *the non-vanishing Lyapunov spectrum of* (T, θ).

Pick $i_0 > 1$ *such that* $\lambda_{i_0} < 0 < \lambda_{i_0 - 1}$. *Then the following assertions hold perfectly in* $\omega \in \Omega$:

There exist stable and unstable subspaces $\{S(\omega), U(\omega)\}$, $\mathcal{F}$*-measurable (into the Grassmannian), such that*

(i) $H = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega)$. *The unstable subspace* $\mathcal{U}(\omega)$ *is finite-dimensional with a fixed non-random dimension, and the stable subspace* $\mathcal{S}(\omega)$ *is closed with a finite non-random codimension. In fact,* $\mathcal{S}(\omega) := E_{i_0}$.

(ii) *(Invariance)*

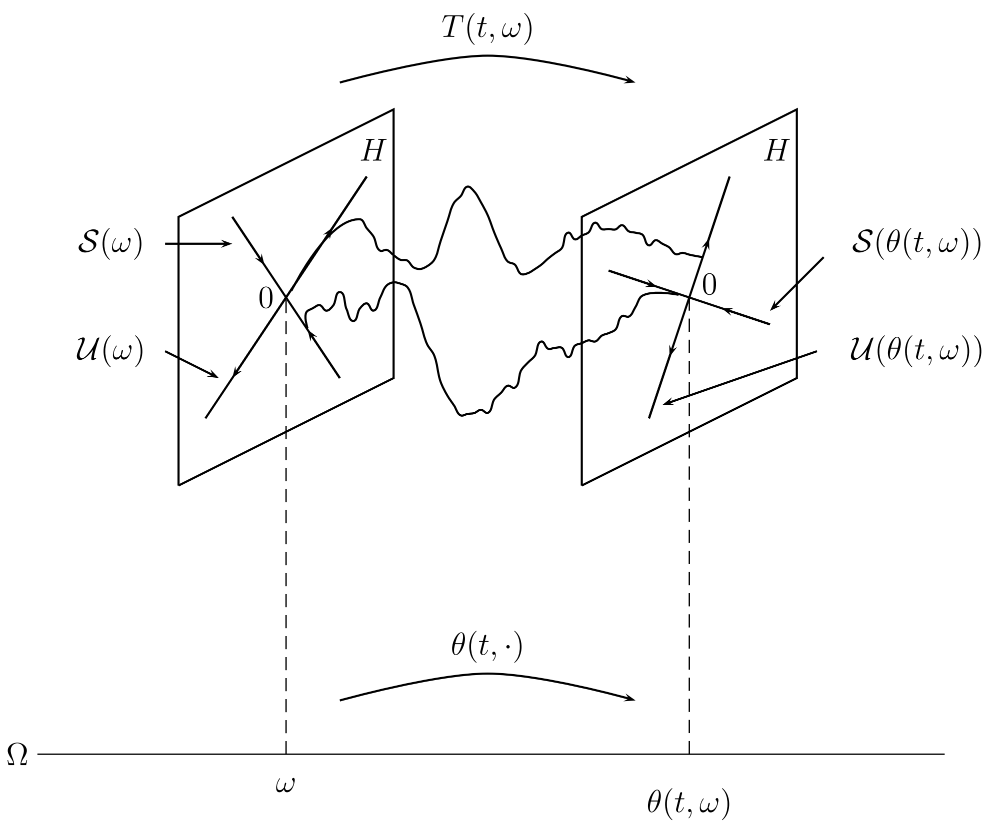
$$T(t,\omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t,\omega)),\ T(t,\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)),$$

for all $t \geq 0$,

(iii) *(Exponential dichotomies)*

$$|T(t,\omega)(x)| \geq |x| e^{\delta_1 t} \quad \textit{for all} \quad t \geq \tau_1^*, x \in \mathcal{U}(\omega),$$
$$|T(t,\omega)(x)| \leq |x| e^{-\delta_2 t} \quad \textit{for all} \quad t \geq \tau_2^*, x \in \mathcal{S}(\omega),$$

where $\tau_i^* = \tau_i^*(x,\omega) > 0, i = 1, 2,$ *are random times and* $\delta_i > 0, i = 1, 2,$ *are fixed.*



We are now in a position to define the concept of *hyperbolicity* for a stationary point Y of the nonlinear cocycle (U, θ):

Definition 3.4. Let (U, θ) be a $C^{k,\epsilon}$ ($k \geq 1, \epsilon \in (0, 1]$) perfect cocycle on a separable Hilbert space H and there exists $t_0 > 0$ such that $U(t, \cdot, \omega) : H \to H$ takes bounded sets into relatively compact sets for each $(t, \omega) \in (t_0, \infty) \times \Omega$. A stationary point $Y : \Omega \to H$ of the cocycle (U, θ) is said to be *hyperbolic* if

(a) For any $a \in (t_0, \infty)$,

$$\int_\Omega \log^+ \sup_{0 \leq t_1, t_2 \leq a} \|DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega))\|_{L(H)} \, dP(\omega) < \infty.$$

(b) The linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$, viz. $\lambda_i \neq 0$ for all $i \geq 1$.

Note that, in Definition 3.4, the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$ has a discrete non-random Lyapunov spectrum because of the compactness hypothesis on (U, θ) and the integrability condition (a). This follows immediately from the Oseledec-Ruelle spectral theorem (Theorem 3.1).

4 The local stable manifold theorem

In this section, we will show that, within a stationary random neighborhood of a hyperbolic stationary point, the long-time asymptotics of the cocycle are characterized by local stable and unstable manifolds. The stable/unstable manifolds are smooth, random and asymptotically forward/backward invariant (viz. stationary) under the non-linear cocycle. Unlike the issue of ergodicity, the quest for hyperbolic behavior is driven by the need to identify generic classes of stochastic dynamical systems. Indeed, our approach is philosophically distinct from the search for uniquely ergodic statistical equilibria in stochastic differential equations, or for globally asymptotically stable critical points for deterministic dynamical systems. There is a considerable volume of current and recent research on the ergodicity of stochastic partial differential equations. See [7], [14] and the references therein. However, little is known regarding generic behavior of stochastic dynamical systems. It is hoped that the results in this article would open the door for further research in this direction.

The main result in this section is the *local stable manifold theorem* (Theorem 4.1 below). This result characterizes the asymptotic behavior of the cocycle (U, θ) in a random neighborhood of a hyperbolic stationary point. The local stable manifold theorem is the main tool that we use to analyze the almost sure stability of cocycles generated by stochastic systems with memory, semilinear see's and spde's. The proof of the theorem is a non-trivial refinement and extension to the continuous-time setting of discrete-time results due to D. Ruelle ([33], [34]). An outline of the main ideas in the proof of Theorem 4.1 is given after the statement of the theorem. For further details the reader may consult ([21], [22] and [27]).

In what follows, we denote by $B(x,\rho)$ the open ball, radius ρ and center $x \in H$, and by $\bar{B}(x,\rho)$ the corresponding closed ball.

Theorem 4.1 (The local stable manifold theorem). *Let (U,θ) be a $C^{k,\epsilon}$ ($k \geq 1, \epsilon \in (0,1]$) perfect cocycle on a separable Hilbert space H such that for each $(t,\omega) \in (0,\infty) \times \Omega$, $U(t,\cdot,\omega) : H \to H$ takes bounded sets into relatively compact sets. For any $\rho \in (0,\infty)$, denote by $\|\cdot\|_{k,\epsilon}$ the $C^{k,\epsilon}$-norm on the Banach space $C^{k,\epsilon}(\bar{B}(0,\rho), H)$. Let Y be a hyperbolic stationary point of the cocycle (U,θ) satisfying the following integrability property:*

$$\int_\Omega \log^+ \sup_{0 \leq t_1, t_2 \leq a} \|U(t_2, Y(\theta(t_1,\omega)) + (\cdot), \theta(t_1,\omega))\|_{k,\epsilon} \, dP(\omega) < \infty \tag{$*$}$$

for any fixed $0 < \rho, a < \infty$ and $\epsilon \in (0,1]$. Denote by $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ the Lyapunov spectrum of the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t,\omega), t \geq 0)$. Define $\lambda_{i_0} := max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite λ_i are positive, set $\lambda_{i_0} := -\infty$. (Thus λ_{i_0-1} is the smallest positive Lyapunov exponent of the linearized cocycle, if at least one $\lambda_i > 0$; when all the λ_i's are negative, set $\lambda_{i_0-1} := \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) *a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$,*

