Anticipating Stochastic $2D$ Navier-Stokes Equations

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Abstract

In this article, we consider the two-dimensional stochastic Navier-Stokes equation (SNSE) on a smooth bounded domain, driven by affine-linear multiplicative white noise and with random initial conditions and Dirichlet boundary conditions. The random initial condition is allowed to anticipate the forcing noise. Our main objective is to prove the existence and uniqueness of the solution to the SNSE under sufficient Malliavin regularity of the initial condition. To this end we employ anticipating calculus ideas.

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1 Introduction. The main result.

Two-dimensional stochastic Navier-Stokes equations (SNSE's) are often used to model the time evolution of the velocity field for an incompressible fluid in a smooth bounded planar domain. Existing models of fluid dynamics employ SNSE's with deterministic initial and boundary conditions.

Our main objective in this article is to obtain a new result on the existence and uniqueness of a variational solution to the SNSE with random initial velocities that may possibly anticipate the driving noise.

The impetus for considering randomness in the initial condition for the SNSE is based on the following considerations:

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- Random measurement errors exist both in the initial velocity and in the parameters for physical models of hydrodynamic fluid movement. Indeed, earlier mathematical models of fluid dynamics (such as stochastic Burgers equations) were motivated by allowing for randomness in the initial velocity distribution.

- Our main result in this article implies that each stationary point (or invariant measure) of the underlying stochastic dynamical system generates a *pathwise* anticipating stationary *solution* of the stratonovich SNSE. In the generic case when the dynamics is hyperbolic, the stationary solution is necessarily anticipating because the unstable manifold is anticipating. Thus uniqueness/ergodicity regimes for the stationary solution are not necessarily generic from a dynamical systems point of view.

- A theoretical motivation for our result is provided by the fact that, near stationary solutions, multiplicative ergodic theory techniques lead to the existence of local random invariant manifolds that necessarily anticipate the driving noise of the SNSE ([MZ]). Thus a dynamic characterization of the semiflow along the invariant manifolds will require analysis of the SNSE with anticipating initial conditions. In particular, the restriction of the cocycle to the anticipating random and finite-dimensional invariant manifolds amounts to a solution of an anticipating SNSE living on the invariant manifold. This is a necessary initial step in the analysis of the regularity of the invariant manifold. However this is a long-term future project.

To establish the existence/uniqueness theorem for the solution of the SNSE, we adopt a dynamical systems approach rather than the more conventional strategy that is often used: Establish the dynamics as a consequence of existence and uniqueness of solutions rather than the other way round. More specifically, we adopt the following strategy:

If one parametrizes the non-anticipating stochastic Navier Stokes equation by its initial condition as an infinite-dimensional parameter, is it possible to "substitute" an arbitrary (anticipating) random variable for the parameter? It is well-known that such a substitution mechanism is possible for finite-dimensional stochastic ordinary differential equations due to Kolomogorov's continuity theorem, which only applies to finite-dimensional random fields. Indeed, within our infinite-dimensional setting, both Kolmogorov and Sobolev embedding theorems fail. In this article, we are able to resolve this issue by imposing Malliavin regularity hypotheses on the initial velocity and appealing to ideas and methods of the Malliavin calculus coupled with apriori estimates on the semiflow of the SNSE and Galerkin approximations. In particular, we develop in Section 4 an infinite-dimensional anticipating version of the product rule in order to address the above substitution issue.

We now proceed to formulate our result.

