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# STOCHASTIC BURGERS EQUATION WITH RANDOM INITIAL VELOCITIES: A MALLIAVIN CALCULUS APPROACH*

SALAH-ELDIN A. MOHAMMED† AND TUSHENG ZHANG‡

**Abstract.** In this article we prove an existence theorem for solutions of the stochastic Burgers equation (SBE) on the unit interval with Dirichlet boundary conditions and anticipating initial velocities. The SBE is driven by affine (additive + linear) noise. In order to establish the existence theorem, we adopt a somewhat counterintuitive perspective in which stochastic dynamical systems ideas lead to the existence of solutions rather than vice versa. More specifically, our approach uses the Malliavin calculus and is based on the existence and regularity of a perfect cocycle on the energy space for the SBE. The proof of the existence theorem requires Malliavin regularity of the infinite-dimensional initial velocity field together with new spatial estimates on the cocycle, its Fréchet and Malliavin derivatives. The existence theorem provides a dynamic characterization of solutions of the *nonanticipating* SBE on its unstable invariant manifolds. Furthermore, as a corollary of the existence theorem, we show that random cocycle-invariant points on the energy space correspond to (possibly nonergodic) stationary pathwise solutions for the SBE.

**Key words.** Malliavin calculus, stochastic semiflow, smooth cocycle, stochastic Burgers equation, anticipating Burgers equation, random initial velocity

**AMS subject classifications.** Primary, 60H10, 60H20; Secondary, 60H25

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## 1. Introduction.

In this article we establish the existence of a solution to the following stochastic Burgers equation (SBE) with affine noise and a random (possibly anticipating) initial velocity:

$$
\left.\begin{aligned}
du(t) &= \nu\Delta u\,dt - u\frac{\partial u}{\partial \xi}\,dt + \gamma u(t)\,dt + \sigma u(t)\,dW(t)\\
&\qquad + \sigma_0\,dW_0(t,\xi), \quad t>0, \quad \xi\in[0,1],\\
u(t,0) &= u(t,1) = 0 \quad \text{for all } t>0,\\
u(0,\xi) &= Y(\xi,\omega),\ \xi\in[0,1].
\end{aligned}\right\} \tag{1.1}
$$

In the above SBE, the multiplicative and additive noise coefficients $\sigma, \sigma_0$ are constants; $W$ is a standard Brownian motion defined on the complete Wiener space $(\Omega, \mathcal{F}, P)$; $W_0(t,\xi)$ is space-time additive noise which is $C^2$ in $\xi$, white in $t$, and independent of $W$; $\gamma u(t)\,dt$ is a deterministic linear drift term with a fixed parameter $\gamma$; the positive constant $\nu$ is the viscosity coefficient; and the initial velocity $Y : [0,1]\times\Omega \to \mathbf{R}$ is an $\mathcal{F}$-measurable real-valued random process on $[0,1]$. Note that the external stochastic forcing in the SBE (1.1) is provided by the linear drift term $\gamma u(t)\,dt$, the linear white noise term $\sigma u(t)\,dW(t)$, and an independent additive space-time noise term $\sigma_0\,dW_0(t,\xi)$. This choice of external forcing allows for the existence of a perfect cocycle (cf. [M-Z.1]). It is not clear if our Theorem 2.2 still holds if we allow for nonlinear external random forcing in the SBE (1.1).

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†Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, IL 62901 (salah@sfde.math.siu.edu, http://sfde.math.siu.edu). The research of this author was supported in part by NSF grant DMS-0705970 and by Alexander von Humboldt Foundation.

‡Department of Mathematics, University of Manchester, Manchester M13 9PL, UK (tzhang@math.man.ac.uk). The research of this author was supported in part by EPSRC grant GR/R91144.

The stochastic Burgers equation has been studied extensively by many authors, mainly due to its significance in modeling turbulence in physics and engineering. The reader may refer to works by [B-C-J], [D-Z], [E-V], [G], [G-N], [L-Z], [Si], [T-Za], [T-Z] and the references therein.

Our motivation for studying the SBE with a random (possibly anticipating) initial velocity is based on the following observations:

(i) Experimental random measurement errors exist in the initial velocities for physical fluid dynamics models that employ the SBE. In fact, starting with Burgers original contribution [Bu], [Bu1], [Bu2], there has been significant interest in the analysis of Burgers hydrodynamic turbulence under random initial velocity regimes. See, e.g., [Be], [T-Za], [R-A], [S-A-F] and the references therein. These contributions were focused on statistical properties of solutions of Burgers equation under deterministic dynamics or additive stochastic forcing together with Gaussian initial velocities. On the other hand, little research has been directed to the case of *nonlinear* multiplicative driving noise and random initial velocities. We believe this is a challenging problem due to the difficulty in constructing the stochastic semiflow. We expect that our existence theorem will open the door for further statistical analysis of solutions of the SBE under *multiplicative* stochastic forcing and possibly *anticipating* initial velocities.

(ii) The study of stationary points is vitally important for understanding the long-time behavior of the random dynamics generated by the SBE. Stationary points depend generically on the whole path of the random forcing and are hence anticipating in nature. In order to show that stationary points generate stationary pathwise solutions of the SBE, it is necessary to consider SBEs with anticipating initial velocities.

(iii) In the neighborhood of a nonergodic stationary solution, it is known that a *nonanticipating* SBE has local random (and stationary) invariant manifolds that are necessarily *anticipating* in nature (cf. [A-I], [M-Z.1]). Indeed, in the hyperbolic case, the nonlinear Oseledec multiplicative ergodic theorem leads to random invariant unstable manifolds that are constructed using forward asymptotics of the stochastic semiflow. The anticipating nature of the invariant manifolds is dictated by the Ruelle–Oseledec operator generated by the forward asymptotics of the linearized cocycle along the stationary solution [M-Z.1, Theorem 4.1]. Our existence theorem (Theorem 2.2) provides a natural dynamic characterization of Burgers stochastic semiflow on the random invariant manifolds. Such a realization of the stochastic semiflow on the finite-dimensional invariant manifolds is a necessary step toward an understanding of the long-term dynamics and regularity of the semiflow on its invariant manifolds.

Throughout the article we denote by $L^2([0,1],\mathbf{R})$ the real Hilbert space of all (Lebesgue) square-integrable functions $f:[0,1]\to\mathbf{R}$ given the Hilbert norm:

$$\|f\|_{L^2} := \left[\int_0^1 |f(\xi)|^2\, d\xi\right]^{1/2}.$$

To prove existence of an anticipating solution for the SBE (1.1), we adopt the following strategy:

- We replace the random initial velocity $Y$ by a fixed (deterministic) function $f$ in $L^2([0,1],\mathbf{R})$. We then describe the stochastic dynamics of the SBE

(1.1) via a perfect locally compacting smooth semiflow (cocycle) $U : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \Omega \to L^2([0,1],\mathbf{R})$ generated by mild solutions of the SBE (1.1) starting at any deterministic $f \in L^2([0,1],\mathbf{R})$. Details of the construction of the cocycle are given in [M-Z.1].

- Given the existence of the cocycle, we view the SBE as a family of stochastic partial differential equations (spde's) parametrized by *fixed* (nonrandom) initial velocities $f$ in $L^2([0,1],\mathbf{R})$.
- The existing estimates on the cocycle $U$ of the SBE (1.1) are inadequate for a direct substitution $f = Y$ in the SBE. So we introduce a suitable truncation of the SBE parametrized by the initial function $f \in L^2([0,1],\mathbf{R})$. We then develop sharp spatial estimates on the semiflow of the truncated SBE and its Fréchet and Malliavin derivatives [Ma], [Nu.1], [Nu.2], [N-P].
- Making the substitution $f = Y$ in the truncated SBE yields an anticipating stochastic partial differential equation with $Y$ as the initial velocity. This substitution scheme employs the spatial estimates on the semiflow of the truncated SBE and Malliavin regularity of $Y$ as well as its finite-dimensional approximations.
- The truncation is then lifted via the local property of the Stratonovich noise term. This gives the required anticipating mild solution of the SBE (1.1) with $Y$ as a random initial velocity.

The main obstruction to justifying the above substitution technique is the inherent infinite-dimensionality of the random initial velocity $Y$ and the dynamics. This is because almost all of the existing substitution theorems require *finite-dimensional* or *compact settings* [Nu.1], [Nu.2], [M-S], [G-Nu-S], [A-I], contrary to the current infinite-dimensional context of the SBE (1.1). In particular, Kolmogorov's continuity theorem as well as Sobolev's inequalities fail in infinite dimensions [Mo.1], [Mo.2], [M-Z-Z]. In this paper, we are able to overcome the above obstruction by using ideas and techniques of the Malliavin calculus. More specifically, the initial velocity $Y$ is assumed to be Malliavin differentiable with Malliavin derivatives having fourth-order moments. Under this single requirement, the SBE (1.1) admits an anticipating solution for all $t > 0$. In the special case when the initial velocity $Y$ is a *stationary equilibrium*, one has the pathwise identity

$$U(t, Y(\omega), \omega) = Y(\theta(t,\omega)), \quad t \geq 0,\ \omega \in \Omega,$$

where $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ is the $P$-preserving Brownian shift

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t, s \geq 0,\ \omega \in \Omega.$$

Note that $U(t, Y(\cdot), \cdot) = Y(\theta(t,\cdot)$, $t \geq 0$, is a stationary process in $L^2([0,1],\mathbf{R})$ because $\theta(t,\cdot)$ is $P$-preserving. As a corollary of the existence theorem, it follows that $Y(\theta(t,\cdot))$, $t \geq 0$, is a *stationary solution* of the SBE (1.1). This fact is especially relevant when $Y$ is a hyperbolic equilibrium for the SBE (1.1); see [M-Z.1] for the existence of local invariant unstable/stable manifolds for the SBE (1.1) near hyperbolic equilibria. The random invariant manifolds are in general anticipating, and so our existence theorem gives via (1.1) a dynamic characterization of the cocycle on these invariant manifolds.

In section 2, we state our main theorem in this article and also summarize known results on the existence of the cocycle generated by mild solutions of the SBE (1.1). Details of the construction of the cocycle may be recovered from the article [M-Z.1]. However, the estimates on the cocycle established in [M-Z.1] are not sufficiently sharp to yield a satisfactory substitution mechanism. To overcome this difficulty, we introduce a suitable truncation of the mild SBE in section 3. For the truncated mild SBE, we develop regularity and spatial moment estimates on its cocycle together with its Malliavin and Fréchet derivatives. These estimates are used to establish an infinite-dimensional substitution theorem for the truncated SBE. In section 4, we show that the substitution theorem holds for the truncated SBE for *n-dimensional* random initial conditions. By a somewhat elaborate approximation argument as $n \to \infty$, we prove in section 5 that the substitution works on the truncated SBE for a full-fledged random initial condition $Y$ which has a fourth-order moment together with its Malliavin derivative $\mathcal{D}Y$ [Nu.1], [Nu.2]. Finally, in view of the local property of the Stratonovich integral, we are able to lift the truncation restriction and obtain the existence of an anticipating solution for the SBE (1.1).