(ii) *$\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1,2$, such that for each $\omega \in \Omega^*$, the following is true:*

There are $C^{k,\epsilon}$ ($\epsilon \in (0,1]$) submanifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) *For $\lambda_{i_0} > -\infty$, $\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that*

$$|U(n,x,\omega) - Y(\theta(n,\omega))| \leq \beta_1(\omega)\, e^{(\lambda_{i_0}+\epsilon_1)n}$$

for all integers $n \geq 0$. If $\lambda_{i_0} = -\infty$, then $\tilde{\mathcal{S}}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|U(n,x,\omega) - Y(\theta(n,\omega))| \leq \beta_1(\omega)\, e^{\lambda n}$$

for all integers $n \geq 0$ and any $\lambda \in (-\infty, 0)$. Furthermore,

$$\limsup_{t\to\infty} \frac{1}{t} \log |U(t,x,\omega) - Y(\theta(t,\omega))| \leq \lambda_{i_0} \tag{4.1}$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. Each stable subspace $\mathcal{S}(\omega)$ of the linearized cocycle $(DU(t, Y(\cdot), \cdot), \theta(t,\cdot))$ is tangent at $Y(\omega)$ to the submanifold $\tilde{\mathcal{S}}(\omega)$, viz. $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$. In particular, codim $\tilde{\mathcal{S}}(\omega)$ = codim $\mathcal{S}(\omega)$, is fixed and finite.

(b) $\displaystyle \limsup_{t\to\infty} \frac{1}{t} \log \left[\sup\left\{ \frac{|U(t,x_1,\omega) - U(t,x_2,\omega)|}{|x_1 - x_2|} : x_1 \neq x_2,\ x_1,\ x_2 \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}.$

(c) *(Cocycle-invariance of the stable manifolds):*
There exists $\tau_1(\omega) \geq 0$ *such that*

$$U(t,\cdot,\omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t,\omega)) \tag{4.2}$$

for all $t \geq \tau_1(\omega)$. *Also*

$$DU(t,Y(\omega),\omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t,\omega)), \quad t \geq 0. \tag{4.3}$$

(d) *For* $\lambda_{i_0-1} < \infty$, $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega),\rho_2(\omega))$ *with the property that there is a discrete-time "history" process* $y(\cdot,\omega) : \{-n : n \geq 0\} \to H$ *such that* $y(0,\omega) = x$ *and for each integer* $n \geq 1$, *one has* $U(1, y(-n,\omega), \theta(-n,\omega)) = y(-(n-1),\omega)$ *and*

$$|y(-n,\omega) - Y(\theta(-n,\omega))| \leq \beta_2(\omega) e^{-(\lambda_{i_0-1}-\epsilon_2)n}.$$

If $\lambda_{i_0-1} = \infty$, $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega),\rho_2(\omega))$ *with the property that there is a discrete-time "history" process* $y(\cdot,\omega) : \{-n : n \geq 0\} \to H$ *such that* $y(0,\omega) = x$ *and for each integer* $n \geq 1$,

$$|y(-n,\omega) - Y(\theta(-n,\omega))| \leq \beta_2(\omega) e^{-\lambda n},$$

for any $\lambda \in (0,\infty)$. *Furthermore, for each* $x \in \tilde{\mathcal{U}}(\omega)$, *there is a unique continuous-time "history" process also denoted by* $y(\cdot,\omega) : (-\infty, 0] \to H$ *such that* $y(0,\omega) = x$, $U(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0, 0 \leq t \leq -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log |y(-t,\omega) - Y(\theta(-t,\omega))| \leq -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized cocycle* $(DU(t,Y(\cdot),\cdot),\theta(t,\cdot))$ *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$. *In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

(e) *Let* $y(\cdot, x_i, \omega), i = 1,2$, *be the history processes associated with* $x_i = y(0,x_i,\omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1,2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{\frac{|y(-t,x_1,\omega) - y(-t,x_2,\omega)|}{|x_1 - x_2|} : x_1 \neq x_2,\ x_i \in \tilde{\mathcal{U}}(\omega), i = 1,2\right\}\right] \leq -\lambda_{i_0-1}.$$

(f) *(Cocycle-invariance of the unstable manifolds):*
There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq U(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega))) \tag{4.4}$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DU(t,\cdot,\theta(-t,\omega))(\mathcal{U}(\theta(-t,\omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction

$$DU(t,\cdot,\theta(-t,\omega))|\mathcal{U}(\theta(-t,\omega)) : \mathcal{U}(\theta(-t,\omega)) \to \mathcal{U}(\omega), \quad t \geq 0,$$

is a linear homeomorphism onto.

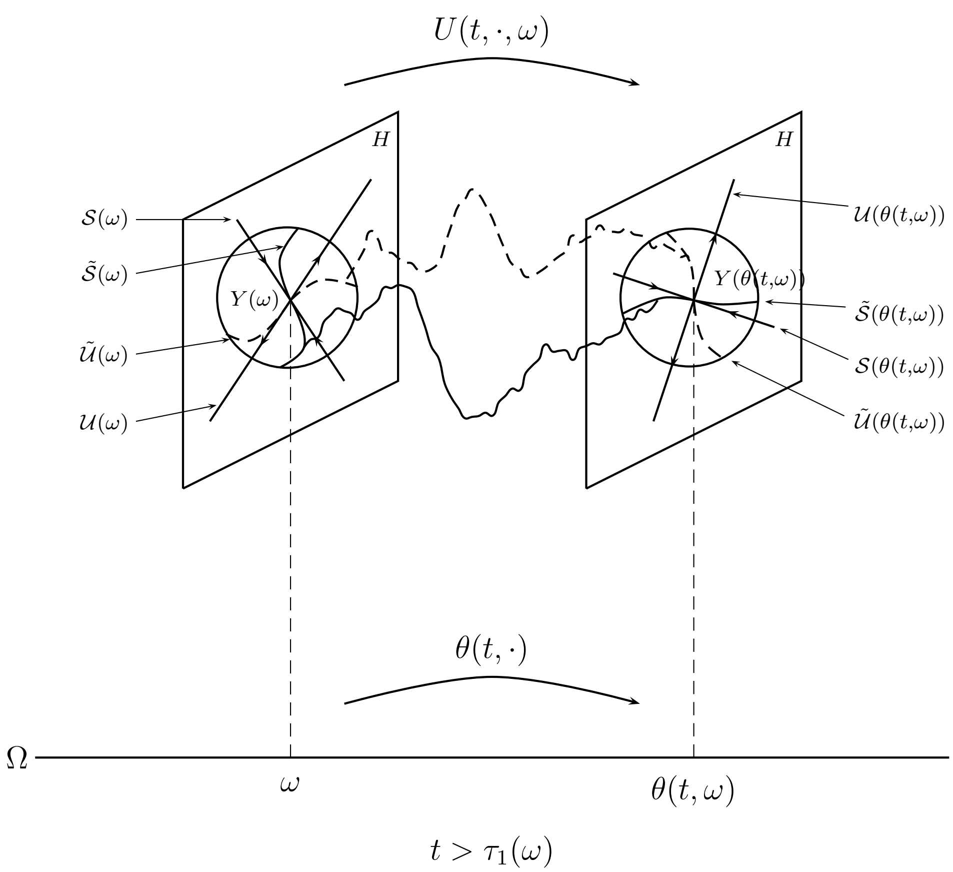
(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$H = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

Assume, in addition, that the cocycle (U,θ) *is* C^∞. *Then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are also* C^∞.

Below is an illustration of the local stable manifold theorem.

The Stable Manifold Theorem



Due to limitations of space, it is not possible to give a complete proof of the local stable manifold theorem (Theorem 4.1). However, we will outline below its main ingredients. For further details, the reader may consult ([21], [22], [27]).

An outline of the proof of Theorem 4.1:

- Develop perfect continuous-time versions of Kingman's subadditive ergodic theorem as well as the ergodic theorem ([27], Lemma 2.3.1 (ii), (iii)). The linearized cocycle $(DU(t,Y),\theta(t))$ at the hyperbolic stationary point Y can be shown to satisfy the hypotheses of these perfect ergodic theorems. As a consequence of the perfect ergodic theorems one obtains stable/unstable subspaces for the linearized cocycle, which will constitute tangent spaces to the local stable and unstable manifolds of the nonlinear cocycle (U,θ).