Consider the following two-dimensional stochastic Navier-Stokes equation (SNSE) with

Dirichlet boundary conditions and a random initial condition:

$$\left.\begin{aligned} du - \nu\triangle u\ dt + (u\cdot\nabla)u\ dt + \nabla p\ dt &= \sum_{k=1}^{\infty}\sigma_k u(t)\ \circ dW_k(t) + \sigma_0\, d\tilde{W}_0(t,x), \\ (div\ u)(t,x) &= 0, \quad x\in D,\, t>0, \\ u(t,x) &= 0, \quad x\in\partial D,\, t>0, \\ u(0,x) &= Y(x), \quad x\in D. \end{aligned}\right\} \tag{1.1}$$

In the above SNSE, D is a bounded domain in $\mathbf{R}^2$ with smooth boundary ∂D; $u(t,x)\in\mathbf{R}^2$ denotes the velocity field at time t and position $x\in D$; $p(t,x)$ denotes the pressure field, and $\nu>0$ the viscosity coefficient. Moreover, the random forcing field is provided by a family of independent one-dimensional standard Brownian motions $W_k, k\geq 1$, together with space-time noise $\tilde{W}_0(t,x)$, independent of $W_k, k\geq 1$, that is Brownian in the time variable $t\in\mathbf{R}^+$ and smooth in the space variable $x\in D$. The random forces are defined on a complete filtered Wiener space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$. We assume that the noise parameters σ_k, $k\geq 0$, are such that $\sigma_0\in\mathbf{R}$ and $\displaystyle\sum_{k=1}^{\infty}\sigma_k^2<\infty$. The initial condition Y is an $\mathcal{F}\otimes\mathcal{B}(D)$-measurable random field on D where $\mathcal{B}(D)$ is the Borel σ-algebra of D.

Under deterministic initial conditions, there is a large amount of literature on the stochastic Navier-Stokes equation and its abstract setting. We will only refer to part of this literature. A good reference for stochastic Navier-Stokes equations driven by additive noise is the book [D-Z.1] and the references therein. The existence and uniqueness of solutions of stochastic $2D$ Navier-Stokes equations with multiplicative noise is established in [Fl.1] and [S-S]. Ergodic properties and invariant measures of stochastic $2D$ Navier-Stokes equations are studied in [Fl.1] and [H-M]. Large deviations under small noise and occupation measures of stochastic $2D$ Navier-Stokes equations are studied in [S-S] and [Gourcy]. The existence of a $C^{1,1}$ cocycle and a multiplicative ergodic theory for the SNSE (1.1) (in its abstract form) is established in [M-Z].

In order to state our main result in this article, we consider the Hilbert space

$$V := \{v\in H_0^1(D,\mathbf{R}^2) : \nabla\cdot v = 0\ a.e.\ \text{in}\ D\},$$

with the norm

$$||v||_V := \Big(\int_D |\nabla v|^2\,dx\Big)^{\frac{1}{2}}$$

and the associated inner product

$$\ll v_1, v_2 \gg_V := \int_D \nabla v_1\cdot\nabla v_2\,dx, \quad v_1,v_2\in V.$$

Denote by H the closure of V in the L^2-norm

$$|v|_H := \Big(\int_D |v|^2 \, dx \Big)^{\frac{1}{2}}.$$

The inner product on H will be denoted by $< \cdot, \cdot >$.

Denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ the standard Wiener shift on the Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$:

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}.$$

Throughout the article, we will denote by $\mathcal{D}$ Malliavin differentiation with respect to the Brownian noise $\{W_k : k \geq 1\}$. In particular, $\mathcal{D}\tilde{W}_0 = 0$.

We will denote by $\mathcal{D}^{1,4}(H)$ the Malliavin Sobolev space of all $\mathcal{F}$-measurable and Malliavin differentiable random variables $\Omega \to H$ with Malliavin derivatives having fourth-order moments. The symbol $\mathcal{D}^{1,4}_{loc}(H)$ will denote all random variable $Y : \Omega \to H$ that are locally in $\mathcal{D}^{1,4}(H)$.