**2. Mild formulation, the cocycle, and the main result.** We view the SBE (1.1) as an evolution equation in the Hilbert space $L^2([0,1],\mathbf{R})$ and consider its mild solutions with respect to the heat semigroup generated by the Dirichlet Laplacian $\nu\Delta$ on $[0,1]$.

Consider the stochastic Burgers equation (1.1) with a *deterministic* initial velocity $f \in L^2([0,1],\mathbf{R})$, viz.,

$$\left.\begin{aligned} du(t) &= \nu\Delta u\,dt - u\frac{\partial u}{\partial \xi}\,dt + \gamma u(t)\,dt + \sigma u(t)\,dW(t) \\ &\qquad + \sigma_0\,dW_0(t,\xi), \quad t>0, \quad \xi \in [0,1], \\ u(t,0) &= u(t,1) = 0 \ \text{ for all } \ t>0, \\ u(0) &= f \in L^2([0,1],\mathbf{R}). \end{aligned}\right\} \tag{2.1}$$

In the above SBE, $\Omega$ is the space of all continuous paths $\omega : \mathbf{R} \to \mathbf{R}$ such that $\omega(0) = 0$ with the compact open topology, $\mathcal{F}$ is its Borel $\sigma$-field, $\mathcal{F}_t$ is the sub-$\sigma$-field of $\mathcal{F}$ generated by all evaluations $\Omega \ni \omega \mapsto \omega(u) \in \mathbf{R}, u \le t$, and $P$ is Wiener measure on $\Omega$. The Brownian motion $W$ is given by $W(t,\omega) := \omega(t), \quad \omega \in \Omega,\ t \in \mathbf{R}$. We denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ the standard $P$-preserving ergodic Wiener shift on $\Omega$:

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}.$$

That is, $(W,\theta)$ is a *helix*:

$$W(t_1+t_2,\omega) - W(t_1,\omega) = W(t_2,\theta(t_1,\omega)), \quad t_1,t_2 \in \mathbf{R},\ \omega \in \Omega.$$

Denote by $T_t : L^2([0,1],\mathbf{R}) \to L^2([0,1],\mathbf{R}),\ t \ge 0$, the strongly continuous heat semigroup generated by the Laplacian $\nu\Delta$ with Dirichlet boundary conditions on $[0,1]$: $T_t = e^{t\nu\Delta},\ t \ge 0$. Pick a complete orthonormal system of eigenvectors $\{f_n : n \ge 1\}$ of $\nu\Delta$ in $L^2([0,1],\mathbf{R})$ with corresponding negative eigenvalues $\{\mu_n = -\nu\pi^2 n^2, n \ge 1\}$.

Denote by $p(t,\xi,y)$ the heat kernel for the heat equation

$$\left.\begin{aligned} \frac{\partial u_0}{\partial t} &= \Delta u_0(t), \quad t>0, \\ u_0(0,\cdot) &= f \in L^2\big([0,1],\mathbf{R}\big), \\ u_0(t,0) &= u_0(t,1) = 0 \quad \text{for all } t \ge 0 \end{aligned}\right\} \tag{2.2}$$

with Dirichlet boundary conditions. Therefore,

$$u_0(t,\xi) = T_t(f)(\xi) = \int_0^1 p(t,\xi,y)f(y)\,dy, \quad t>0,\ \xi\in[0,1].$$

The Hilbert space of all Hilbert–Schmidt operators $S : L^2([0,1],\mathbf{R}) \to L^2([0,1],\mathbf{R})$ is denoted by $L_2(L^2([0,1],\mathbf{R}))$ and carries the Hilbert–Schmidt norm

$$\|S\|_{L_2(L^2)} := \left(\sum_{n=1}^{\infty} \|S(f_n)\|_{L^2}^2\right)^{1/2}.$$

We now write the SBE (2.1) in its mild Stratonovich form,

(2.3)
$$\left.\begin{aligned} u(t) = T_t(f) - \int_0^t T_{t-s}\left[u(s)\frac{\partial u(s)}{\partial \xi}\right] ds + \gamma \int_0^t T_{t-s}u(s)\,ds + \sigma \int_0^t T_{t-s}u(s)\circ dW(s) \\ - \frac{1}{2}\sigma^2 \int_0^t T_{t-s}u(s)\,ds + \sigma_0 \int_0^t T_{t-s}\,dW_0(s,\cdot), \quad t\ge 0, \end{aligned}\right\}$$

and the equivalent integral form,

$$\left.\begin{aligned} u(t) &= T_t(f) + \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)u^2(s)(y)\,dy\,ds + \gamma \int_0^t T_{t-s}u(s)\,ds \\ &\quad + \sigma \int_0^t T_{t-s}u(s)\circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s}u(s)\,ds \\ &\quad + \sigma_0 \int_0^t T_{t-s}\,dW_0(s,\cdot), \quad t\ge 0. \end{aligned}\right\} \tag{2.4}$$

A *mild solution* of the SBE (2.1) is a family of $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{F}, \mathcal{B}(L^2([0,1],\mathbf{R})))$-measurable, $(\mathcal{F}_t)_{t\ge 0}$-adapted processes $u(\cdot,f,\cdot) : \mathbf{R}^+\times\Omega \to L^2([0,1],\mathbf{R}),\ f\in L^2([0,1],\mathbf{R})$, satisfying the stochastic integral equation (2.3) or (2.4).

Many of the statements in this article will turn out to hold *perfectly in* $\omega$ in the following sense: A family of propositions $\{P(\omega) : \omega\in\Omega\}$ is said to *hold perfectly in* $\omega$ if there is a sure event $\Omega^*\in\mathcal{F}$ such that $\theta(t,\cdot)(\Omega^*) = \Omega^*$ for all $t\in\mathbf{R}$ and $P(\omega)$ is true *for every* $\omega\in\Omega^*$.

We are now in a position to quote the following result from [M-Z.1]. This result shows that the family of mild solutions $u(\cdot,f,\cdot) : \mathbf{R}^+\times\Omega\to L^2([0,1],\mathbf{R}),\ f\in L^2([0,1],\mathbf{R})$, of the stochastic Burgers equation (2.1) generate a $C^\infty$ jointly measurable perfect cocycle

$$\begin{aligned} U : \mathbf{R}^+\times L^2\big([0,1],\mathbf{R}\big)\times\Omega &\longrightarrow L^2\big([0,1],\mathbf{R}\big), \\ (t,f,\omega) &\longmapsto U(t,f,\omega)\equiv u(t,f,\omega) \end{aligned}$$

on $L^2([0,1],\mathbf{R})$.

THEOREM 2.1. *The family of mild solutions* $u(\cdot,f,\cdot) : \mathbf{R}^+\times\Omega\to L^2([0,1],\mathbf{R}),\ f\in L^2([0,1],\mathbf{R})$, *of the SBE* (2.1) *has a version* $\mathbf{R}^+\times L^2([0,1],\mathbf{R})\times\Omega \ni (t,f,\omega)\mapsto U(t,f,\omega)\in L^2([0,1],\mathbf{R})$ *with the following properties:*

(i) $U : \mathbf{R}^+\times L^2\big([0,1],\mathbf{R}\big)\times\Omega\to L^2\big([0,1],\mathbf{R}\big)$ *is jointly measurable.*

(ii) *For each* $f \in L^2([0,1],\mathbf{R})$, $U(\cdot,f,\cdot):\mathbf{R}^+\times\Omega\to L^2([0,1],\mathbf{R})$ *is the mild solution of the SBE* (2.1)*:*

$$\left.\begin{aligned} U(t,f) &= T_t(f) + \frac{1}{2}\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)[U(s,f)(y)]^2\,dy\,ds \\ &\quad + \gamma\int_0^t T_{t-s}U(s,f)\,ds \\ &\quad + \sigma\int_0^t T_{t-s}U(s,f)\circ dW(s) - \frac{1}{2}\sigma^2\int_0^t T_{t-s}U(s,f)\,ds \\ &\quad + \sigma_0\int_0^t T_{t-s}\,dW_0(s,\cdot),\ t\geq 0. \end{aligned}\right\} \tag{2.4$'$}$$

(iii) $(U,\theta)$ *is a perfect cocycle, viz,*

$$U(t_1+t_2,f,\omega) = U(t_2,U\big(t_1,f,\omega),\theta(t_1,\omega))$$

*perfectly in* $\omega\in\Omega$ *for all* $t_1,t_2\geq 0$ *and all* $f\in L^2([0,1],\mathbf{R})$.

(iv) *For fixed* $t>0$ *and* $\omega\in\Omega$, *the map* $U(t,\cdot,\omega):L^2([0,1],\mathbf{R})\to L^2([0,1],\mathbf{R})$ *is (Fréchet)* $C^\infty$ *and takes bounded sets into relatively compact sets in* $L^2([0,1],\mathbf{R})$.

(v) *For each* $(t,f,\omega)\in\mathbf{R}^+\times L^2([0,1],\mathbf{R})\times\Omega$, *the Fréchet derivative* $DU(t,f,\omega)\in L(L^2([0,1],\mathbf{R}))$ *is Hilbert–Schmidt, and the map*

$$\begin{aligned} \mathbf{R}^+\times L^2\big([0,1],\mathbf{R}\big)\times\Omega &\longrightarrow L_2\big(L^2\big([0,1],\mathbf{R}\big)\big),\\ (t,f,\omega) &\longmapsto DU(t,f,\omega) \end{aligned}$$

*is strongly measurable.*

(vi) *The a priori estimate*

$$\sup_{0\leq t\leq a}\|U\big(t,f,\omega\big)\|_{L^2}\leq c_1(\omega)\big[\|f\|_{L^2}+c_2(\omega)\big] \tag{2.5}$$

*holds perfectly in* $\omega\in\Omega$ *for all* $f\in L^2([0,1],\mathbf{R})$ *and* $a>0$, *where* $c_i, i=1,2$, *are random positive constants independent of* $f$ *(but may depend on* $a$*) and are such that* $Ec_1<\infty$ *and* $E\log c_2<\infty$.

For a proof of Theorem 2.1, the reader may consult the proofs of Theorem 2.2 and Proposition 3.1 in [M-Z.1].

The following remark indicates the shortcoming of the estimate (2.5).

*Remark.* Observe that the estimate (2.5) *does not imply*

$$E\sup_{\substack{0\leq t\leq a\\ f\in L^2}}\frac{\|U(t,f,\cdot)\|_{L^2}^2}{1+\|f\|_{L^2}^2}<\infty. \tag{2.6}$$

This is because the random constant $c_2$ in (2.5) has only a logarithmic moment: $E\log c_2<\infty$ [M-Z.1]. As will be apparent in due course, this presents serious difficulties in the substitution scenario.