- The non-linear cocycle (U,θ) may be "centered" around the hyperbolic equilibrium $Y(\theta(t))$ by using the auxiliary perfect cocycle (Z,θ):

  $$Z(t,\cdot,\omega) := U(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)),\ t\in\mathbf{R}^+,\omega\in\Omega.$$

  Hence $0\in H$ becomes a fixed hyperbolic equilibrium for the auxiliary cocycle (Z,θ). We then use hyperbolicity of Y, the continuous-time integrability condition $(*)$ on the cocycle and perfect versions of the ergodic and subadditive ergodic theorems to show the existence of local stable/unstable manifolds for the discrete auxiliary cocycle $(Z(n,\cdot,\omega),\theta(n,\omega))$ near 0 (cf. [34], Theorems 5.1 and 6.1). These manifolds are random objects and are perfectly defined for $\omega\in\Omega$. Local stable/unstable manifolds for the discrete cocycle $U(n,\cdot,\omega)$ near the equilibrium Y are then obtained via translating the corresponding local manifolds for Z by the stationary point $Y(\omega)$. Using interpolation between discrete times and the (continuous-time) integrability condition $(*)$, it can be shown that the above manifolds for the discrete-time cocycle $(U(n,\cdot,\omega),\theta(n,\omega)), n\geq 1$, also serve as perfectly defined local stable/unstable manifolds for the *continuous-time* cocycle (U,θ) near Y (see [21], [22], [27], [34]).

- Using the integrability condition $(*)$ on the nonlinear cocycle and its Fréchet derivatives, it is possible to control the excursions of the continuous-time cocycle (U,θ) between discrete times. In view of the perfect subadditive ergodic theorem, these estimates show that the local stable manifolds are asymptotically invariant under the non-linear cocycle. The asymptotic invariance of the unstable manifolds is obtained via the concept of a *stochastic history process* for the cocycle. The existence of a stochastic history process is needed because the cocycle is not invertible.

This completes the outline of the proof of Theorem 4.1. □

5 Stochastic systems with finite memory

In order to formulate the stochastic dynamics of systems with finite memory (sfde's), we will first describe the class of *regular* sfde's which admit locally compact smooth cocycles.

It is important to note that not all sfde's are regular: Indeed, consider the simple one-dimensional linear stochastic delay differential equation (sdde):

$$\left.\begin{aligned} dx(t) &= x(t-1)\,dW(t), \quad t>0, \\ (x(0),x_0) &= (v,\eta)\in \mathbf{R}\times L^2([-1,0],\mathbf{R}), \end{aligned}\right\} \tag{5.1}$$

with initial condition $(v,\eta)\in \mathbf{R}\times L^2([-1,0],\mathbf{R})$. In (5.1), we use the symbol ${}^{(v,\eta)}x_t \in L^2([-1,0],\mathbf{R})$ to represent the *segment* (or *slice*) of the solution path ${}^{(v,\eta)}x : [-1,\infty)\times\Omega\to\mathbf{R}$ at time $t\geq 0$, viz.: ${}^{(v,\eta)}x_t(s) :={}^{(v,\eta)} x(t+s)$, $s\in[-1,0]$, $t\geq 0$. The trajectory $\{{}^{(v,\eta)}x_t : t\geq 0, v\in\mathbf{R}, \eta\in L^2([-1,0],\mathbf{R})\}$ of (5.1) does not admit a measurable version $\mathbf{R}^+\times\mathbf{R}\times L^2([-1,0],\mathbf{R})\times\Omega\to L^2([-1,0],\mathbf{R})$ that is pathwise continuous (or even linear) in $\eta\in L^2([-1,0],\mathbf{R})$ ([17], pp. 144–149, [18]). Sfde's such as (5.1) above, which do not admit continuous stochastic semiflows, are called *singular*.

At this point, we should note that in spite of the easy estimate

$$E\|{}^{(0,\eta_1)}x_1 -{}^{(0,\eta_2)} x_1\|_2^{2p} \leq C\|\eta_1-\eta_2\|_2^{2p}, \quad \eta_1,\eta_2\in L^2([-1,0],\mathbf{R}),\ p\geq 1,$$

Kolmogorov's continuity theorem fails to yield a pathwise continuous version of the random field $\{{}^{(0,\eta)}x_1 : \eta\in L^2([-1,0],\mathbf{R})\}$.

Due to the pathological behavior of infinite-dimensional stochastic dynamical systems such as (5.1), it is imperative that one should address perfection issues for such systems with due care.

The construction of the cocycle for regular sfde's is based on the theory of stochastic flows for stochastic *ordinary* differential equations (sode's) in finite dimensions. Once the cocycle is established, we then identify sufficient regularity and growth conditions on the coefficients of the sfde that will allow us to apply the local stable manifold theorem (Theorem 4.1). This yields the existence of local stable/unstable manifolds near hyperbolic stationary solutions of the regular sfde.

Existence of cocycles for regular sfde's:

Let $(\Omega,\mathcal{F},P)$ be Wiener space where $\Omega := C(\mathbf{R},\mathbf{R}^p;0)$ is the space of all continuous paths $\omega:\mathbf{R}\to\mathbf{R}^p$ with $\omega(0)=0$, $\mathcal{F}$ is the Borel σ-field generated by the topology of uniform convergence on compacta, and P is Wiener measure on $C(\mathbf{R}^+,\mathbf{R}^p;0)$. Denote by $\bar{\mathcal{F}}$ the P-completion of $\mathcal{F}$, and by $\mathcal{F}_t$ the P-completion of the sub-σ-algebra of $\mathcal{F}$ generated by all evaluations $\Omega\ni\omega\to\omega(u)\in\mathbf{R}^p$, $u\leq t$. Thus $(\Omega,\bar{\mathcal{F}},(\mathcal{F}_t)_{t\geq 0},P)$ is a complete filtered probability space satisfying the usual conditions ([32]). Fix an arbitrary delay $r>0$ and a positive integer dimension d.

Consider the stochastic functional differential equation (sfde):

$$\left.\begin{aligned} dx(t) &= H(x(t),x_t)\,dt + G(x(t))\,dW(t), \quad t\geq 0 \\ x(0) &= v\in\mathbf{R}^d, \quad x_0=\eta\in L^2([-r,0],\mathbf{R}^d). \end{aligned}\right\} \tag{5.2}$$

A solution of (5.2) is a process $x : [-r, \infty) \times \Omega \to \mathbf{R}^d$ whereby x_t denotes the segment

$$x_t(\cdot, \omega)(s) := x(t+s, \omega), \quad s \in [-r, 0],\ \omega \in \Omega,\ t \geq 0,$$

and (5.2) holds a.s. The state space for (5.2) is the Hilbert space $M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ endowed with the norm

$$\|(v, \eta)\|_{M_2} := (|v|^2 + \|\eta\|_{L^2}^2)^{1/2}, \qquad v \in \mathbf{R}^d, \eta \in L^2([-r, 0], \mathbf{R}^d).$$

The drift is a globally bounded $C^{k,\delta}$ functional $H : M_2 \to \mathbf{R}^d$, the noise coefficient is a $C_b^{k+1,\delta}$ mapping $G : \mathbf{R}^d \to \mathbf{R}^{d \times p}$, and W is p-dimensional Brownian motion on $(\Omega, \bar{\mathcal{F}}, (\mathcal{F}_t)_{t \geq 0}, P)$:

$$W(t, \omega) := \omega(t), \quad t \in \mathbf{R}^+, \omega \in \Omega.$$

Denote by $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ the P-preserving ergodic Brownian shift

$$\theta(t, \omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

It is known that the sfde (5.2) admits a unique family of trajectories $\{({}^{(v,\eta)}x(t), {}^{(v,\eta)}x_t) : t \geq 0, (v, \eta) \in M_2\}$ ([17], [20]). In our next result, we will show that the ensemble of all these trajectories can be viewed as a $C^{k,\epsilon}$ $(0 < \epsilon < \delta)$ locally compact cocycle (U, θ) on M_2 satisfying $U(t, (v, \eta), \cdot) = ({}^{(v,\eta)}x(t), {}^{(v,\eta)}x_t)$ for all $(v, \eta) \in M_2$ and $t \geq 0$, a.s. (Definition 2.1). The cocycle property is still maintained if H and G are allowed to be stationary, or if the diffusion coefficient $G(x(t))$ is replaced by a smooth memory-dependent term of the form $G(x(t), g(x_t))$ where the path $\mathbf{R}^+ \ni t \mapsto g(x_t) \in \mathbf{R}^d$ is locally of bounded variation. More general noise terms such as Kunita-type spatial semimartingales may also be allowed ([16]). The construction of the cocycle uses the finite-dimensional stochastic flow for the diffusion term coupled with a non-linear variational technique. The non-linear variational approach reduces the sfde (5.2) to a *random* (pathwise) neutral functional integral equation.