Our main result is the following existence and uniqueness theorem for solutions of the SNSE (1.1):

Theorem 1.1. *In the SNSE (1.1), assume that the initial random field Y belongs to the Malliavin Sobolev space $\mathcal{D}^{1,4}(H)$. Then the SNSE (1.1) has a unique weak global solution $u(\cdot, Y)$ with initial condition Y and having the following properties: $u(\cdot, Y) \in C([0, T], H)$ and $u(t, Y) \in \mathcal{D}^{1,4}_{loc}(H)$ for all $t \in [0, T]$ and any $T \in (0, \infty)$. In the special case when $Y \in \mathcal{D}^{1,4}(H)$ is a stationary point of the stochastic dynamical system generated by the SNSE (1.1) ([M-Z]), the process $Y(\theta(t, \cdot))$, $t \in \mathbf{R}^+$, is a pathwise weak solution of the SNSE (1.1).*

2 Abstract formulation.

To establish an abstract framework for the dynamics of the stochastic Navier-Stokes equation (1.1), we denote by P_H the Helmholtz-Hodge projection of the Hilbert space $L^2(D, \mathbf{R}^2)$ onto the energy space H introduced in Section 1. Consider the (Stokes) operator A in H defined by the formula

$$Au := -\nu P_H \triangle u, \quad u \in H^2(D, \mathbf{R}^2) \cap V,$$

and the bilinear operator B given by

$$B(u, v) := P_H\big((u \cdot \nabla) v\big),$$

whenever u, v are such that $(u \cdot \nabla v)$ belongs to the space $L^2(D, \mathbf{R}^2)$. We will often adopt the short notation $B(u) := B(u, u)$. We also define the projection of the space-time noise $\tilde{W}_0(t, \cdot)$ by

$$W_0(t, \cdot) := P_H(\tilde{W}_0(t, \cdot)) \in H$$

for $t \in \mathbf{R}^+$.

By applying the operator P_H to each term of the SNSE (1.1), we can rewrite the latter equation in the following abstract form:

$$du(t) + Au(t)\,dt + B(u(t))\,dt = \sum_{k=1}^{\infty} \sigma_k u(t) \circ dW_k(t) + \sigma_0\,dW_0(t), \quad t > 0, \tag{2.1}$$

in V' with the initial condition

$$u(0) = u_0 \in H. \tag{2.2}$$

Here V' stands for the dual of V.

Our approach is to identify the Hilbert space H with its dual H' and consider the stochastic Navier-Stokes equation (2.1) in the framework of the Gelfand triple:

$$V \subset H \cong H' \subset V'.$$

Thus, we may consider the Stokes operator A as a bounded linear map from V into V'. Moreover, we also denote by $< \cdot, \cdot >: V \times V' \to \mathbf{R}$, the canonical bilinear pairing between V and V'. Hence, using integration by parts, we have

$$< Au, w >= \nu \sum_{i,j=1}^{2} \int_D \partial_i u_j \partial_i w_j dx = \nu \ll u, w \gg \tag{2.3}$$

for $u := (u_1, u_2) \in V$, $w = (w_1, w_2) \in V$.

Define the real-valued trilinear form b on $H \times H \times H$ by setting

$$b(u, v, w) := \sum_{i,j}^{2} \int_D u_i \partial_i v_j w_j dx, \tag{2.4}$$

for $v := (v_1, v_2) \in H$ and whenever the integral in (2.4) exists. In particular, if $u, v, w \in V$, then

$$b(u, v, w) =< B(u, v), w >=< (u \cdot \nabla)v, w >= \sum_{i,j}^{2} \int_D u_i \partial_i v_j w_j\,dx.$$

Using integration by parts, it is easy to see that

$$b(u, v, w) = -b(u, w, v), \tag{2.5}$$

for all $u, v, w \in V$. Thus,

$$b(u, v, v) = 0 \tag{2.6}$$

for all $u, v \in V$.