We now conclude this section by stating the main theorem in this article.

Denote by $\mathbb{D}^{1,4}(\Omega,L^2([0,1],\mathbf{R}))$ the Sobolev space of all $\mathcal{F}$-measurable random vectors $Y:\Omega\to L^2([0,1],\mathbf{R})$ which have fourth-order moments together with their Malliavin derivatives $\mathcal{D}Y$.

Theorem 2.2. *Let* $Y\in\mathbb{D}^{1,4}(\Omega,L^2([0,1],\mathbf{R}))$ *be a random variable, and* $U:\mathbf{R}^+\times L^2([0,1],\mathbf{R})\times\Omega\to L^2([0,1],\mathbf{R})$ *be the* $C^\infty$ *cocycle generated by all mild solutions* *of the Stratonovich SBE* (2.1)*. Then* $u(t) := U(t, Y)$*,* $t \geq 0$*, is a pathwise continuous* $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(L^2([0,1], \mathbf{R})))$*-measurable mild solution* $u : \mathbf{R}^+ \times \Omega \to L^2([0,1], \mathbf{R})$ *of the (anticipating) SBE*

$$\left.\begin{aligned} du(t) &= \nu \Delta u\, dt - u \frac{\partial u}{\partial \xi}\, dt + \gamma u(t)\, dt + \sigma u(t) \circ dW(t) \\ &\qquad - \frac{1}{2}\sigma^2 u(t)\, dt + \sigma_0\, dW_0(t, \cdot),\ t > 0, \\ u(t,0) &= u(t,1) = 0,\ t > 0, \\ u(0) &= Y. \end{aligned}\right\} \tag{2.7}$$

*The solution* $u$ *of* (2.7) *satisfies the estimate*

$$\sup_{0 \leq t \leq a} \|u(t,\omega)\|_{L^2} \leq c_1(\omega)\big[\|Y(\omega)\|_{L^2} + c_2(\omega)\big]$$

*for a.a.* $\omega \in \Omega$ *and* $a > 0$*, where* $c_i, i = 1, 2$*, are random positive constants (which may depend on* $a$*) and are such that* $Ec_1 < \infty$ *and* $E \log c_2 < \infty$*.*

*The process* $v(t) := DU(t, Y) \in L_2(L^2([0,1], \mathbf{R}))$*,* $t > 0$*, is a mild solution of the following linearized anticipating SBE:*

$$\left.\begin{aligned} dv(t) &= \nu \Delta v(t)\, dt - \frac{1}{2}\frac{\partial}{\partial \xi}\Big[v(t) U(t,Y)\Big]\, dt \\ &\qquad + \gamma v(t)\, dt + \sigma v(t) \circ dW(t) - \frac{1}{2}\sigma^2 v(t)\, dt,\ t > 0, \\ v(0) &= id_{L^2}. \end{aligned}\right\} \tag{2.8}$$

*If* $Y \in \mathbb{D}^{1,4}(\Omega, L^2([0,1], \mathbf{R}))$ *is a stationary point of the SBE* (2.1)*, then* $Y(\theta(t))$*,* $t \geq 0$*, is a stationary mild solution of the (anticipating) SBE* (2.7)*, and the process* $v(t) := DU(t, Y)$*,* $t > 0$*, is a mild solution of the linear anticipating spde* (2.8) *with* $U(t, Y)$ *replaced by the stationary coefficient* $Y(\theta(t))$*.*

The remaining sections in this article will be devoted to a proof of the above theorem. The main idea behind the proof of Theorem 2.2 is to justify making the substitution $f = Y$ in the parametrized mild integral equation (2.4′) of Theorem 2.1. However, such a substitution is a nontrivial undertaking as indicated in the subsequent computations in this article. As a first step in our strategy, we introduce a suitable truncation of the stochastic integral equation (2.4′) in the next section and then develop appropriate moment estimates on the cocycle generated by the solution field of the truncated integral equation.

**3. Truncation of the SBE and moment estimates.** We recall the mild stochastic integral equation (2.4′):

(2.4′)

$$\left.\begin{aligned} U(t,f) &= T_t(f) + \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y} p(t-s,\cdot,y)[U(s,f)(y)]^2\, dy\, ds + \gamma \int_0^t T_{t-s} U(s,f)\, ds \\ &\quad + \sigma \int_0^t T_{t-s} U(s,f) \circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s} U(s,f)\, ds \\ &\quad + \sigma_0 \int_0^t T_{t-s}\, dW_0(s,\cdot),\ t \geq 0. \end{aligned}\right\}$$

Fix any integer $m \geq 1$. Truncate the nonlinear (viz., quadratic) term in (2.4′) using a smooth bump function $\phi_m \in C_b^\infty(L^2([0,1],\mathbf{R}),\mathbf{R})$ satisfying

$$\phi_m(f) := \begin{cases} 1 & \text{if } \|f\|_{L^2} \leq m, \\ 0 & \text{if } \|f\|_{L^2} \geq m+1, \end{cases}$$

and $|\phi_m(f)| \leq 1$ for all $f \in L^2([0,1],\mathbf{R})$. Let $F_m : L^2([0,1],\mathbf{R}) \to L^1([0,1],\mathbf{R})$ denote the mapping

$$F_m(f) := \phi_m(f)^2 f^2 = \begin{cases} f^2 & \text{if } \|f\|_{L^2} \leq m, \\ 0 & \text{if } \|f\|_{L^2} \geq m+1. \end{cases} \tag{3.1}$$

It is easy to see that $F_m$ is $C^\infty$, globally bounded, and has globally bounded Fréchet derivatives, i.e., $F_m \in C_b^\infty(L^2, L^1)$. Denote by $U_m : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \Omega \to L^2([0,1],\mathbf{R})$ the $C_b^\infty$ cocycle generated by the truncated integral equation

$$\left.\begin{aligned} U_m(t,f) = T_t(f) &+ \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y} p(t-s,\cdot,y) F_m(U_m(s,f))(y)\,dy\,ds \\ &+ \gamma \int_0^t T_{t-s} U_m(s,f)\,ds \\ &+ \sigma \int_0^t T_{t-s} U_m(s,f) \circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s} U_m(s,f)\,ds \\ &+ \sigma_0 \int_0^t T_{t-s}\,dW_0(s,\cdot),\ t \geq 0. \end{aligned}\right\} \tag{2.4′(m)}$$

The existence of the smooth cocycle $(U_m, \theta)$ for the above truncated equation follows by mimicking the contraction mapping argument and the relevant estimates in the proofs of Proposition 2.2 and Theorem 2.1 in [M-Z.1].

Our objective in sections 4 and 5 is to show that if $Y \in \mathbb{D}^{1,4}(\Omega, L^2([0,1],\mathbf{R}))$, then we are able to make the substitution $f = Y$ in the truncated integral equation (2.4′(m)); thus we obtain

$$\left.\begin{aligned} U_m(t,Y) = T_t(Y) &+ \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y} p(t-s,\cdot,y) F_m(U_m(s,Y))(y)\,dy\,ds \\ &+ \gamma \int_0^t T_{t-s} U_m(s,Y)\,ds \\ &+ \sigma \int_0^t T_{t-s} U_m(s,Y) \circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s} U_m(s,Y)\,ds \\ &+ \sigma_0 \int_0^t T_{t-s}\,dW_0(s,\cdot),\ t \geq 0. \end{aligned}\right\} \tag{3.2}$$

In order to justify (3.2), we first develop sufficiently sharp spatial moment estimates on the cocycle $U_m$, its Fréchet derivative $DU_m(t,f,\cdot)$, and its Malliavin derivatives $\mathcal{D}_u U_m(t,f,\cdot)$ for $u,t \in [0,a]$ and $f \in L^2([0,1],\mathbf{R})$. As indicated in section 2, such moment estimates are not available for the cocycle $U : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \Omega \to L^2([0,1],\mathbf{R})$ of the original SBE. The derivations of the moment estimates for the cocycle $U_m$ are based on regularity properties of the heat kernel, Gronwall-type estimates, and the fact that $W$ has independent increments.

We begin by stating a useful lemma.

LEMMA 3.1. *Fix $a \in (0,\infty)$. Let $z, \eta : [0,a]\times\Omega \to \mathbf{R}^+$ be nonnegative $(\mathcal{B}([0,a])\otimes\mathcal{F}, \mathcal{B}(\mathbf{R}^+))$-measurable processes and $\psi : [0,a]\times[0,a]\times\Omega \to \mathbf{R}^+$ a $(\mathcal{B}([0,a]\times[0,a])\otimes\mathcal{F}, \mathcal{B}(\mathbf{R}^+))$-measurable random field satisfying the following hypotheses:*

(i) *For a.a. $\omega\in\Omega$, the paths $z(\cdot,\omega), \eta(\cdot,\omega), \psi(\cdot,\cdot,\omega)$ are bounded on $[0,a]$.*

(ii) *The process $z$ is $(\mathcal{F}_t)_{t\in[0,a]}$-adapted; and whenever $0 < s < t \le a$, the random variables $\psi(t,s,\cdot)$ are measurable with respect to the $\sigma$-algebra generated by the Brownian increments $W(s_2) - W(s_1), s \le s_1 \le s_2 \le t$.*

(iii) $\sup_{0\le t\le a} E\eta(t,\cdot) + \sup_{0\le s,t\le a} E\psi(t,s,\cdot) < \infty$; *and*

$$z(t,\cdot) \le \eta(t,\cdot) + \int_0^t \frac{\psi(t,s,\cdot)}{(t-s)^{3/4}} z(s,\cdot)\, ds \tag{3.3}$$

*a.s. for all $t\in[0,a]$.*

*Then* $\sup_{0\le t\le a} Ez(t,\cdot)$ *is finite and there exist positive constants $K_1, K_2$ such that*

$$\sup_{0\le s\le t} Ez(s,\cdot) \le K_1 e^{K_2 t} \tag{3.4}$$

*for all $t\in[0,a]$.*

*Proof.* In view of condition (iii), set

$$A_1 := \sup_{0\le t\le a} E\eta(t,\cdot), \ A_2 := \sup_{0\le s,t\le a} E\psi(t,s,\cdot).$$

For each integer $N \ge 1$ and any $s\in[0,a]$, define the events

$$\Omega_{s,N} := \left(\sup_{0\le s'\le s} z(s',\cdot) < N\right). \tag{3.5}$$

Since $z$ is $(\mathcal{F}_t)_{t\in[0,a]}$-adapted, $\Omega_{s,N}\in\mathcal{F}_s$ for all $s\in[0,a], N\ge 1$. Furthermore,

$$\Omega_{t,N} \subseteq \Omega_{s,N}, \quad 0\le s\le t,\ N\ge 1,$$

and

$$1_{\Omega_{t,N}} \le 1_{\Omega_{s,N}}, \quad 0\le s\le t,\ N\ge 1. \tag{3.6}$$

Since $z$ has a.a. sample-paths bounded on $[0,a]$, we have

$$\bigcup_{N\ge 1} \Omega_{s,N} = \Omega \tag{3.7}$$

for each $s\in[0,a]$.