Stability issues for linear versions of the sfde (5.2) are studied in [26], [19], [20], [23]-[25].

For general white-noise, an invariant measure on M_2 for the one-point motion of the sfde (5.2) gives a stationary point of the cocycle (U, θ) by enlarging the probability space. On the other hand, if $Y : \Omega \to M_2$ is a stationary random point for (U, θ) independent of the Brownian motion $W(t)$, $t \geq 0$, then the distribution $\rho := P \circ Y^{-1}$ of Y is an invariant measure for the one-point motion of (5.2). This is because Y and W are independent ([22], Part II).

Theorem 5.1. *Under the given assumptions on the coefficients H and G, the trajectories of the sfde (5.2) induce a locally compact $C^{k,\epsilon}$ $(0 < \epsilon < \delta)$ perfect cocycle (U, θ) on M_2, where $U : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ satisfies the following conditions:*

(i) *For each $\omega \in \Omega$ and $t \geq r$ the map $U(t, \cdot, \omega) : M_2 \to M_2$ carries bounded sets into relatively compact sets. In particular, each Fréchet derivative, $DU(t, (v, \eta), \omega) : M_2 \to M_2$, of $U(t, \cdot, \omega)$ with respect to $(v, \eta) \in M_2$, is a compact linear map for $t \geq r, \omega \in \Omega$.*

(ii) *The map* $DU : \mathbf{R}^+ \times M_2 \times \Omega \to L(M_2)$ *is* $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(M_2) \otimes \mathcal{F}, \mathcal{B}_s(L(M_2)))$*-measurable. Furthermore, the function*

$$\mathbf{R}^+ \times M_2 \times \Omega \ni (t, (v, \eta), \omega) \mapsto \|DU(t, (v, \eta), \omega)\|_{L(M_2)} \in \mathbf{R}^+$$

is $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(M_2) \otimes \mathcal{F}, \mathcal{B}(\mathbf{R}^+))$*-measurable.*

(iii) *If* $Y : \Omega \to M_2$ *is a stationary point of* (U, θ) *such that* $E(\|Y\|_{M_2}^{\epsilon_0}) < \infty$ *for some* $\epsilon_0 > 0$*, then the following integrability condition*

$$\int_\Omega \log^+ \sup_{0 \le t_1, t_2 \le a} \|U(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega))\|_{k,\epsilon} \, dP(\omega) < \infty \qquad (5.3)$$

holds for any fixed $0 < \rho, a < \infty$ *and* $\epsilon \in (0, \delta)$*.*

Idea of proof

The construction and regularity of the cocycle (U, θ) is based on the following observation: The sfde (5.2) is equivalent to the random neutral integral equation:

$$\zeta(t, x(t, \omega), \omega) = v + \int_0^t F(u, \zeta(u, x(u, \omega), \omega), x(u, \omega), x_u(\cdot, \omega), \omega) \, du, \qquad (5.4)$$

$t \ge 0, (v, \eta) \in M_2$. In the above integral equation, $F : [0, \infty) \times \mathbf{R}^d \times M_2 \times \Omega \to \mathbf{R}^d$ is given by

$$F(t, z, v, \eta, \omega) := \{D\psi(t, z, \omega)\}^{-1} H(v, \eta)$$

$t \ge 0, z, (v, \eta) \in M_2, \omega \in \Omega$; and $\zeta : [0, \infty) \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is the inverse flow defined by

$$\zeta(t, x, \omega) := \psi(t, \cdot, \omega)^{-1}(x), \quad t \ge 0, \, x \in \mathbf{R}^d, \, \omega \in \Omega,$$

where ψ is the $C^{k+1,\epsilon}$ $(0 < \epsilon < \delta)$ perfect cocycle of the sode

$$\left.\begin{aligned} d\psi(t) &= G(\psi(t)) \, dW(t), \quad t \ge 0, \\ \psi(0) &= x \in \mathbf{R}^d. \end{aligned}\right\}$$

([16], [15]). The existence, perfection and regularity properties of the cocycle (U, θ) may be read from the integral equation (5.4). The integrability property (5.3) also follows from (5.4) coupled with spatial estimates on the finite-dimensional flows ψ and ζ ([22]).

Example

Consider the affine linear sfde

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t) \, dt + G \, dW(t), \quad t > 0 \\ x(0) &= v \in \mathbf{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (5.5)$$

where $H : M_2 \to \mathbf{R}^d$ is a continuous linear map, G is a fixed $(d \times p)$-matrix, and W is p-dimensional Brownian motion. Assume that the linear deterministic $(d \times d)$-matrix-valued fde

$$dy(t) = H \circ (y(t), y_t)\, dt \tag{5.6}$$

has a semiflow $T_t : L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d)) \to L(\mathbf{R}^d) \times L^2([-r, 0], L(\mathbf{R}^d))$, $t \geq 0$, which is uniformly asymptotically stable ([13]). Set

$$Y := \int_{-\infty}^{0} T_{-u}(I, 0) G\, dW(u) \tag{5.7}$$

where I is the identity $(d \times d)$-matrix. It is easy to see that the trajectories of the affine sfde (5.5) admit an affine linear cocycle $U : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$. Integration by parts and the *helix property*

$$W(t_2, \theta(t_1, \omega)) = W(t_2 + t_1, \omega) - W(t_1, \omega), \quad t_1, t_2 \in \mathbf{R},\, \omega \in \Omega, \tag{5.8}$$

imply that Y has a measurable version satisfying the perfect identity

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega)), \quad t \in \mathbf{R}^+,\, \omega \in \Omega.$$

Note that the stationary point Y (as given by (5.7)) is Gaussian and thus has finite moments of all orders. See ([17], Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, if the semigroup generated by the linear fde (5.6) is hyperbolic, then the sfde (5.5) has a stationary point ([17], [26]).

Theorem 5.2 ([22]) (The stable manifold theorem for sfde's). *Assume the given regularity hypotheses on H and G in the sfde (5.2). Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the sfde (5.2) such that $E(\|Y(\cdot)\|_{M_2}^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$. Then the cocycle (U, θ) of (5.2) satisfies the conclusions of the stable manifold theorem (Theorem 4.1) with $H = M_2$. If, in addition, the coefficients H, G of (5.2) are C_b^∞, then the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ are C^∞, perfectly in ω.*

Outline of proof of theorem 5.2.

In view of the integrability property (5.3), the local compactness of $U(t, \cdot, \omega)$, $t \geq r$, and the ergodicity of the Brownian shift θ, it is possible to define hyperbolicity for the stationary point $Y : \Omega \to M_2$. The conditions of the local stable manifold theorem (Theorem 4.1) now apply to the cocycle (U, θ). So Theorem 5.2 follows from Theorem 4.1 with $H = M_2$ and $x = (v, \eta) \in M_2$. □

6 Semilinear see's

In this section, we will first address the question of the existence of a regular cocycle for semilinear stochastic evolution equations in Hilbert space. Using the cocycle together with suitable integrability estimates, we will establish a local stable manifold theorem near hyperbolic stationary points for these equations.

The existence of local stable/unstable manifolds for nonlinear stochastic evolution equations (see's) and stochastic partial differential equations (spde's) has been an open problem since the early nineties ([10], [2], [3], [8], [9]). The analysis in this section will be carried out in the spirit of Section 5, although the construction of the cocycle will require entirely different techniques. Further details are made available in the forthcoming article by the author with T.S. Zhang and H. Z. Zhao ([27]).