Throughout the paper, we will denote various generic positive constants by the same letter c, although the constants may differ from line to line. We now list some well-known estimates for b which will be used frequently in the sequel (see [Te], [Ro] for example):

$$|b(u,v,w)| \leq c\|u\|_V \cdot \|v\|_V \cdot \|w\|_V, \quad u,v,w \in V, \tag{2.7}$$
$$|b(u,v,w)| \leq c|u|_H \cdot \|v\|_V \cdot |Aw|_H, \quad u \in H, v \in V, w \in D(A), \tag{2.8}$$
$$|b(u,v,w)| \leq c\|u\|_V \cdot |v|_H \cdot |Aw|_H, \quad u \in V, v \in H, w \in D(A), \tag{2.9}$$
$$|b(u,v,w)| \leq 2\|u\|_V^{\frac{1}{2}} \cdot |u|_H^{\frac{1}{2}} \cdot \|w\|_V^{\frac{1}{2}} \cdot |w|_H^{\frac{1}{2}} \cdot \|v\|_V, \quad u,v,w \in V. \tag{2.10}$$

Moreover, combining (2.3) and (2.8), we obtain

$$|B(u,w)|_{V'} = \sup_{\|v\|_V \leq 1} |b(u,w,v)| = \sup_{\|v\|_V \leq 1} |b(u,v,w)| \leq 2\|u\|_V^{\frac{1}{2}} \cdot |u|_H^{\frac{1}{2}} \cdot \|w\|_V^{\frac{1}{2}} \cdot |w|_H^{\frac{1}{2}} \tag{2.11}$$

for all $u, w \in V$.

3 Malliavin differentiability of the SNSE

In this section, we will show that the solutions of the SNSE (with non-random initial conditions) are Malliavin differentiable. Our approach is to use a variational technique which transforms the SNSE (2.1) into a *random* Navier-Stokes equation that we can then analyze using a combination of Galerkin approximations and apriori estimates (cf. [Te], [Ro]).

Consider the SNSE

$$\begin{cases} du(t,f) + Au(t,f)dt + B(u(t,f))dt = \sum_{k=1}^{\infty} \sigma_k u(t,f) \circ dW_k(t) + \sigma_0\, dW_0(t),\ t > 0, \\ u(0,f) = f \in H, \end{cases} \tag{3.1}$$

with a deterministic initial condition $f \in H$. It is known that for each $f \in H$, the SNSE (3.1) admits a unique (probabilistically) strong solution $u(\cdot, f) \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T]; V)$ ([B-C-F]). Writing (3.1) in integral form, we have

$$u(t,f) = f - \int_0^t Au(s,f)\,ds - \int_0^t B(u(s,f))\,ds + \sum_{k=1}^{\infty} \int_0^t \sigma_k u(s,f) \circ dW_k(s) + \sigma_0\, W_0(t), \tag{3.2}$$

for all $t \in [0,T]$.

Let $Q : [0,\infty) \times \Omega \to \mathbf{R}$ be the solution of the one-dimensional linear sode

$$\left.\begin{aligned} dQ(t) &= \sum_{k=1}^{\infty} \sigma_k Q(t) \circ dW_k(t), \quad t \geq 0, \\ Q(0) &= 1. \end{aligned}\right\} \tag{3.3}$$

By Itô's formula, we have

$$Q(t) = \exp\left\{\sum_{k=1}^{\infty} \sigma_k W_k(t)\right\}, \quad t \geq 0.$$

This implies that

$$E\|Q\|_\infty < \infty$$

where

$$\|Q\|_\infty \equiv \|Q(\cdot,\omega)\|_\infty := \sup_{0\leq t\leq T} Q(t,\omega), \quad \omega \in \Omega,$$

for any finite positive T.

Let $Z : \mathbf{R}^+ \times D \times \Omega \to \mathbf{R}^2$ be the unique solution of the stochastic linear heat equation:

$$\left.\begin{aligned} dZ(t) &= -AZ(t)\,dt + \sigma_0 Q(t)^{-1}\,dW_0(t), \quad t \geq 0,\\ Z(0) &= 0,\\ Z(t,x) &= 0,\ x \in \partial D,\ t \geq 0. \end{aligned}\right\} \tag{3.4}$$

Define

$$v(t,f) := u(t,f)Q^{-1}(t) - Z(t), \quad t \geq 0. \tag{3.5}$$

Applying Itô's formula to the relation $u(t,f) = Q(t)[v(t,f) + Z(t)]$, $t \geq 0$, and using (3.3) and (3.4), it is easy to see that $v(t) \equiv v(t,f)$ satisfies the random NSE