Define

$$z_N(t,\cdot) := z(t,\cdot)\cdot 1_{\Omega_{t,N}}, \quad 0\le t\le a,\ N\ge 1. \tag{3.8}$$

Clearly $|z_N(t,\cdot)| \le N$ a.s. and $Ez_N(t,\cdot)| \le N$ for all $t\in[0,a]$ and all $N\ge 1$.

In (3.3), $z(s)$ is $\mathcal{F}_s$-measurable and $\psi(t,s,\cdot)$ is measurable with respect to the $\sigma$-algebra generated by the Brownian increments $W(s_2)-W(s_1), s \le s_1 \le s_2 \le t$. Therefore $z(s)$ and $\psi(t,s)$ are independent. Now multiply both sides of (3.3) by $1_{\Omega_{t,N}}$, use (3.6), take expectations, and use hypothesis (iii) together with the independence of $z(s,\cdot)\cdot 1_{\Omega_{s,N}}$ and $\psi(t,s,\cdot)$ to obtain

$$Ez_N(t,\cdot) \le A_1 + A_2 \int_0^t \frac{1}{(t-s)^{3/4}} Ez_N(s,\cdot)\,ds, \quad 0 \le t \le a, \tag{3.9}$$

for all $N \ge 1$, where the positive constants $A_1, A_2$ in (3.9) are *independent* of $N$.

Using the above inequality and Lemma 15 in [D] (or the arguments in [W]), it follows that

$$Ez_N(t,\cdot) \le K_1 e^{K_2 t}, \quad 0 \le t \le a, \tag{3.10}$$

for all $N \ge 1$. The constants $K_1, K_2$ in the above inequality are also independent of $N$. Letting $N \to \infty$ in (3.10), using the fact that

$$\lim_{N\to\infty} z_N(t,\cdot) = z(t,\cdot)$$

a.s., and applying the monotone convergence theorem, we obtain the required inequality

$$\sup_{0\le t'\le t} Ez(t',\cdot) \le K_1 e^{K_2 t}$$

for all $0 \le t \le a$. This proves (3.4). □

*Remark.* From the proof of the above lemma it is easy to see that when $\eta \equiv 0$ a.s., $z(t) = 0$ a.s. for all $t \in [0,a]$. This follows from the fact that the constant $K_1$ in (3.4) is a linear multiple of $A_1$.

We are now in a position to develop a series of moment estimates on the cocycle $U_m$ of the truncated integral equation (2.4′(m)). These estimates are needed in order to facilitate the substitution $f = Y$ in (2.4′(m)).

THEOREM 3.2. *Fix any integers $m, p \ge 1$ and let $a \in (0,\infty)$. Let $U_m : \mathbf{R}^+ \times L^2([0,1],\mathbf{R}) \times \Omega \to L^2([0,1],\mathbf{R})$ be the $C_b^\infty$ cocycle generated by the truncated integral equation* (2.4′(m)). *Then the following estimates hold:*

(i)

$$\sup_{0\le t\le a} E \sup_{\substack{f_1,f_2\in L^2\\ f_1\ne f_2}} \left\{ \frac{\|U_m(t,f_1,\cdot)-U_m(t,f_2,\cdot)\|_{L^2}^{2p}}{\|f_1-f_2\|_{L^2}^{2p}} \right\} < \infty; \tag{3.11}$$

(ii)

$$\sup_{0\le t\le a} E\left\{ \sup_{f\in L^2} \frac{\|U_m(t,f,\cdot)\|_{L^2}^{2p}}{(1+\|f\|_{L^2}^{2p})} \right\} < \infty; \tag{3.12}$$

(iii)

$$\sup_{0\le t\le a} E\left\{ \sup_{f\in L^2} \|D^{(j)}U_m(t,f,\cdot)\|_{L^{(j)}(L^2)}^{2p} \right\} < \infty,\ j=1,2; \tag{3.13}$$

*in the above estimate, $D^{(j)}$ stands for the $j$th Fréchet derivative of $U$ in the variable $f$, and $L^{(j)}(L^2)$ is the Banach space of all continuous $j$-multilinear mappings $A : L^2 \times L^2 \times \cdots L^2 \to L^2$ given the uniform norm*

$$\|A\|_{L^{(j)}(L^2)} := \sup\{\|A(f_1, f_2, \ldots, f_j)\|_{L^2} : f_k \in L^2, \|f_k\|_{L^2} \le 1, 1 \le k \le j\};$$

(iv)

$$\sup_{0\le t\le a} E\Big\{ \sup_{f\in L^2} \frac{\|\mathcal{D}U_m(t,f,\cdot)\|_{L^2}^{2p}}{(1+\|f\|_{L^2}^{2p})} \Big\} < \infty, \tag{3.14}$$

*where $\mathcal{D}$ stands for the Malliavin derivative;*

(v)

$$\int_0^a \int_u^a E\Big\{ \sup_{f\in L^2} \frac{\|\mathcal{D}_u D U_m(t,f,\cdot)\|_{L_2(L^2)}^{2p}}{(1+\|f\|_{L^2}^{2p})} \Big\} \, dt\, du < \infty. \tag{3.15}$$

*Proof.* The proof is based on the following well-known estimates on the Dirichlet heat kernel $p(t, \xi, y)$:

$$\left| \frac{\partial p(t,\xi,y)}{\partial y} \right| \le \frac{c_1}{t} e^{\frac{-(\xi-y)^2}{2c_2 t}}, \quad t > 0,\ \xi, y \in [0,1], \tag{3.16}$$

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2c_2 t}} \, dy \le c_3 \sqrt{t}, \quad t > 0, \tag{3.17}$$

with positive constants $c_1, c_2, c_3$.

Fix any integers $m, p \ge 1$.

To simplify the computations, we will assume that $\gamma = \sigma_0 = 0$ in the truncated integral equation (2.4′(m)). Under this assumption, we then replace (2.4′(m)) by its random equivalent form

(3.18)

$$U_m(t,f) = Q(t)T_t(f) + \frac{1}{2} \int_0^t \int_0^1 Q(t)Q(s)^{-1} \frac{\partial p(t-s,\cdot,y)}{\partial y} F_m(U_m(s,f))(y)\, dy\, ds,$$

where

$$Q(t) := \exp\left\{ \sigma W(t) - \frac{\sigma^2}{2} t \right\}$$

for all $t \in [0, a]$. The relation (3.18) above follows from equations (2.4) and (2.6) in [M-Z.1].

We will first prove the estimate (3.11) in (i). Using the truncated integral equation (3.18), the above two heat kernel estimates, and the global Lipschitz property of

$F_m : L^2([0,1],\mathbf{R}) \to L^1([0,1],\mathbf{R})$, we get the following inequalities for any $f_1, f_2 \in L^2([0,1],\mathbf{R})$:

$$
\begin{aligned}
&\|U_m(t,f_1) - U_m(t,f_2)\|_{L^2}^2 \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 \\
&\qquad + \frac{1}{2}\int_0^1 \left( \int_0^t \int_0^1 Q(t)Q(s)^{-1} \left| \frac{\partial p(t-s,\xi,y)}{\partial y} \right| \right. \\
&\qquad\qquad \left. \times |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)|\, dy\, ds \right)^2 d\xi \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 + C_1 \int_0^1 \Bigg[ \int_0^t \frac{Q(t)Q(s)^{-1}}{(t-s)^{1/8}} \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} \\
&\qquad\qquad \times |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)| \\
&\qquad\qquad dy \frac{1}{(t-s)^{3/8}}\, ds \Bigg]^2 d\xi \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 + C_1 \int_0^1 \Bigg\{ \int_0^t \frac{Q(t)^2Q(s)^{-2}}{(t-s)^{1/4}} \\
&\qquad\qquad \times \Bigg( \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} |F_m(U_m(s,f_1))(y) \\
&\qquad\qquad - F_m(U_m(s,f_2))(y)|\, dy \Bigg)^2 ds \Bigg\} d\xi \int_0^t \frac{ds}{(t-s)^{3/4}} \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 + C \int_0^1 \int_0^t \frac{Q(t)^2Q(s)^{-2}}{(t-s)^{3/4}} \int_0^1 e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} \\
&\qquad \times |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)|\, dy \\
&\qquad \times \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)|\, dy\, ds\, d\xi \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 \\
&\qquad + C \int_0^1 \int_0^t \frac{Q(t)^2Q(s)^{-2}}{(t-s)^{3/4}} \left( \int_0^1 |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)|\, dy \right) \\
&\qquad \times \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)|\, dy\, ds\, d\xi \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 \\
&\qquad + C \int_0^t \frac{Q(t)^2Q(s)^{-2}}{(t-s)^{3/4}} \|F_m(U_m(s,f_1)) - F_m(U_m(s,f_2))\|_{L^1} \\
&\qquad \times \int_0^1 \left( \int_0^1 \frac{1}{\sqrt{t-s}} e^{\frac{-(\xi-y)^2}{2c_2(t-s)}} d\xi \right) |F_m(U_m(s,f_1))(y) - F_m(U_m(s,f_2))(y)|\, dy\, ds \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 + C \int_0^t \frac{Q(t)^2Q(s)^{-2}}{(t-s)^{3/4}} \|F_m(U_m(s,f_1)) - F_m(U_m(s,f_2))\|_{L^1}^2\, ds \\
&\quad \le 2Q(t)^2\|f_1 - f_2\|_{L^2}^2 + C_m \int_0^t \frac{Q(t)^2Q(s)^{-2}}{(t-s)^{3/4}} \|U_m(s,f_1)) - U_m(s,f_2))\|_{L^2}^2\, ds
\end{aligned}
\tag{3.19}
$$

for $0 \le t \le a$. By (3.19) and Hölder's inequality, there exist positive constants

$C_p, C^i_{m,p}, i = 1, 2$ such that

$$
\begin{aligned}
&\|U_m(t, f_1) - U_m(t, f_2)\|^{2p}_{L^2} \\
&\quad \le 2^p Q(t)^{2p} \|f_1 - f_2\|^{2p}_{L^2} \\
&\quad\quad + 2^{p-1} C^p_m \left[ \int_0^t \frac{Q(t)^2 Q(s)^{-2}}{(t-s)^{3/4}} \|U_m(s, f_1)) - U_m(s, f_2))\|^2_{L^2}\, ds \right]^p \\
&\quad \le 2^p Q(t)^{2p} \|f_1 - f_2\|^{2p}_{L^2} \\
&\quad\quad + C^1_{m,p} \int_0^t \frac{Q(t)^{2p} Q(s)^{-2p}}{(t-s)^{3/4}} \|U_m(s, f_1)) - U_m(s, f_2))\|^{2p}_{L^2}\, ds,
\end{aligned}
\tag{3.20}
$$

a.s. for all $t \in [0, a]$.