In [10], the existence of a random evolution operator and its Lyapunov spectrum is established for a linear stochastic heat equation on a bounded Euclidean domain, driven by finite-dimensional white noise. For linear see's with finite-dimensional white noise, a stochastic semi-flow (i.e. a random evolution operator) is obtained in [2]. A multiplicative ergodic theorem for hyperbolic spde's is developed in [11]. Subsequent work on the dynamics of nonlinear spde's has focused mainly on the question of existence of *continuous* semiflows and the existence and uniqueness of invariant measures and/or stationary solutions. Existence of global invariant, stable/unstable manifolds (through a fixed point) for semilinear see's is established in ([4], [5]), when the global Lipschitz constant is relatively small with respect to the spectral gaps of the second-order term.

The main objectives in this section are to:

- construct a Fréchet differentiable, locally compact cocycle for mild/weak trajectories of the semilinear see;
- derive appropriate estimates on the cocycle of the see so as to guarantee applicability of the local stable manifold theorem (Theorem 4.1);
- show the existence of local stable/unstable manifolds near a hyperbolic stationary point, in the spirit of Theorem 4.1.

Smooth cocycles for semilinear see's and spde's:

As was indicated at the beginning of Section 5, there are no general techniques which give the existence of infinite-dimensional smooth cocycles. In this section we will use a combination of lifting techniques, chaos-type expansion and variational methods in order to construct smooth cocycles for semilinear see's.

The problem of existence of semiflows for see's (and spde's) is a nontrivial one, mainly due to the well-established fact that finite-dimensional methods for constructing (even continuous) stochastic flows break down in the infinite-dimensional setting of spde's and see's. More specifically, for see's in Hilbert space, our construction employs a "chaos-type" representation in the Hilbert-Schmidt operators, using the linear terms of the see ([27], Theorems 1.2.1-1.2.4). This technique bypasses the need for Kolmogorov's continuity theorem. A variational technique is then employed in order to handle the nonlinear terms. Applications to specific classes of spde's are given in Section 7.

It should be noted that the case of *nonlinear* multiplicative noise is largely open: It is not known to us if see's driven by nonlinear *multidimensional* white noise admit perfect (smooth, or even continuous) cocycles.

We now formulate the set-up for the class of semilinear see's we wish to consider.

Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ the complete filtered Wiener space of all continuous paths $\omega : \mathbf{R} \to E$, $\omega(0) = 0$, where E is a real separable Hilbert space, $\Omega := C(\mathbf{R}^+, E; 0)$ has the compact open topology, $\mathcal{F}$ is the Borel (completed) σ-field of Ω; $\mathcal{F}_t$ is the sub-σ-field of $\mathcal{F}$ generated by all evaluations $\Omega \ni \omega \mapsto \omega(u) \in E, u \leq t$; and P is Wiener measure on Ω. Define the group of P-preserving ergodic Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$ by

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \, \omega \in \Omega.$$

Let H be a real (separable) Hilbert space, with norm $|\cdot|_H$. Recall that $\mathcal{B}(H)$ is its Borel σ-algebra, and $L(H)$ the Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$.

Let W denote E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, a Hilbert-Schmidt embedding. Write

$$W(t) := \sum_{k=1}^{\infty} W^k(t) f_k, \;\; t \in \mathbf{R},$$

where $\{f_k : k \geq 1\}$ is a complete orthonormal basis of K; $W^k, k \geq 1$, are standard independent one-dimensional Wiener processes ([6], Chapter 4). The series $\sum_{k=1}^{\infty} W^k(t) f_k$ converges absolutely in E but not necessarily in K. Note that (W, θ) is a helix:

$$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega)), \quad t_1, t_2 \in \mathbf{R}, \, \omega \in \Omega.$$

Denote by $L_2(K, H)$ the Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, furnished with the norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2 \right]^{1/2}.$$

Consider the semilinear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt + Bu(t,x)\,dW(t), \quad t > 0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{6.1}$$

in H.

In the above see, $A : D(A) \subset H \to H$ is a closed linear operator on H. Assume A has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding positive eigenvalues $\{\mu_n : n \geq 1\}$; i.e., $Ae_n = \mu_n e_n$, $n \geq 1$. Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geq 0$. Assume that $F : H \to H$ is (Fréchet) $C_b^{k,\epsilon}$ $(k \geq 1, \epsilon \in (0,1])$; thus F has a continuous and globally bounded Fréchet derivative $F : H \to L(H)$. Suppose $B : H \to L_2(K,H)$ is a bounded linear operator. The stochastic Itô integral in the see (6.1) is defined in the following sense ([6], Chapter 4):

Let $\psi : [0,a] \times \Omega \to L_2(K,H)$ be jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted and $\int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt < \infty$. Define the Itô integral

$$\int_0^a \psi(t)\, dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k)\, dW^k(t)$$

where the H-valued Itô integrals on the right hand side are with respect to the one-dimensional Wiener processes W^k, $k \geq 1$. The above series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E\left| \int_0^a \psi(t)(f_k)\, dW^k(t)\right|^2 = \int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt < \infty.$$

The following standing hypotheses will be invoked throughout this section.

Hypothesis (A): $\displaystyle\sum_{n=1}^{\infty} \mu_n^{-1}\|B(e_n)\|^2_{L_2(K,H)} < \infty.$

Hypothesis (B): Assume that $B : H \to L_2(K,H)$ extends to a bounded linear operator $B \in L(H, L(E,H))$, and the series $\displaystyle\sum_{k=1}^{\infty} \|B_k\|^2$ converges, where $B_k \in L(H)$ is defined by $B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$.

Observe that Hypothesis (A) is implied by the following two requirements:

(a) The operator $B : H \to L_2(K,H)$ is Hilbert-Schmidt.

(b) $\liminf_{n\to\infty} \mu_n > 0$.

The requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions. Moreover, Hypothesis (A) does not place any restriction on the dimension of M.

A *mild solution* of the semilinear see (6.1) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H$, $x \in H$, satisfying the following Itô stochastic integral equation:

$$u(t,x,\cdot) = T_t x + \int_0^t T_{t-s}F(u(s,x,\cdot))\, ds + \int_0^t T_{t-s}Bu(s,x,\cdot)\, dW(s), \quad t \geq 0, \tag{6.2}$$

([6], [7]).

Theorem 6.1. *Under Hypotheses (A) and (B), the see (6.1) admits a perfect* $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$*-measurable* $C^{k,\epsilon}$ *cocycle* (U, θ) *with* $U : \mathbf{R}^+ \times H \times \Omega \to H$*. Furthermore, if* $Y : \Omega \to H$ *is a stationary point of* (U, θ) *such that* $E|Y|^{\epsilon_0} < \infty$ *for some* $\epsilon_0 > 0$*, then the integrability estimate* $(*)$ *of Theorem 4.1 holds. Indeed, the following (stronger) spatial estimates hold*

$$E\left\{\sup_{\substack{0\le t_1,t_2\le a\\ x\in H}} \frac{|U(t_2, x, \theta(t_1, \cdot))|^{2p}}{(1+|x|^{2p})}\right\} < \infty, \quad p \ge 1,$$

and

$$E \sup_{\substack{0\le t_1,t_2\le a\\ x\in H,\, 1\le j\le k}} \left\{\|D^{(j)}U(t_2, x, \theta(t_1, \cdot))\|_{L^{(j)}(H,H)}\right\} < \infty.$$

Sketch of Proof of Theorem 6.1.

We will only sketch the proof of Theorem 6.1. For more details of the arguments involved the reader may consult ([27], Theorem 1.2.6, [28]).