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt - Q(t)B\big(v(t) + Z(t)\big)\,dt, \quad t \geq 0,\\ v(0) &= f \in H\,. \end{aligned}\right\} \tag{3.6}$$

By adapting the corresponding estimates from [M-Z], we obtain the following two propositions:

Propostion 3.1. *For any $f \in H$ and $\omega \in \Omega$, let $v(\cdot, f, \omega) \in C\big([0,T], H\big) \cap L^2\big([0,T], V\big)$ be a solution of (3.6) on $[0,T]$ for some $T > 0$. Then for each $\omega \in \Omega$ and any $f \in H$, the following apriori estimates hold*

$$\sup_{0\leq t\leq T} |v(t,f,\omega)|_H^2 \leq \left[|f|_H^2 + \int_0^T Q(s)|Z(s)\cdot\nabla Z(s)|_H^2\,ds\right] \exp\left\{\int_0^T Q(s)[1 + \|\nabla Z(s)\|_\infty]\,ds\right\} \tag{3.7}$$

and

$$\int_0^T \|v(t,f,\omega)\|_V^2 dt \leq \frac{1}{2\nu}\left[|f|_H^2 + \int_0^T Q(s)|Z(s)\cdot\nabla Z(s)|_H^2\,ds\right] \exp\left\{\int_0^T Q(s)[1+\|\nabla Z(s)\|_\infty]\,ds\right\}, \tag{3.8}$$

where $\|\cdot\|_\infty$ denotes the sup norm with respect to the space parameter $x \in D$. Moreover, for each $\omega \in \Omega$, the map $H \ni f \mapsto v(\cdot, f, \omega) \in C\big([0,T], H\big) \cap L^2\big([0,T], V\big)$ is Lipschitz on bounded sets in H.

Propostion 3.2. *The solution map*

$$H \ni f \to v(t, f, \omega) \in H$$

of the random NSE (3.6) is $C^{1,1}$ *for each* $\omega \in \Omega$ *and* $t \geq 0$*, and has Lipschitz Fréchet derivatives on bounded sets in* H*. Furthermore, the Fréchet derivative* $[0, \infty) \ni t \to Dv(t, f, \omega) \in L(H)$ *is continuous in* t *and the following estimate holds a.s.:*

$$\sup_{0 \leq t \leq T} \|Dv(t, f)\|_{L(H)} \leq C_1 \exp\left(C_2 |f|_H^2\right), \tag{3.9}$$

where $L(H)$ *denotes the space of bounded linear operators from* H *into* H *and* C_1, C_2 *are positive random constants such that* $EC_2^p < \infty$ *for all* $p \geq 1$ *and* $E \log^+ C_1 < \infty$*.*

For simplicity, introduce the following random constants:

$$D_1(\omega) := \int_0^T Q(s)|Z(s) \cdot \nabla Z(s)|_H^2 \, ds, \quad D_2(\omega) := \exp\left\{ \int_0^T Q(s)[1 + \|\nabla Z(s)\|_\infty] \, ds \right\} \tag{3.10}$$

$$D_3(\omega) := \sup_{0 \leq s \leq T} |Z(s)|_H, \quad D_4(\omega) := \sup_{0 \leq s \leq T} \|Z(s)\|_V, \quad D_5(\omega) := \sup_{0 \leq s \leq T} |AZ(s)|_H \tag{3.11}$$

$$D_6(\omega) := \sup_{0 \leq s \leq T} |\mathcal{D}_u Z(s)|_H, \quad D_7(\omega) := \sup_{0 \leq s \leq T} \|\mathcal{D}_u Z(s)\|_V \tag{3.12}$$

The following result is an apriori energy estimate in the space V for the random NSE (3.1).