Now set

$$
z(t) := \sup_{\substack{f_1, f_2 \in L^2 \\ f_1 \neq f_2}} \frac{\|U_m(t, f_1, \cdot) - U_m(t, f_2\|^{2p}_{L^2}}{\|f_1 - f_2\|^{2p}_{L^2}}, \ 0 \le t \le a.
$$

Then the inequality (3.20) implies

$$
z(t) \le 2^p Q(t)^{2p} + C^1_{m,p} \int_0^t \frac{Q(t)^{2p} Q(s)^{-2p}}{(t-s)^{3/4}} z(s)\, ds, \tag{3.21}
$$

a.s. for all $t \in [0, a]$. Now $z$ has a.a. sample paths bounded on $[0, a]$ and measurable. Furthermore, $\sup_{0 \le t \le a} EQ(t)^{2p} < \infty$, $z$ is $(\mathcal{F}_t)_{t \in [0,a]}$-adapted, and $Q(t)^{2p} Q(s)^{-2p}$ is measurable with respect to the $\sigma$-algebra generated by the Brownian increments $W(s_2) - W(s_1), s \le s_1 \le s_2 \le t$. So the requirements of Lemma 3.1 are fulfilled by the inequality (3.21). Therefore, the estimate (3.11) in (i) follows directly from (3.21).

We next prove the estimate (3.12). Let $f \in L^2([0, 1], \mathbf{R})$. Starting with (3.18) and using an argument similar to that used in obtaining (3.20), we get

$$
\|U_m(t, f)\|^{2p}_{L^2} \le 2^p Q(t)^{2p} \|f\|^{2p}_{L^2} + C^2_{m,p} \int_0^t \frac{Q(t)^{2p} Q(s)^{-2p}}{(t-s)^{3/4}} \|F_m(U_m(s, f))\|^{2p}_{L^1}\, ds, \tag{3.22}
$$

a.s. for $t \in [0, a]$. Define the process $z$ by

$$
z(t) := \sup_{f \in L^2} \frac{\|U_m(t, f, \cdot)\|^{2p}_{L^2}}{1 + \|f\|^{2p}_{L^2}}, \quad 0 \le t \le a.
$$

Since $F_m : L^2([0, 1], \mathbf{R}) \to L^1([0, 1], \mathbf{R})$ is globally bounded (and $U_m(t, \cdot)$ is continuous on $L^2$), $z$ has a.a. sample paths (measurable and) bounded on $[0, a]$ and the following inequality holds a.s.

$$
z(t) \le 2^p Q(t)^{2p} + C^3_{m,p} \int_0^t \frac{Q(t)^{2p} Q(s)^{-2p}}{(t-s)^{3/4}}\, ds, \quad 0 \le t \le a. \tag{3.23}
$$

The above inequality gives $\sup_{0 \le t \le a} Ez(t) < \infty$ and (3.12) follows.

To prove (3.13), use the fact that $F_m \in C_b^1(L^2, L^1)$ and take Fréchet derivatives in the integral equation (3.18) to get the following:

$$DU_m(t,f) = Q(t)T_t + \frac{1}{2}\int_0^t \int_0^1 Q(t)Q(s)^{-1}\frac{\partial p(t-s,\cdot,y)}{\partial y} \times DF_m(U_m(s,f))DU_m(s,f)(\cdot)(y)\,dy\,ds, \tag{3.24}$$

a.s. for all $f \in L^2([0,1],\mathbf{R})$, $t \in [0,a]$. Since $DF_m$ is globally bounded, then by similar arguments as before, we have

$$\|DU_m(t,f)\|_{L(L^2)}^{2p} \le 2^p Q(t)^{2p} + C_{m,p}^3 \int_0^t \frac{Q(t)^{2p}Q(s)^{-2p}}{(t-s)^{3/4}}\|DU_m(s,f)\|_{L(L^2)}^{2p}\,ds, \tag{3.25}$$

a.s. for $t \in [0,a]$. Taking $\sup_{f\in L^2}$ in (3.25) and applying Lemma 3.1 to the resulting inequality gives the required inequality (3.13) for $j=1$. The corresponding estimate for $D^{(2)}U_m(t,f)$ follows by differentiating (3.24) once more, using the boundedness of $DU_m(t,f), DF_m, D^2F_m$ and Lemma 3.1.

We now prove (3.14). Let $0 \le u,t \le a, 0 \le s \le t$ and consider the Malliavin derivatives

$$\mathcal{D}_u Q(t) = \mathcal{D}_u \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\} = \sigma Q(t) 1_{[0,t]}(u), \tag{3.26}$$

$$\begin{aligned}\mathcal{D}_u[Q(t)Q(s)^{-1}] &= \mathcal{D}_u \exp\left\{\sigma[W(t)-W(s)] - \frac{1}{2}\sigma^2(t-s)\right\}\\ &= \sigma Q(t)Q(s)^{-1}1_{[s,t]}(u).\end{aligned} \tag{3.27}$$

Next, take Malliavin derivatives of both sides of (3.18) to obtain

$$\begin{aligned}\mathcal{D}_u U_m(t,f) = \mathcal{D}_u Q(t)T_t &+ \frac{1}{2}\int_0^t\int_0^1 Q(t)Q(s)^{-1}\frac{\partial p(t-s,\cdot,y)}{\partial y}\\ &\times [DF_m(U_m(s,f))\mathcal{D}_u U_m(s,f)](y)\,dy\,ds\\ &+ \frac{1}{2}\int_0^t\int_0^1 \mathcal{D}_u[Q(t)Q(s)^{-1}]\frac{\partial p(t-s,\cdot,y)}{\partial y}F_m(U_m(s,f))(y)\,dy\,ds,\end{aligned} \tag{3.28}$$

a.s. for $f \in L^2([0,1],\mathbf{R})$, $t \in [0,a]$. Substituting from (3.26), (3.27) into (3.28), and using Hölder's inequality (as in (3.20)) and the boundedness of $F_m$ and $DF_m$, we get

$$\begin{aligned}\|\mathcal{D}_u U_m(t,f)\|_{L^2}^{2p} &\le C_p Q(t)^{2p} + C_{m,p}^3\int_0^t \frac{Q(t)^{2p}Q(s)^{-2p}}{(t-s)^{3/4}}\|\mathcal{D}_u U_m(s,f)\|_{L^2}^{2p}\,ds\\ &\quad + C_{m,p}^4 \int_0^t \frac{Q(t)^{2p}Q(s)^{-2p}}{(t-s)^{3/4}}\|F_m(U_m(s,f))\|_{L^1}\,ds\\ &\le C_p\|Q\|_\infty^{2p} + C_{m,p}^5\|Q\|_\infty^{2p}\|Q^{-1}\|_\infty^{2p}\int_0^t \frac{1}{(t-s)^{3/4}}\,ds\\ &\quad + C_{m,p}^3\int_0^t \frac{Q(t)^{2p}Q(s)^{-2p}}{(t-s)^{3/4}}\|\mathcal{D}_u U_m(s,f)\|_{L^2}^{2p}\,ds\\ &\le C_p\|Q\|_\infty^{2p} + C_{m,p}^6\|Q\|_\infty^{2p}\|Q^{-1}\|_\infty^{2p}\\ &\quad + C_{m,p}^3\int_0^t \frac{Q(t)^{2p}Q(s)^{-2p}}{(t-s)^{3/4}}\|\mathcal{D}_u U_m(s,f)\|_{L^2}^{2p}\,ds,\end{aligned} \tag{3.29}$$

a.s. for $f \in L^2([0,1],\mathbf{R})$, $t \in [0,a]$. In the above inequality $\|Q\|_\infty := \sup_{0\le t\le a} Q(t)$, $\|Q^{-1}\|_\infty := \sup_{0\le t\le a} Q(t)^{-1}$. Now divide both sides of (3.29) by $(1+\|f\|_{L^2}^{2p})$, take $\sup_{f\in L^2}$, and apply Lemma 3.1 to get (3.14).

To prove the estimate (3.15) in (v), we need to estimate the Malliavin derivative $\mathcal{D}_u DU_m(t,f,\cdot)$ for $t>0$ and any $f \in L^2([0,1],\mathbf{R})$. To this end, we must first show that $DU_m(t,f) : L^2([0,1],\mathbf{R}) \to L^2([0,1],\mathbf{R})$ is Hilbert–Schmidt for any $t>0$ and $f \in L^2([0,1],\mathbf{R})$. In fact, we will establish the following preliminary estimate:

$$\int_0^a E\left\{\sup_{f\in L^2} \|DU_m(t,f,\cdot)\|_{L_2(L^2)}^{2p}\right\} dt < \infty. \tag{3.30}$$

To justify the above estimate, let $f \in L^2([0,1],\mathbf{R})$. Recall that $\{f_k,\, k\ge 1\}$ is the complete orthonormal system of eigenfunctions of the Dirichlet Laplacian $\nu\Delta$ with eigenvalues $\{\mu_k = -\nu\pi^2 k^2,\, k \ge 1\}$. Then using (3.24) we obtain

$$\begin{aligned} DU_m(t,f)(f_k) = Q(t)T_t(f_k) + \frac{1}{2}\int_0^t \int_0^1 Q(t)Q(s)^{-1} \frac{\partial p(t-s,\cdot,y)}{\partial y} \\ \times DF_m(U_m(s,f))DU_m(s,f)(f_k)(y)\,dy\,ds, \end{aligned} \tag{3.31}$$

a.s. for $f \in L^2([0,1],\mathbf{R})$, $t \in [0,a]$, $k \ge 1$. Since $DF_m$ is globally bounded, then by employing a similar argument to the one used for (3.19), we get the following estimates:

$$\begin{aligned} \sum_{k=1}^N \|DU_m(t,f)(f_k)\|_{L^2}^2 \le 2\|Q\|_\infty^2 \|T_u\|_{L_2(L^2)}^2 \\ + C_m^4 \int_0^t \frac{Q(t)^2 Q(s)^{-2}}{(t-s)^{3/4}} \sum_{k=1}^N \|DU_m(s,f)(f_k)\|_{L^2}^2\, ds, \end{aligned} \tag{3.32}$$

a.s. for all $t \in [u,a]$, $u \in (0,a]$ and all integers $N \ge 1$. The constant $C_m^4$ is *independent* of $N$. Taking $\sup_{f\in L^2}$ in the above inequality and applying Lemma 3.1 gives

$$E \sup_{f\in L^2} \sum_{k=1}^N \|DU_m(t,f)(f_k)\|_{L^2}^2 \le C_1 \|T_u\|_{L_2(L^2)}^2 \exp\{C_2 t\} \tag{3.33}$$

for all $t \ge u > 0$ and all $N \ge 1$. Now let $N \to \infty$ in the above inequality and apply the monotone convergence theorem to see that $DU_m(t,f) \in L_2(L^2([0,1],\mathbf{R}))$ for $t \ge u > 0$ and

$$E \sup_{f\in L^2} \|DU_m(t,f)\|_{L_2(L^2)}^2 \le C_1 \|T_u\|_{L_2(L^2)}^2 \exp\{C_2 t\}. \tag{3.34}$$

In view of the fact that $\int_0^a \|T_u\|_{L_2(L^2)}^2\, du < \infty$, the inequality (3.30) (for $p=1$) follows immediately from (3.34). The case $p \ge 2$ in (3.30) is treated similarly.