Step 1:

We first construct an $L(H)$-valued linear cocycle for mild solutions of the following associated linear see ($F \equiv 0$ in (6.1)):

$$\left.\begin{aligned} &du(t, x, \cdot) = -Au(t, x, \cdot)\, dt + Bu(t, x, \cdot)\, dW(t), \quad t > 0,\\ &u(0, x, \omega) = x \in H. \end{aligned}\right\} \tag{6.3}$$

A *mild solution* of the above linear see is a family of jointly measurable, $(\mathcal{F}_t)_{t\ge 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H,\ x \in H$, such that

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} Bu(s, x, \cdot)\, dW(s), \quad t \ge 0.$$

The above integral equation holds x*-almost surely*, for each $x \in H$. The crucial question here is whether $u(t, x, \omega)$ is pathwise continuous linear in x perfectly in ω? In view of the failure of Kolmogorov's continuity theorem in infinite dimensions (as pointed out in Section 5), we will use a chaos-type expansion technique to show that $u(t, \cdot, \omega) \in L(H)$ perfectly in $\omega \in \Omega$, for all $t \ge 0$. In order to do this, we first lift the linear see (6.3) to the Hilbert space $L_2(H)$ of all Hilbert-Schmidt operators $H \to H$. This is achieved as follows:

- Lift the semigroup $T_t : H \to H, t \ge 0$, to a strongly continuous semigroup of bounded linear operators $\tilde{T}_t : L_2(K, H) \to L_2(K, H), t \ge 0$, defined by the composition $\tilde{T}_t(C) := T_t \circ C,\ C \in L_2(K, H)$, $t \ge 0$.
- Lift the Itô stochastic integral $\displaystyle\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s),\ x \in H, t \ge 0$, to $L_2(H)$ for adapted square-integrable $v : \mathbf{R}^+ \times \Omega \to L_2(H)$. Denote the lifting by $\displaystyle\int_0^t T_{t-s} Bv(s)\, dW(s) \in L_2(H)$. That is:

$$\left[\int_0^t T_{t-s} B v(s)\, dW(s)\right](x) = \int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$$

for all $t \geq 0$, x-a.s.

Step 2:

Next we solve the "lifted" linear see using the following "chaos-type" series expansion in $L_2(H)$ for its solution $\Phi(t,\omega) \in L_2(H), t > 0, \omega \in \Omega$:

$$\Phi(t,\cdot) = T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n}\, dW(s_n) \cdots dW(s_2)\, dW(s_1). \tag{6.4}$$

In the above expansion, the iterated Itô stochastic integrals are lifted integrals in $L_2(H)$. More specifically, denote by $\Psi^n(t) \in L_2(H)$ the general term in the series (6.4), viz.

$$\Psi^n(t) := \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n}\, dW(s_n) \cdots dW(s_2)\, dW(s_1),$$

for $t \geq 0, n \geq 1$. Observe that

$$\left.\begin{aligned} \Psi^n(t) &= \int_0^t T_{t-s_1} B \Psi^{n-1}(s_1)\, dW(s_1), \quad n \geq 2, \\ \Psi^1(t) &= \int_0^t T_{t-s_1} B T_{s_1}\, dW(s_1), \end{aligned}\right\} \tag{6.5}$$

for $t \geq 0$. Using Hypotheses (A) and (B) and induction on $n \geq 1$, one may obtain the following estimate from (6.5):

$$E \sup_{0 \leq s \leq t} \|\Psi^n(s)\|^2_{L_2(H)} \leq K_1 \frac{(K_2 t)^{n-1}}{(n-1)!}, \quad t \in [0,a],$$

for fixed $a > 0$ and for all integers $n \geq 1$, where K_1, K_2 are positive constants depending only on a. The above estimate implies that the series on the right-hand-side of (6.4) converges absolutely in $L^2(\Omega, L_2(H))$ for any fixed $t > 0$.

Step 3:

We now approximate the Brownian noise W in (6.3) by a sequence of smooth helices

$$W_n(t,\omega) := n \int_{t-1/n}^t W(u,\omega)\, du - n \int_{-1/n}^0 W(u,\omega)\, du, \quad t \geq 0,\ \omega \in \Omega.$$

Thus we obtain a perfect linear cocycle $\Phi(t,\omega) \in L_2(H)$, $t > 0$, $\omega \in \Omega$, for (6.3):

$$\Phi(t+s,\omega) = \Phi(t,\theta(s,\omega)) \circ \Phi(s,\omega), \quad s,t \geq 0,\ \omega \in \Omega$$

satisfying the estimate

$$\sup_{0\le s\le t\le a} \|\Phi(t-s,\theta(s,\omega))\|_{L(H)} < \infty$$

for any $\omega \in \Omega$ and any fixed $a \in (0,\infty)$.

Step 4:

Now we consider the semilinear Itô see (6.1). Since the linear cocycle (Φ,θ) is a mild solution of (6.3), it is not hard to see that solutions of the random integral equation

$$U(t,x,\omega) = \Phi(t,\omega)(x) + \int_0^t \Phi(t-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds, \quad t\ge 0,\, x\in H, \tag{6.6}$$

give a version of the mild solution of the see (6.1). Using successive approximations on the above integral equation together with the cocycle property for (Φ,θ), we obtain a $C^{k,\epsilon}$ perfect cocycle (U,θ) for mild solutions of the semilinear see (6.1).

Step 5:

The integrability estimate $(*)$ of Theorem 4.1, as well as the two estimates in Theorem 6.1, follow from the random integral equation (6.6) and a “Gronwall-type” argument using Lemma 2.1 in [28]. Cf. proof of Theorem 2.2 in [28]. □

We may now state the stable manifold theorem for the semilinear see (6.1). It is a direct consequence of Theorems 4.1 and 6.1.

Theorem 6.2 ([27]) (The stable manifold theorem for semilinear see's). *In the see (6.1) assume Hypotheses (A) and (B) and let F be $C_b^{k,\epsilon}$. Let $Y:\Omega\to H$ be a hyperbolic stationary point of (6.1) such that $E(\|Y(\cdot)\|_H^{\epsilon_0})<\infty$ for some $\epsilon_0>0$. Then the local stable manifold theorem (4.1) holds for the cocycle (U,θ) of (6.1). If F is C_b^∞, the local stable and unstable manifolds $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ of (6.1) are C^∞, perfectly in ω.*

7 Examples: Semilinear spde's

In this section, we will examine applications of the ideas in Section 6 to two classes of semilinear spde's: *Semilinear parabolic spde's with Lipschitz nonlinearity* and *stochastic reaction diffusion equations with dissipative nonlinearity.* In particular, we obtain smooth globally defined stochastic semiflows for semilinear spde's driven by cylindrical Brownian motion. In constructing such semiflows, it turns out that in addition to smoothness of the nonlinear terms, one requires some level of dissipativity or Lipschitz continuity of the nonlinear terms. A discussion of the stochastic semiflow for Burgers equations with additive infinite-dimensional noise is given in [27].

Consider the semilinear spde

$$\left.\begin{aligned} du(t) &= \frac{1}{2}\Delta u(t)dt + f(u(t))dt + \sum_{i=1}^{\infty}\sigma_i u(t)\,dW^i(t), \quad t>0,\\ u(0) &= \psi \in H_0^k(\mathcal{D}). \end{aligned}\right\} \tag{7.1}$$

In the above spde, Δ is the Laplacian $\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial\xi_i^2}$ on a bounded domain $\mathcal{D}$ in $\mathbf{R}^d$, with a smooth boundary $\partial\mathcal{D}$ and Dirichlet boundary conditions. The nonlinearity in (7.1) is given by a C_b^∞ function $f:\mathbf{R}\to\mathbf{R}$. We consider weak solutions of (7.1) with initial conditions ψ in the Sobolev space $H_0^k(\mathcal{D})$, the completion of $C_0^\infty(\mathcal{D},\mathbf{R})$ under the Sobolev norm

$$||u||^2_{H_0^k(\mathcal{D})} := \sum_{|\alpha|\leq k}\int_{\mathcal{D}} |D^\alpha u(\xi)|^2\, d\xi,$$

with $d\xi$ Lebesgue measure on $\mathbf{R}^d$. The noise in (7.1) is given by a family $W^i, i\geq 1$, of independent one-dimensional standard Brownian motions with $W^i(0)=0$ defined on the canonical complete filtered Wiener space $(\Omega,\bar{\mathcal{F}},(\mathcal{F}_t)_{t\in\mathbf{R}},P)$. The Brownian shift on $\Omega := C(\mathbf{R},\mathbf{R}^\infty;0)$ is denoted by θ. Furthermore, we assume that $\sigma_i\in H_0^s(\mathcal{D})$ for all $i\geq 1$, and the series $\sum_{i=1}^{\infty}\|\sigma_i\|^2_{H_0^s}$ converges, where $s > k+\frac{d}{2} > d$. Note also that f induces the C_b^∞ (Nemytskii) map $F: H_0^k(\mathcal{D})\to H_0^k(\mathcal{D})$, $F(\psi) := f\circ\psi$, $\psi\in H_0^k(\mathcal{D})$.