Propostion 3.3. *Let* $v(t, f)$ *be the solution to equation (3.6). Then for any* $f \in V$*, we have*

$$\begin{aligned} &\sup_{0 \leq t \leq T} ||v(t, f)||_V^2 + \int_0^T |Av(s, f)|_H^2 ds \\ \leq \; &\{||f||_V^2 + c[((|f|_H^2 + D_1)D_2 + D_3^2)D_4^4 \int_0^T Q^4(s)ds + (((|f|_H^2 + D_1)D_2)^{\frac{1}{2}} + D_3)D_4^2 D_5 \int_0^T Q^2(s)ds\} \\ &\times exp\{c((|f|_H^2 + D_1)D_2 + D_3^2) \sup_{0 \leq s \leq T} Q^4(s) + c((|f|_H^2 + D_1)D_2)^{\frac{1}{2}} + D_3)D_5 \int_0^T Q^2(s)ds\}, \end{aligned} \tag{3.13}$$

a.s.

Proof. By the chain rule, it follows that

$$\begin{aligned} ||v(t, f)||_V^2 &= ||f||_V^2 - 2 \int_0^t |Av(s, f)|_H^2 \, ds \\ &\quad -2 \int_0^t Q(s) < B(v(s, f) + Z(s)), Av(s, f) > \, ds. \end{aligned} \tag{3.14}$$

Recall that for $v \in H^2 \cap V$, one has

$$|B(v)|_H \le |v|_H^{\frac{1}{2}} ||v||_V |Av|_H^{\frac{1}{2}}; \tag{3.15}$$

(See Lemma 3.8 in [Te1].) Thus

$$\begin{aligned}
& 2|Q(s) < B(v(s,f) + Z(s)), Av(s,f) > | \\
\le\ & 2Q(s)|v(s,f) + Z(s)|_H^{\frac{1}{2}} ||v(s,f) + Z(s)||_V |Av(s,f) + AZ(s)|_H^{\frac{1}{2}} |Av(s,f)|_H \\
\le\ & 2Q(s)|v(s,f) + Z(s)|_H^{\frac{1}{2}} ||v(s,f) + Z(s)||_V |Av(s,f)|_H^{\frac{3}{2}} \\
& +2Q(s)|v(s,f) + Z(s)|_H^{\frac{1}{2}} ||v(s,f) + Z(s)||_V |AZ(s)|_H^{\frac{1}{2}} |Av(s,f)|_H \\
\le\ & \frac{1}{2}|Av(s,f)|_H^2 + cQ^4(s)|v(s,f) + Z(s)|_H^2 (||v(s,f)||_V^4 + \|Z(s)\|^4) \\
& +\frac{1}{2}|Av(s,f)|_H^2 + cQ^2(s)|v(s,f) + Z(s)|_H (||v(s,f)||_V^2 + \|Z(s)\|^2)|AZ(s)|_H.
\end{aligned} \tag{3.16}$$

By relations (3.14) and (3.16), we obtain

$$\begin{aligned}
||v(t,f)||_V^2 \ \le\ & ||f||_V^2 - \int_0^t |Av(s,f)|_H^2 ds \\
& +c\int_0^t Q^4(s)|v(s,f) + Z(s)|_H^2 ||v(s,f)||_V^2 ||v(s,f)||_V^2 ds \\
& +c\int_0^t Q^2(s)|v(s,f) + Z(s)|_H |AZ(s)|_H ||v(s,f)||_V^2 ds \\
& +c\int_0^t Q^4(s)|v(s,f) + Z(s)|_H^2 ||Z(s)||_V^4 ds \\
& +c\int_0^t Q^2(s)|v(s,f) + Z(s)|_H |AZ(s)|_H ||Z(s)||_V^2 ds
\end{aligned} \tag{3.17}$$