We now proceed to prove the final estimate (3.15) (for $p = 1$) of the theorem. We start by taking Malliavin derivatives $\mathcal{D}_u$ in (3.31). This gives

$$\begin{aligned}
\mathcal{D}_u DU_m(t,f)(f_k) &= \mathcal{D}_u Q(t)T_t(f_k) + \frac{1}{2}\int_0^t \int_0^1 \mathcal{D}_u[Q(t)Q(s)^{-1}]\frac{\partial p(t-s,\cdot,y)}{\partial y} \\
&\quad \times DF_m(U_m(s,f))DU_m(s,f)(f_k)(y)\,dy\,ds \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 Q(t)Q(s)^{-1}\frac{\partial p(t-s,\cdot,y)}{\partial y} D^2F_m(U_m(s,f)) \\
&\quad \times DU_m(s,f)(f_k)(y)\mathcal{D}_u U_m(s,f)\,dy\,ds \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 Q(t)Q(s)^{-1}\frac{\partial p(t-s,\cdot,y)}{\partial y} DF_m(U_m(s,f)) \\
&\quad \times \mathcal{D}_u DU_m(s,f)(f_k)(y)\,dy\,ds \\
&= \sigma Q(t)T_t(f_k)1_{[0,t]}(u) \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 \sigma Q(t)Q(s)^{-1}1_{[s,t]}(u)\frac{\partial p(t-s,\cdot,y)}{\partial y} \\
&\quad \times DF_m(U_m(s,f))DU_m(s,f)(f_k)(y)\,dy\,ds \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 Q(t)Q(s)^{-1}\frac{\partial p(t-s,\cdot,y)}{\partial y} D^2F_m(U_m(s,f)) \\
&\quad \times DU_m(s,f)(f_k)(y)\mathcal{D}_u U_m(s,f)\,dy\,ds \\
&\quad + \frac{1}{2}\int_0^t \int_0^1 Q(t)Q(s)^{-1}\frac{\partial p(t-s,\cdot,y)}{\partial y} \\
&\quad \times DF_m(U_m(s,f))\mathcal{D}_u DU_m(s,f)(f_k)(y)\,dy\,ds, \qquad (3.35)
\end{aligned}$$

a.s. for all $f \in L^2([0,1],\mathbf{R})$, $0 < u \le t \le a$, $k \ge 1$. Following the argument used to obtain (3.34) from (3.31), we get positive deterministic constants $C_3, C_4$ such that

$$E \sup_{f\in L^2} \frac{\|\mathcal{D}_u DU_m(t,f)\|^2_{L_2(L^2)}}{(1+\|f\|^2_{L^2})} \le C_3\Bigl(1+\|T_u\|^2_{L_2(L^2)}\Bigr)\exp\{C_4 t\} \qquad (3.36)$$

for $a \ge t \ge u > 0$. This implies (3.15) (for $p = 1$). The case $p \ge 2$ is treated similarly. The proof of Theorem 3.2 is now complete. □

**4. Finite-dimensional initial velocities.** This section considers the truncated integral equation (2.4′(m)) in the special scenario when the initial function $f$ is to be replaced by a *finite-dimensional* random variable.

Let $Y \in \mathbb{D}^{1,4}(\Omega, L^2([0,1],\mathbf{R}))$. Recall the complete orthonormal system of eigenfunctions $\{f_k,\, k \ge 1\}$ of the Dirichlet Laplacian $\nu\Delta$ on $[0,1]$. For each integer $n \ge 1$, denote by $H_n$ the $n$-dimensional linear subspace of $L^2([0,1],\mathbf{R})$ spanned by $\{f_k\}_{k=1}^n$, and let $P_n : L^2([0,1],\mathbf{R}) \to H_n$ be the projection

$$P_n(f) := \sum_{k=1}^n \langle f, f_k, \rangle e_k, \quad f \in L^2\bigl([0,1],\mathbf{R}\bigr). \qquad (4.1)$$

Define $Y_n : \Omega \to H_n$ by

$$Y_n := P_n \circ Y, \quad n \ge 1. \qquad (4.2)$$

Clearly $Y_n \in \mathbb{D}^{1,4}(\Omega, H_n)$ for each $n \ge 1$ and $Y_n \to Y$ as $n \to \infty$ a.s.

Our main objective in this section is to replace $Y$ by its finite-dimensional projection $Y_n$ in the truncated mild equation (3.2). More specifically, we have the following.

THEOREM 4.1. *Suppose* $Y \in \mathbb{D}^{1,4}(\Omega, L^2([0,1], \mathbf{R}))$. *Then*

$$\left.\begin{aligned} U_m(t,Y_n) &= T_t(Y_n) + \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y} p(t-s,\cdot,y) F_m(U_m(s,Y_n))(y)\,dy\,ds \\ &\quad + \gamma \int_0^t T_{t-s} U_m(s,Y_n)\,ds \\ &\quad + \sigma \int_0^t T_{t-s} U_m(s,Y_n) \circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s} U_m(s,Y_n)\,ds \\ &\quad + \sigma_0 \int_0^t T_{t-s}\,dW_0(s,\cdot),\ t \geq 0, \end{aligned}\right\} \tag{4.3}$$

*a.s. for each* $m, n \geq 1$.

*Remark.* In addition to employing finite-dimensional selection theorems, the proof of the above result still requires Malliavin calculus techniques. This is basically due to the underlying *infinite-dimensional* semigroup dynamics in $\{T_t\}_{t\geq 0}$.

*Proof of Theorem* 4.1. We continue to assume for simplicity that $\mu = \sigma_0 = 0$ in (2.4′(m)). So it is sufficient to prove the following identity:

$$\left.\begin{aligned} U_m(t,Y_n) &= T_t(Y_n) + \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y} p(t-s,\cdot,y) F_m(U_m(s,Y_n))(y)\,dy\,ds \\ &\quad + \sigma \int_0^t T_{t-s} U_m(s,Y_n) \circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s} U_m(s,Y_n)\,ds,\ t \geq 0, \end{aligned}\right\} \tag{4.4}$$

a.s. for each $m, n \geq 1$.

The main difficulty in proving (4.4) is to substitute $f = Y_n$ in the Stratonovich integral $\int_0^t T_{t-s} U_m(s,f) \circ dW(s)$ in (2.4′(m)). To do this, we further project the integral onto $H_N$ for each integer $N \geq 1$. Fix $t \in (0,a]$ and define the family of finite-dimensional random fields $I_m^N : \Omega \times H_n \to H_N$ by

$$I_m^N(f) := \int_0^t T_{t-s} P_N U_m(s,f) \circ dW(s), \quad f \in H_n. \tag{4.5}$$

Note that the above random field takes values in $H_N$ because the latter subspace is invariant under the semigroup $\{T_t\}_{t\geq 0}$. We now apply finite-dimensional substitution theorems to (4.5) [Nu.2]. Note that, by the estimate (3.11), there exists a positive deterministic constant $C_1$, independent of $N$, but it may depend on $m, p$, such that

$$E\|T_{t-s} P_N U_m(s,g_1) - T_{t-s} P_N U_m(s,g_2)\|^{2p} \leq C_1 \|g_1 - g_2\|_{L^2}^{2p} \tag{4.6}$$

for all $g_1, g_2 \in H_n$, $s < t$. The above estimate implies that the random field $I_m^N$ has a continuous version which gives

$$\int_0^t T_{t-s} P_N U_m(s,f) \circ dW(s)\bigg|_{f=Y_n} = \int_0^t P_N T_{t-s} U_m(s,Y_n) \circ dW(s), \tag{4.7}$$

a.s. for all $N \geq 1$. Next we let $N \to \infty$ (with fixed $n \geq 1$) in the above relation. To do this, define

$$I_m(f) := \int_0^t T_{t-s} U_m(s,f) \circ dW(s), \quad f \in H_n. \tag{4.8}$$

Then rewrite (4.5) and (4.8) in the Itô form

$$I_m^N(f) := \int_0^t T_{t-s} P_N U_m(s,f)\, dW(s) + \frac{1}{2}\int_0^t T_{t-s} P_N U_m(s,f)\, ds, \tag{4.9}$$

$$I_m(f) := \int_0^t T_{t-s} U_m(s,f)\, dW(s) + \frac{1}{2}\int_0^t T_{t-s} U_m(s,f)\, ds \tag{4.9$'$}$$

for all $f \in H_n$. Note that (4.9) holds because the projection $P_N = P_N^2$ commutes with $T_{t-s}$, $0 \le s \le t$. Using (4.9) and (4.9′), we will show that

$$\lim_{N\to\infty} I_m^N(f) = I_m(f) \tag{4.10}$$

in probability for all $f \in H_n$. So fix $f \in H_n$ and consider the easy estimate

$$\begin{aligned} E\left\| \int_0^t T_{t-s} P_N U_m(s,f)\, dW(s) - \int_0^t T_{t-s} U_m(s,f)\, dW(s)\right\|_{L^2}^{2p} \\ \le C_2 \int_0^t E\|P_N U_m(s,f) - U_m(s,f)\|_{L^2}^{2p}\, ds. \end{aligned} \tag{4.11}$$

Since $\|P_N U_m(s,f) - U_m(s,f)\|_{L^2}^{2p}\|_{L^2} \to 0$ as $N \to \infty$, then by the above estimate, (3.12), and the dominated convergence theorem, it follows that the left-hand side of (4.11) tends to 0 as $N \to \infty$ for each $f \in H_n$:

$$\lim_{N\to\infty} E\| \int_0^t T_{t-s} P_N U_m(s,f)\, dW(s) - \int_0^t T_{t-s} U_m(s,f)\, dW(s)\|_{L^2}^{2p} = 0. \tag{4.12}$$

By a similar argument, it follows that

$$\lim_{N\to\infty} E\left\| \int_0^t T_{t-s} P_N U_m(s,f)\, ds - \int_0^t T_{t-s} U_m(s,f)\, ds\right\|_{L^2}^{2p} = 0. \tag{4.13}$$

Using (4.12) and (4.13), (4.10) then follows. The latter relation implies that

$$\lim_{N\to\infty} I_m^N(Y_n) = I_m(Y_n) \tag{4.14}$$

in probability for each $n \ge 1$ ([Nu.2, Lemma 5.3.1] and [M-S, Lemma A.1]).