Under these conditions and using similar ideas to those in Section 6, one can show that the random field of weak solutions of the initial-value problem (7.1) yields a perfect smooth cocycle (U,θ) on the Sobolev space $H_0^k(\mathcal{D})$ which satisfies the integrability estimate $(*)$ of Theorem 4.1 with $H := H_0^k(\mathcal{D})$. Suppose $Y:\Omega\to H_0^k(\mathcal{D})$ is a hyperbolic stationary point of the cocycle (U,θ) of (7.1) such that $E\log^+\|Y\|_{H_0^k}<\infty$. Then the local stable manifold theorem (Theorem 4.1) applies to the cocycle (U,θ) in a neighborhood of Y. Indeed, we have:

Theorem 7.1. *Assume the above hypotheses on the coefficients of the spde (7.1). Then the weak solutions of (7.1) induce a C^∞ perfect cocycle $U:\mathbf{R}^+\times H_0^k(\mathcal{D})\times\Omega\to H_0^k(\mathcal{D})$. Suppose the cocycle (U,θ) of (7.1) has a hyperbolic stationary point $Y:\Omega\to H_0^k(\mathcal{D})$ such that $E\log^+\|Y\|_{H_0^k}<\infty$. Then (U,θ) has a perfect family of C^∞ local stable and unstable manifolds in $H_0^k(\mathcal{D})$ satisfying all the assertions of Theorem (4.1) with $H := H_0^k(\mathcal{D})$.*

For further details on the proof of Theorem (7.1), see [27].

We close this section by discussing the dynamics of the following stochastic reaction diffusion equation with dissipative nonlinearity:

$$\left.\begin{aligned} du &= \nu\Delta u\, dt + u(1-|u|^\alpha)\, dt + \sum_{i=1}^{\infty}\sigma_i u(t)\, dW^i(t), \quad t>0,\\ u(0) &= \psi\in L^2(\mathcal{D}), \end{aligned}\right\} \tag{7.2}$$

defined on a bounded domain $\mathcal{D}\subset\mathbf{R}^d$ with a smooth boundary $\partial\mathcal{D}$. In (7.2), $\mathcal{D}$ and the $W^i, i\geq 1$ are as in (7.1), and the series $\sum_{i=1}^{\infty}\|\sigma_i\|^2_{H_0^s}$ converges for $s>2+\frac{d}{2}$.

For weak solutions of (7.2), one can construct a C^1 cocycle (U, θ) on the Hilbert space $H := L^2(\mathcal{D})$ ([27]).

Under appropriate choice of the diffusion parameter ν, a unique stationary solution of (7.2) exists ([7]).

The following local stable manifold theorem holds for (7.2) ([27]).

Theorem 7.2. *Assume the above hypotheses on the coefficients of the spde (7.2). Let* $\alpha < \dfrac{4}{d}$. *Then the weak solutions of (7.2) generate a* C^1 *cocycle* $U : \mathbf{R}^+ \times L^2(\mathcal{D}) \times \Omega \to L^2(\mathcal{D})$. *Suppose* $Y : \Omega \to L^2(\mathcal{D})$ *is a hyperbolic stationary point of the cocycle* (U, θ) *such that* $E \log^+ \|Y\|_{L^2} < \infty$. *Then* (U, θ) *has a perfect family of* C^1 *local stable and unstable manifolds in* $L^2(\mathcal{D})$ *satisfying the assertions of Theorem 4.1 with* $H := L^2(\mathcal{D})$.

A proof of Theorem (7.2) is given in [27].

8 Applications: anticipating semilinear systems

In this section we give dynamic representations of infinite-dimensional cocycles on their stable/unstable manifolds at stationary points. This is done via substitution theorems which provide pathwise solutions of semilinear sfde's or see's when the initial conditions are random, anticipating and sufficiently regular in the Malliavin sense. The need for Malliavin regularity of the substituting initial condition is dictated by the infinite-dimensionality of the stochastic dynamics. Indeed, existing substitution theorems ([12], [1]) do not apply in our present context because the substituting random variable may not take values in a relatively compact or σ-compact space.

Anticipating semilinear sfde's:

Consider the following Stratonovich version of the sfde (5.2) of Section 5

$$\left.\begin{aligned} dx(t) &= H(x(t), x_t)\, dt - \frac{1}{2}\sum_{k=1}^{p} G_k^2(x(t))\, dt + G(x(t)) \circ dW(t), \quad t > 0, \\ (x(0), x_0) &= Y, \end{aligned}\right\} \tag{8.1}$$

with anticipating random initial condition $Y : \Omega \to M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ and with linear noise coefficient $G : \mathbf{R}^d \to \mathbf{R}^{d \times p}$. Using a coordinate basis $\{f_k\}_{k=1}^p$ of $\mathbf{R}^p$, write the p-dimensional Brownian motion W in the form $W(t) = \sum_{k=1}^{p} W^k(t) f_k, t \geq 0$, where the W^k, $1 \leq k \leq p$, are independent standard one-dimensional Wiener processes. The linear maps $G_k \in L(\mathbf{R}^d)$, $1 \leq k \leq p$, are defined by $G_k(v) := G(v)(f_k)$, $v \in \mathbf{R}^d$, $1 \leq k \leq p$. Assume the rest of the conditions in Section 5.

The following theorem establishes the existence of a solution to (8.1) when Y is sufficiently regular in the Malliavin sense; that is $Y \in \mathbb{D}^{1,4}(\Omega, M_2)$, the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which have fourth-order moments together with their Malliavin derivatives $\mathcal{D}Y$ ([29], [30]). Throughout this section, we will denote Fréchet derivatives by D and Malliavin derivatives by $\mathcal{D}$.

Theorem 8.1. *In the semilinear sfde (8.1) assume that H is C_b^1 and $Y \in \mathbb{D}^{1,4}(\Omega, M_2)$. Then (8.1) has a solution $x \in L^\infty([0,a], \mathbb{D}^{1,2}(\Omega, \mathbf{R}^d))$ satisfying*

$$\sup_{t\in[0,a]} |x(t,\omega)| \le K(\omega)\big[1 + \|Y(\omega)\|_{M_2}\big], \quad a.a.\ \omega \in \Omega,$$

for any $a \in (0,\infty)$, where K is a positive random variable having moments of all orders. When H is C_b^2, a similar substitution result holds for the linearized version of (8.1).

Sketch of Proof of Theorem 8.1.

Denote by $\Psi(t,\cdot,\omega) \in L(\mathbf{R}^d), t \in \mathbf{R}^+$, $\omega \in \Omega$, the linear cocycle for the linear Itô sode

$$\left.\begin{aligned} d\Psi(t) &= G \circ \Psi(t)\, dW(t), \quad t \ge 0, \\ \Psi(0) &= I \in L(\mathbf{R}^d). \end{aligned}\right\}$$

From the construction in Section 5, the semilinear sfde (8.1) has a perfect cocycle $U : \mathbf{R}^+ \times M_2 \times \Omega \to M_2$ satisfying the following random functional integral equation:

$$\left.\begin{aligned} p_1(U(t,(v,\eta),\omega)) &= \Psi(t,\omega)(v) + \int_0^t \Psi(t-u,\theta(u,\omega))(H(U(u,(v,\eta),\omega)))\, du, \\ p_2(U(0,(v,\eta),\omega)) &= \eta \in L^2([-r,0],\mathbf{R}^d), \end{aligned}\right\} \tag{8.2}$$

for each $\omega \in \Omega$, $t \ge 0$, $(v,\eta) \in M_2$. In (8.2), $p_1 : M_2 \to \mathbf{R}^d$, $p_2 : M_2 \to L^2([-r,0],\mathbf{R}^d)$ denote the projections onto the first and second factors respectively.