Applying Gronwall's lemma to (3.17), we get

$$\begin{aligned}
& \sup_{0\le t\le T} ||v(t,f)||_V^2 + \int_0^T |Av(s,f)|_H^2\, ds \\
\le\ & \{||f||_V^2 + c\int_0^T Q^4(s)|v(s,f) + Z(s)|_H^2 ||Z(s)||_V^4 ds \\
& \quad +c\int_0^T Q^2(s)|v(s,f) + Z(s)|_H |AZ(s)|_H ||Z(s)||_V^2 ds\} \\
& \times \exp\left\{ \int_0^T Q^4(s)|v(s,f) + Z(s)|_H^2 ||v(s,f)||_V^2 ds \right. \\
& \quad \left. +c\int_0^T Q^2(s)|v(s,f) + Z(s)|_H |AZ(s)|_H |ds \right\}
\end{aligned} \tag{3.18}$$

In view of (3.7) and (3.8), it follows from (3.18) that

$$
\begin{aligned}
&\sup_{0\le t\le T} ||v(t,f)||_V^2 + \int_0^T |Av(s,f)|_H^2\, ds \\
\le\ & \{||f||_V^2 + c[((|f|_H^2 + D_1)D_2 + D_3^2)D_4^4 \int_0^T Q^4(s)ds + (((|f|_H^2 + D_1)D_2)^{\frac{1}{2}} + D_3)D_4^2 D_5 \int_0^T Q^2(s)ds\} \\
& \times \exp\{c((|f|_H^2 + D_1)D_2 + D_3^2) \sup_{0\le s\le T} Q^4(s) + c(((|f|_H^2 + D_1)D_2)^{\frac{1}{2}} + D_3)D_5 \int_0^T Q^2(s)ds\} \qquad (3.19)
\end{aligned}
$$

□

Remark 3.1. *Set* $\sigma := \sqrt{\sum_{k=1}^{\infty} \sigma_k^2}$. *Define*

$$W(t) := \frac{1}{\sigma}\sum_{k=1}^{\infty} \sigma_k W_k(t), \quad t \ge 0.$$

Then $W(t), t \ge 0$, *is a new one-dimensional standard Brownian motion and*

$$\sum_{k=1}^{\infty} \sigma_k u(t,f) \circ dW_k(t) = \sigma u(t,f) \circ dW(t).$$

Thus, from now on and without loss of generality, we will assume that the SNSE (2.1) is driven by a Brownian motion W *(with* $\sigma = 1$*) together with the space-time noise* W_0*. In particular,* $Q(t) = \exp(W(t))$, $t \ge 0$.

We next develop Malliavin regularity for solutions of the random NSE (3.6).

Propostion 3.4. *For each* $f \in V$ *and* $t \ge 0$, *the solution map*

$$\Omega \ni \omega \to v(t,f,\omega) \in H$$

of (3.6) is Malliavin differentiable. Its Malliavin derivative $\mathcal{D}_u v(t,f)$, *solves the following random evolution equation:*

$$
\begin{aligned}
\mathcal{D}_u v(t,f) \ =\ & -\int_0^t A\mathcal{D}_u v(s,f)\, ds - \int_0^t Q(s)(\mathcal{D}_u v(s,f) + \mathcal{D}_u Z(s)) \cdot \nabla)[v(s,f) + Z(s)]\, ds \\
& - \int_0^t Q(s)(v(s,f) + Z(s)) \cdot \nabla)(\mathcal{D}_u v(s,f) + \mathcal{D}_u Z(s))ds \\
& - \int_0^t \mathcal{D}_u Q(s)(v(s,f) + Z(s)) \cdot \nabla)(v(s,f) + Z(s))\, ds
\end{aligned}
\qquad (3.20)
$$

for all $t \in [0,T]$, *a.s.*.

Proof. We will show that $v(t,f) \in \mathcal{D}^{1,2}_{loc}(H)$. By the uniqueness of the solution of the random NSE (3.6), we have $v(t,f) = v^N(t,f)$ on $\Omega_N = \{\sup_{0\le s\le T} |W(s)| \le N\}$, where $v^N(t,f)$ is the solution of an equation similar to (3.6) replacing $Q(s)$ there by $Q_N(s) := \exp(W(s)\wedge N)$. Thus it is sufficient to prove that $v^N(t,f) \in \mathcal{D}^{1,2}(H)$ for every fixed N. For this reason, we assume implicitly in the rest of this proof that $Q = Q_N$. To continue, we appeal to Galerkin approximations: Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of H that consists of eigenvectors of the operator $-A$ under Dirichlet boundary conditions with corresponding eigenvalues $\{\mu_i\}_{i=1}^{\infty}$; that is $A(e_i) = -\mu_i e_i$, $e_i|_{\partial D} = 0$, $i \ge 1$. Let H_n denote the n-dimensional subspace of H spanned by $\{e_1, e_2, ..., e_n\}$. Define $f_n \in H_n$ by