Since $Y_n \in \mathbb{D}^{1,4}(\Omega, H_n)$, then using the estimate (3.12) and a similar argument to the one used in obtaining relation (5.8) in the next section, it follows that the process $[0,t) \ni s \mapsto T_{t-s} U_m(s, Y_n) \in L^2([0,1], \mathbf{R})$ belongs to $\mathbb{L}^{1,2}$ and is therefore Stratonovich integrable; furthermore,

$$\lim_{N\to\infty} \int_0^t T_{t-s} P_N U_m(s, Y_n) \circ dW(s) = \int_0^t T_{t-s} U_m(s, Y_n) \circ dW(s) \tag{4.15}$$

in probability for fixed $n, m \ge 1$.

Using the estimate (3.11) and Kolomogorov's continuity theorem, it is easy to see that the random field $I_m$ has a continuous version $I_m : \Omega \times H_n \to L^2([0,1], \mathbf{R})$).

Putting things together, we get the following equalities from (4.10), (4.7), and (4.15):

$$\begin{aligned}
&\int_0^t T_{t-s}U_m(s,f)\circ dW(s)\Big|_{f=Y_n}\\
&= \left\{\lim_{N\to\infty}\int_0^t T_{t-s}P_N U_m(s,f)\circ dW(s)\right\}\Big|_{f=Y_n}\\
&= \lim_{N\to\infty}\left\{\int_0^t T_{t-s}P_N U_m(s,f)\circ dW(s)\Big|_{f=Y_n}\right\}\\
&= \lim_{N\to\infty}\int_0^t T_{t-s}P_N U_m(s,Y_n)\circ dW(s)\\
&= \int_0^t T_{t-s}U_m(s,Y_n)\circ dW(s),
\end{aligned} \tag{4.16}$$

a.s. for all $n \geq 1$. To complete the proof of Theorem 4.1, it is easy to see that we can substitute $f = Y_n$ in all the remaining (nonstochastic) terms in the mild truncated integral equation (2.4′(m)). Hence (4.3) holds and the proof of Theorem 4.1 is complete. ◻

**5. Proof of the existence theorem.** The main issue in this section is to complete the argument in the proof of our main existence theorem (viz., Theorem 2.2 of section 2). So far, we are able to perform finite-dimensional substitutions in the mild truncated integral equation (2.4′(m)) of section 3. The rest of the proof of Theorem 2.2 will address the following two steps:

(i) infinite-dimensional substitution in the mild truncated equation (2.4′(m));

(ii) lifting the truncation from (2.4′(m)) to obtain an anticipating mild solution of the SBE (2.7) of section 2.

*Proof of Theorem* 2.2.

*Step* 1. We show that one can substitute $f = Y \in \mathbb{D}^{1,4}(\Omega, L^2([0,1],\mathbf{R}))$ in (2.4′(m)), that is,

$$\left.\begin{aligned}
U_m(t,Y) &= T_t(Y) + \frac{1}{2}\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)F_m(U_m(s,Y))(y)\,dy\,ds\\
&\quad + \gamma\int_0^t T_{t-s}U_m(s,Y)\,ds\\
&\quad + \sigma\int_0^t T_{t-s}U_m(s,Y)\circ dW(s) - \frac{1}{2}\sigma^2\int_0^t T_{t-s}U_m(s,Y)\,ds\\
&\quad + \sigma_0\int_0^t T_{t-s}\,dW_0(s,\cdot),
\end{aligned}\right\} \tag{5.1}$$

a.s. for $t > 0$. To prove (5.1) we pass to the limit as $n \to \infty$ in relation (4.3) of Theorem 4.1. We will only sketch the proof. The reader may fill in the details.

Taking limits as $n \to \infty$ in (4.3), the following a.s. limits in $L^2([0,1],\mathbf{R})$ are easy to see:

$$\left.\begin{aligned}
\lim_{n\to\infty} U_m(t,Y_n) &= U_m(t,Y),\\
\lim_{n\to\infty} T_t(Y_n) &= T_t(Y),\\
\lim_{n\to\infty}\int_0^t T_{t-s}U_m(s,Y_n)\,ds &= \int_0^t T_{t-s}U_m(s,Y)\,ds
\end{aligned}\right\} \tag{5.2}$$

for $t > 0$.

To prove the a.s. convergence

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)F_m(U_m(s,Y_n))(y)\,dy\,ds \\ =\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)F_m(U_m(s,Y))(y)\,dy\,ds\end{aligned}\tag{5.3}$$

for $t>0$, we consider the a.s. inequality

$$\begin{aligned}&\left\|\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)F_m(U_m(s,Y_n))(y)\,dy\,ds\right.\\ &\qquad\left.-\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)F_m(U_m(s,Y))(y)\,dy\,ds\right\|_{L^2}^2\\ &\quad\le C_{m,p}\|Q\|_\infty^2\|Q^{-1}\|_\infty^2\int_0^t\frac{1}{(t-s)^{3/4}}\|U_m(s,Y_n))-U_m(s,Y))\|_{L^2}^2\,ds,\ 0\le t\le a.\end{aligned}\tag{5.4}$$

Now

$$\lim_{n\to\infty}\|U_m(s,Y_n))-U_m(s,Y))\|_{L^2}^2=0,\tag{5.5}$$

and

$$\begin{aligned}\|U_m(s,Y_n))-U_m(s,Y))\|_{L^2}^2&\le 2\|U_m(s,Y_n))\|_{L^2}^2+2\|U_m(s,Y))\|_{L^2}^2\\ &\le K(1+\|Y\|_{L^2}^2),\end{aligned}\tag{5.6}$$

a.s. for all $0\le s\le t$, $n\ge 1$. The existence of the random positive constant $K$ (*independent* of $s\in[0,a]$) in the above inequality follows from the fact that

$$\sup_{\substack{f\in L^2\\ 0\le s\le a}}\frac{\|U_m(s,f)\|^2}{(1+\|f\|_{L^2}^2)}<\infty,\tag{5.7}$$

a.s. The proof of (5.7) follows along similar lines to that of Lemma 3.1. By the dominated convergence theorem, it follows from (5.4), (5.5), and (5.6) that

$$\lim_{n\to\infty}\int_0^t\frac{1}{(t-s)^{3/4}}\|U_m(s,Y_n))-U_m(s,Y))\|_{L^2}^2\,ds=0$$

a.s. This implies (5.3).

To complete the proof of (5.1) it remains to prove the following limit in probability:

$$\lim_{n\to\infty}\int_0^t T_{t-s}U_m(s,Y_n)\circ dW(s)=\int_0^t T_{t-s}U_m(s,Y)\circ dW(s)\tag{5.8}$$

for fixed $m\ge 1$.

First, we denote by $\mathbb{L}^{1,2}$ the set of all processes $v:[0,t]\times\Omega\to L^2([0,1],\mathbf{R})$ such that $v\in L^2([0,t]\times\Omega,L^2([0,1],\mathbf{R}))$, $v(s,\cdot)\in\mathbb{D}^{1,2}(\Omega,L^2([0,1],\mathbf{R}))$ for almost all

$s \in [0,t]$, and $\int_0^t \int_0^t E\|\mathcal{D}_u v(s,\cdot)\|_{L^2}^2 \, du \, ds < \infty$. Next, we will show that the processes

$$[0,t) \ni s \mapsto T_{t-s} U_m(s, Y_n) \in L^2\big([0,1], \mathbf{R}\big),$$
$$[0,t) \ni s \mapsto T_{t-s} U_m(s, Y) \in L^2\big([0,1], \mathbf{R}\big)$$

belong to $\mathbb{L}^{1,2}$ and are therefore Stratonovich integrable. To see this, define

$$\theta(s) := T_{t-s} U_m(s, Y), \quad \theta_n(s) := T_{t-s} U_m(s, Y_n), \quad s \in [0,t]. \tag{5.9}$$

We will show only that $\theta \in \mathbb{L}^{1,2}$. The argument for $\theta_n$ is very similar. Using the estimate (3.12), define the process $K_p : [0,t] \times \Omega \to \mathbf{R}^+$, $p = 1, 2$, by

$$K_p(s) := \sup_{f \in L^2} \left\{ \frac{\|U_m(s,f,\cdot)\|_{L^2}^{2p}}{(1 + \|f\|_{L^2}^{2p})} \right\} < \infty. \tag{5.10}$$

By (3.12), it is clear that

$$EK_1(s)^2 \leq EK_2(s) < \infty \tag{5.11}$$

for all $s \in [0,t]$. Therefore by (5.11) and Hölder's inequality, we get

$$\begin{aligned} \int_0^t E\|\theta(s)\|_{L^2}^2 \, ds &\leq \sup_{s \in [0,t]} EK_1(s)(1 + \|Y\|^2) \\ &\leq \sqrt{2} \sup_{s \in [0,t]} \{EK_1(s)^2\}^{1/2} (1 + E\|Y\|^4)^{1/2} \\ &\leq \sqrt{2} \sup_{s \in [0,t]} \{EK_2(s)\}^{1/2} (1 + E\|Y\|^4)^{1/2} < \infty \end{aligned} \tag{5.12}$$

because $Y \in L^4(\Omega, L^2([0,1], \mathbf{R})$. Therefore, $\theta \in L^2([0,t] \times \Omega, L^2([0,1], \mathbf{R}))$. Next, consider the Malliavin derivatives

$$\mathcal{D}_u \theta_n(s) = T_{t-s}[\mathcal{D}_u U_m(s, Y_n) + DU_m(s, Y_n) \mathcal{D}_u Y_n], \tag{5.13}$$
$$\mathcal{D}_u \theta(s) = T_{t-s}[\mathcal{D}_u U_m(s, Y) + DU_m(s, Y) \mathcal{D}_u Y] \tag{5.14}$$

for all $s \in [0,t], u \in [0,t]$. Using (3.13), (3.14), a similar argument to the one employed in (5.12), and the fact that $Y \in \mathbb{D}^{1,4}(\Omega, L^2([0,1], \mathbf{R})$, it follows that

$$\int_0^t \int_0^t E\|\mathcal{D}_u \theta(s)\|_{L^2}^2 \, ds \, du < \infty.$$

Therefore, $\theta, \theta_n \in \mathbb{L}^{1,2}$, $n \geq 1$, and the Stratonovich integrals in (5.8) are well defined. Using the continuity of $U_m(s,\cdot)$, it follows that $\theta_n(s) \to \theta(s)$ a.s. as $n \to \infty$. Thus by the dominated convergence theorem, the continuity of $\mathcal{D}_u U_m(s,\cdot)$ (from (3.28)), and the estimates (3.12), (3.13), (3.14), it is not hard to see that

$$\lim_{n \to \infty} \int_0^T \int_0^T E\|\theta_n(s) - \theta(s)\|_{L^2}^2 \, du \, ds = 0, \tag{5.15}$$

and

$$\lim_{n \to \infty} \int_0^T \int_0^T E\|\mathcal{D}_u \theta_n(s) - \mathcal{D}_u \theta(s)\|_{L^2}^2 \, du \, ds = 0. \tag{5.16}$$