We will show that $U(t,Y)$, $t \ge 0$, is a solution of (8.1) satisfying the conclusion of the theorem. To this aim, it is sufficient to show that Y can be substituted in place of the parameter (v,η) in the semilinear Stratonovich integral equation

$$\left.\begin{aligned} p_1U(t,(v,\eta)) &= v + \int_0^t H(U(u,(v,\eta)))\, du - \frac{1}{2}\sum_{k=1}^p \int_0^t G_k^2(p_1U(u,(v,\eta)))\, du \\ &\qquad + \int_0^t G(p_1U(u,(v,\eta))) \circ dW(u), \quad t > 0, \\ U(0,(v,\eta)) &= (v,\eta) \in M_2. \end{aligned}\right\} \tag{8.3}$$

One can easily make the substitution $(v,\eta) = Y$ in the two Lebesgue integrals on the right hand side of (8.3). So it is sufficient to show that a similar substitution also works for the Stratonovich integral; that is

$$\int_0^t G(p_1U(u,(v,\eta))) \circ dW(u)\Big|_{(v,\eta)=Y} = \int_0^t G(p_1U(u,Y)) \circ dW(u) \tag{8.4}$$

a.s. for all $t \geq 0$. We will establish (8.4) in two steps: First, we show it holds if Y is replaced by its finite-dimensional projections $Y_n : \Omega \to H_n$, $n \geq 1$, where H_n is the linear subspace spanned by $\{e_i : 1 \leq i \leq n\}$ from a complete orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of M_2; secondly, we pass to the limit as n goes to ∞ (in (8.5) below). Denote $g(t,(v,\eta) := G(p_1U(t,(v,\eta)))$, $t \geq 0$, $(v,\eta) \in H_n$. Using martingale estimates for $p_1U(t,(v,\eta))$ it is easy to see that g satisfies all requirements of Theorem 5.3.4 in [29]. Therefore,

$$\int_0^t G(p_1U(u,(v,\eta))) \circ dW(u)\bigg|_{(v,\eta)=Y_n} = \int_0^t G(p_1U(u,Y_n)) \circ dW(u) \tag{8.5}$$

a.s. for all $n \geq 1$ and $t \geq 0$. The next step is to establish the a.s. limit

$$\lim_{n\to\infty} \int_0^t G(p_1U(s,Y_n)) \circ dW(s) = \int_0^t G(p_1U(s,Y)) \circ dW(s),\ t \geq 0. \tag{8.6}$$

Set $f(s) := G(p_1U(s,Y))$, $f_n(s) := G(p_1U(s,Y_n))$, $s \geq 0$, $n \geq 1$. To prove (8.6), we will show first that f and f_n are sufficiently regular to allow for the following representations of the Stratonovich integrals in terms of Skorohod integrals:

$$\int_0^t f(s) \circ dW(s) = \int_0^t f(s)\, dW(s) + \frac{1}{2}\int_0^t \nabla f(s)\, ds \tag{8.7}$$

and

$$\int_0^t f_n(s) \circ dW(s) = \int_0^t f_n(s)\, dW(s) + \frac{1}{2}\int_0^t \nabla f_n(s)\, ds, \tag{8.8}$$

where

$$\nabla f(s) := (\mathcal{D}^+ f)(s) + (\mathcal{D}^- f)(s),\ (\mathcal{D}^+ f)(s) := \lim_{t\to s+} \mathcal{D}_s f(t),\ (\mathcal{D}^- f)(s) := \lim_{t\to s-} \mathcal{D}_s f(t), \tag{8.9}$$

for any $s \geq 0$. In view of the expression

$$\mathcal{D}_s f(t) = Gp_1\mathcal{D}_s U(t,Y) + Gp_1 DU(t,Y)\mathcal{D}_s Y, \quad s,t \geq 0, \tag{8.10}$$

the integrability estimates (8.11) below, and the fact that $Y \in \mathbb{D}^{1,4}(\Omega, M_2)$, it can be shown that

$$\int_0^a \int_0^a E|\mathcal{D}_s f(t)|^2\, dt\, ds < \infty.$$

This implies that $f \in \mathbb{L}^{1,2}$, and so the Stratonovich integral in (8.7) is well-defined. Similarly for $f_n \in \mathbb{L}^{1,2}$, $n \geq 1$. The following integrability estimates on the cocycle U of the semilinear sfde are obtained using the integral equation (8.2) and a Gronwall-type lemma in ([28], Lemma 2.1; cf proofs of Theorems 2.2, 2.3):

$$\left.\begin{aligned} &E \sup_{\substack{0\le t\le a\\ (v,\eta)\in M_2}} \frac{|U(t,(v,\eta),\cdot)|^{2p}}{(1+\|(v,\eta)\|_{M_2}^{2p})} < \infty, \quad E \sup_{\substack{0\le t\le a\\ (v,\eta)\in M_2}} \|DU(t,x,\cdot)\|^{2p} < \infty, \\ &E \sup_{\substack{0\le t\le a\\ (v,\eta)\in M_2}} \|D^2U(t,(v,\eta),\cdot)\|^{2p} < \infty, \quad E\Big[\sup_{s,u\le t\le a} \|\mathcal{D}_s\Psi(t-u,\theta(u,\cdot))\|_{L(\mathbf{R}^d)}^{2p}\Big] < \infty, \\ &E\Big[\sup_{\substack{0\le t\le a\\ (v,\eta)\in H}} \frac{|\mathcal{D}U(t,(v,\eta),\cdot)|_{M_2}^{2p}}{(1+\|(v,\eta)\|_{M_2}^{2p})}\Big] < \infty, \end{aligned}\right\} \tag{8.11}$$

for any $0 < a < \infty$ and $p \ge 1$.

Using the estimates (8.11) again, together with (8.10) and the integral equation (8.2), a lengthy computation shows that

$$\lim_{l\to\infty} \int_0^a \sup_{s<t\le s+(1/l)} E(|\mathcal{D}_s f(t) - (\mathcal{D}^+ f)(s)|)\, ds = 0 \tag{8.12}$$

and

$$\lim_{l\to\infty} \int_0^a \sup_{0\vee[s-(1/l)]\le t<s} E(|\mathcal{D}_s f(t) - (\mathcal{D}^- f)(s)|)\, ds = 0 \tag{8.13}$$

Similar statements also hold for each f_n, $n \ge 1$. This justifies (8.7) and (8.8).

To complete the proof of (8.6), we take limits as $n \to \infty$ in (8.8) and note that

$$\lim_{n\to\infty} \int_0^a \int_0^a E|\mathcal{D}_s f_n(t) - \mathcal{D}_s f(t)|^2\, dt\, ds = 0 \tag{8.14}$$

and

$$\lim_{n\to\infty} \int_0^a \nabla f_n(s)\, ds = \int_0^a \nabla f(s)\, ds. \tag{8.15}$$

Relations (8.14) and (8.15) follow from (8.10), the fact that $Y \in \mathbb{D}^{1,4}(\Omega, M_2)$, the estimates (8.11) and the dominated convergence theorem. This completes the proof of the substitution formula (8.4). □

In the second part of this section, we describe a similar substitution formula for the semilinear see of Section 6.

Anticipating semilinear see's:

Here we adopt the setting and hypotheses of Section 6. Specifically, we consider the following Stratonovich version of the see (6.1):

$$\left.\begin{aligned} &du(t,x) = -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,x)\,dt + Bu(t,x)\circ dW(t), \ t>0, \\ &u(0,x) = x \in H. \end{aligned}\right\} \tag{8.16}$$

The following result is obtained using similar techniques to those used in the proof of Theorem 8.1:

Theorem 8.2. *Assume that the see (8.16) satisfies all the conditions of Section 6. Let* $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *be a random variable, and* $U : \mathbf{R}^+ \times H \times \Omega \to H$ *be the* C^1 *cocycle generated by all mild solutions of the Stratonovich see (8.16). Then* $U(t, Y)$, $t \geq 0$, *is a mild solution of the (anticipating) Stratonovich see*

$$\left.\begin{aligned} dU(t,Y) &= -AU(t,Y)\,dt + F\big(U(t,Y)\big)\,dt - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 U(t,Y)\,dt + BU(t,Y)\circ dW(t),\ t>0,\\ U(0,Y) &= Y. \end{aligned}\right\} \tag{8.17}$$

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (8.16), then* $U(t, Y) = Y\big(\theta(t)\big)$, $t \geq 0$, *is a stationary solution of the (anticipating) Stratonovich see*

$$\left.\begin{aligned} dY(\theta(t)) &= -AY(\theta(t))\,dt + F\big(Y(\theta(t))\big)\,dt - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 Y(\theta(t))\,dt + BY(\theta(t))\circ dW(t), t>0,\\ Y(\theta(0)) &= Y. \end{aligned}\right\} \tag{8.18}$$

Details of the proof of the above result are given in [28].

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