From (5.15) and (5.16), it follows that

$$\lim_{n\to\infty}\int_0^t \theta_n(s)\,dW(s) = \int_0^t \theta(s)\,dW(s) \tag{5.17}$$

in probability for the Skorohod integrals. Therefore, to prove (5.8) it is sufficient to express the Stratonovich integrals in (5.8) in terms of Skorohod integrals, that is, we need to show that

$$\int_0^t \theta_n(s)\circ dW(s) = \int_0^t \theta_n(s)dW(s) + \frac{1}{2}\int_0^t (\nabla\theta_n)(s)\,ds, \quad n\geq 1, \tag{5.18}$$

and

$$\int_0^t \theta(s)\circ dW(s) = \int_0^t \theta(s)dW(s) + \frac{1}{2}\int_0^t (\nabla\theta)(s)\,ds \tag{5.19}$$

a.s., where

$$\begin{aligned}
(\nabla\theta_n)(u) &:= (\mathcal{D}_+\theta_n)(u) + (\mathcal{D}_-\theta_n)(u),\ (\nabla\theta)(u) := (\mathcal{D}_+\theta)(u) + (\mathcal{D}_-\theta)(u),\\
(\mathcal{D}_+\theta_n)(u) &:= \lim_{s\to u+} \mathcal{D}_u\theta_n(s),\ (\mathcal{D}_+\theta)(u) := \lim_{s\to u+}\mathcal{D}_u\theta(s),\\
(\mathcal{D}_-\theta_n)(u) &:= \lim_{s\to u-} \mathcal{D}_u\theta_n(s),\ (\mathcal{D}_-\theta)(u) := \lim_{s\to u-}\mathcal{D}_u\theta(s).
\end{aligned}$$

From the above relations, (5.13), and (5.14), it is easy to see that the following a.s. equalities hold:

$$(\mathcal{D}_+\theta_n)(u) = T_{t-u}[(\mathcal{D}_+U_m)_u(Y_n) + DU_m(u,Y_n)\mathcal{D}_uY_n], \tag{5.20}$$

$$(\mathcal{D}_-\theta_n)(u) = T_{t-u}[DU_m(u,Y_n)\mathcal{D}_uY_n], \tag{5.21}$$

$$(\mathcal{D}_+\theta)(u) = T_{t-u}[(\mathcal{D}_+U_m)_u(Y) + DU_m(u,Y)\mathcal{D}_uY], \tag{5.22}$$

$$(\mathcal{D}_-\theta)(u) = T_{t-u}[DU_m(u,Y)\mathcal{D}_uY], \tag{5.23}$$

$$\begin{aligned}
\lim_{n\to\infty}(\nabla\theta_n)(u) &= \lim_{n\to\infty}[(\mathcal{D}_+\theta_n)(u) + (\mathcal{D}_-\theta_n)(u)]\\
&= (\mathcal{D}_+\theta)(u) + (\mathcal{D}_-\theta)(u),\\
&= (\nabla\theta)(u).
\end{aligned} \tag{5.24}$$

Note that (5.21) and (5.23) hold because $(\mathcal{D}_-U_m)_u(Y_n) = (\mathcal{D}_-U_m)_u(Y) = 0$ due to the fact that $U_m(t,f,\cdot)$ is $\mathcal{F}_t$-measurable for each $t>0$ and any $f\in L^2([0,1],\mathbf{R})$. The proofs of the equalities (5.18) and (5.19) follow by appealing to the estimates (3.13) and (3.14). This completes the proof of the substitution result (5.1) for the truncated mild equation.

*Step* 2. We now lift the truncation from (5.1) in order to obtain the following mild version of (2.7):

$$\left.\begin{aligned}
U(t,Y) = T_t(Y) &+ \frac{1}{2}\int_0^t\int_0^1 \frac{\partial}{\partial y}p(t-s,\cdot,y)U(s,Y)^2(y)\,dy\,ds + \gamma\int_0^t T_{t-s}U(s,Y)\,ds\\
&+ \sigma\int_0^t T_{t-s}U(s,Y)\circ dW(s) - \frac{1}{2}\sigma^2\int_0^t T_{t-s}U(s,Y)\,ds\\
&+ \sigma_0\int_0^t T_{t-s}\,dW_0(s,\cdot),
\end{aligned}\right\} \tag{5.25}$$

a.s. for $t>0$.

To prove (5.25), we first show that there is a family of measurable sets $\{\Omega_{l,m} : l,m \geq 1\} \subset \mathcal{F}$ such that $\Omega_{l,m} \uparrow \Omega$ as $m \to \infty$, and

$$U(t,f,\omega) = U_m(t,f,\omega) \tag{5.26}$$

for all $f \in L^2([0,1],\mathbf{R})$ such that $\|f\|_{L^2} \leq l, 0 \leq t \leq a$, all $\omega \in \Omega_{l,m}$, and each $l \geq 1$.

Define

$$\Omega_{l,m} := \{\omega : \omega \in \Omega, \sup_{\substack{\|f\|_{L^2} \leq l \\ 0 \leq t \leq a}} \|U(t,f,\omega)\|_{L^2} \leq m\} \tag{5.27}$$

for $l,m \geq 1$. Observe that $\sup_{\substack{\|f\|_{L^2} \leq l \\ 0 \leq t \leq a}} \|U(t,f,\omega)\|_{L^2}$ is finite for a.a. $\omega \in \Omega$, by Theorem 2.1 (v). Therefore, $P(\bigcup_{m=1}^{\infty} \Omega_{l,m}) = 1$ for each $l \geq 1$. Now fix $\omega \in \Omega_{l,m}$ and let $f \in L^2([0,1],\mathbf{R})$ be such that $\|f\|_{L^2} \leq l$ and $t \in [0,a]$. Then

$$\begin{aligned}
&\|U(t,f,\omega) - U_m(t,f,\omega)\|_{L^2}^2 \\
&\leq \frac{1}{4}\int_0^1 \left(\int_0^t \int_0^1 Q(t,\omega)Q(s,\omega)^{-1}\left|\frac{\partial p(t-s,\xi,y)}{\partial y}\right| \right.\\
&\qquad\qquad \left. \times\, |U(s,f,\omega)(y)^2 - F_m(U_m(s,f,\omega))(y)|\, dy\, ds\right)^2 d\xi \\
&= \frac{1}{4}\int_0^1 \left(\int_0^t \int_0^1 Q(t)Q(s)^{-1}\left|\frac{\partial p(t-s,\xi,y)}{\partial y}\right| \right.\\
&\qquad\qquad \left. \times\, |F_m(U(s,f,\omega)(y) - F_m(U_m(s,f))(y)|\, dy\, ds\right)^2 d\xi \\
&\leq C_m^2 \|Q\|_\infty^2 \|Q^{-1}\|_\infty^2 \int_0^t \frac{1}{(t-s)^{3/4}} \|U(s,f) - U_m(s,f)\|_{L^2}^2\, ds.
\end{aligned} \tag{5.28}$$

Iterating the above inequality twice (cf. Lemma 3.1) implies that

$$\|U(t,f,\omega) - U_m(t,f,\omega)\|_{L^2}^2 = 0$$

for $t \in [0,a]$, $\|f\|_{L^2} \leq l$, $\omega \in \Omega_{l,m}$. Hence (5.26) holds.

Next define $\Omega_l := \{\omega : \omega \in \Omega, \|Y(\omega)\|_{L^2} \leq l\}$ for each $l \geq 1$. Then $\bigcup_{l=1}^{\infty} \Omega_l = \Omega$. Also by (5.26), we have

$$U(t,Y(\omega),\omega) = U_m(t,Y(\omega),\omega) \tag{5.29}$$

for all $\omega \in \Omega_l \cap \Omega_{l,m}$ and $t \in [0,a]$. So by the local property of the Stratonovich integral, it follows that the process $[0,t) \ni s \mapsto T_{t-s}U(s,Y) \in L^2([0,1],\mathbf{R})$ is Stratonovich integrable and

$$\int_0^t T_{t-s}U_m(s,Y) \circ dW(s) = \int_0^t T_{t-s}U(s,Y) \circ dW(s) \tag{5.30}$$

on $\Omega_l \cap \Omega_{l,m}$ for fixed $l,m \geq 1$. Furthermore, we have the equality

$$F_m(U(s,Y)) = U(s,Y)^2 \tag{5.31}$$

on $\Omega_l \cap \Omega_{l,m}$ for all $l,m \geq 1$, $s \in [0,t]$. The equality (5.31) holds because for each $\omega \in \Omega_l \cap \Omega_{l,m}$, $\|Y(\omega)\|_{L^2} \leq l$, so $\|U(s,Y(\omega),\omega)\|_{L^2} \leq m$ for all $s \in [0,t]$.

Using (5.29), (5.30), (5.31), and (5.1), the following mild integral equation holds on $\Omega_l \cap \Omega_{l,m}$ for all $l, m \geq 1$ and $t \in [0, a]$:

$$\left.\begin{aligned} U(t,Y) = U_m(t,Y) = T_t(Y) + \frac{1}{2}\int_0^t \int_0^1 \frac{\partial}{\partial y} p(t-s,\cdot,y) U(s,Y)^2(y)\, dy\, ds \\ + \gamma \int_0^t T_{t-s} U(s,Y)\, ds \\ + \sigma \int_0^t T_{t-s} U(s,Y) \circ dW(s) - \frac{1}{2}\sigma^2 \int_0^t T_{t-s} U(s,Y)\, ds \\ + \sigma_0 \int_0^t T_{t-s}\, dW_0(s,\cdot). \end{aligned}\right\} \tag{5.32}$$

Since $\bigcup_{l,m=1}^{\infty}(\Omega_l \cap \Omega_{l,m}) = \Omega$, (5.32) implies that (5.25) holds a.s. for all $t > 0$. This completes the proof of assertion (2.7) of Theorem 2.2.

The second and third assertions of Theorem 2.2 follow from Theorem 2.1(v) and assertion (2.7). The last assertion (2.8) of Theorem 2.2 follows along similar lines to the proof of (2.7). This completes the proof of Theorem 2.2. ☐

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