$$f_n := \sum_{j=1}^{n} \langle f, e_j\rangle e_j\,.$$

Clearly, the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f in H. Now for every integer $n \ge 1$, let v_n be unique solution of the random NSE

$$\left.\begin{aligned} dv_n(t,f_n) &= -Av_n(t,f_n)\,dt - Q(t)B\big(v_n(t,f_n)+Z(t)\big)\,dt, \quad t>0,\\ v_n(0,f_n) &= f_n,\\ v_n(t,f_n)\,|_{\partial D} &= 0, \quad t>0, \end{aligned}\right\} \tag{3.21}$$

such that

$$v_n(t,f_n) := \sum_{j=1}^{n} g_{jn}(t)e_j, \quad t \ge 0,$$

for appropriate choice of the real-valued random processes g_{jn}. It was shown in [M-Z] that v_n converges to v and

$$\lim_{n\to\infty} E[\int_0^T |v_n(s,f_n) - v(s,f)|_H^2 ds] = 0. \tag{3.22}$$

As $v_n(t,f_n)$ is a solution of the finite dimensional random ordinary differential equation (3.21), it is known (see e.g. [N]) that $v_n(t,f_n)$ is Malliavin differentiable and the corresponding Malliavin derivative $\mathcal{D}_u v_n(t,f_n)$ satisfies the following random ODE:

$$\begin{aligned} \mathcal{D}_u v_n(t,f_n) \;=\; &-\int_0^t A\mathcal{D}_u v_n(s,f_n)ds - \int_0^t Q(s)(\mathcal{D}_u v_n(s,f_n) + \mathcal{D}_u Z(s))\cdot\nabla)(v_n(s,f_n)+Z(s))\,ds\\ &-\int_0^t Q(s)(v_n(s,f_n)+Z(s))\cdot\nabla)(\mathcal{D}_u v_n(s,f_n)+\mathcal{D}_u Z(s))\,ds\\ &-\int_0^t \mathcal{D}_u Q(s)(v_n(s,f_n)+Z(s))\cdot\nabla)(v_n(s,f_n)+Z(s))\,ds \end{aligned} \tag{3.23}$$

for all $t \in [0, T]$. Let $Y_u(t, f)$ be the solution of the following random evolution equation:

$$\begin{aligned}
Y_u(t,f) &= -\int_0^t AY_u(s,f)ds - \int_0^t Q(s)(Y_u(s,f) + \mathcal{D}_u Z(s)) \cdot \nabla)(v(s,f) + Z(s))\, ds \\
&\quad - \int_0^t Q(s)(v(s,f) + Z(s)) \cdot \nabla)(Y_u(s,f) + \mathcal{D}_u Z(s))\, ds \\
&\quad - \int_0^t \mathcal{D}_u Q(s)(v(s,f) + Z(s)) \cdot \nabla)(v(s,f) + Z(s))\, ds \qquad (3.24)
\end{aligned}$$

for all $t \in [0, T]$. The existence of the solution of the above equation can be obtained by a similar method to the one used for (3.6) (see [M-Z]). Since the Malliavin derivative operator $\mathcal{D}$ is closed, to prove the theorem it suffices to show that

$$\lim_{n\to\infty} \sup_{0\le u\le t} E[|\mathcal{D}_u v_n(t, f_n) - Y_u(t,f)|_H^2] = 0. \qquad (3.25)$$

Now,

$$\begin{aligned}
& |\mathcal{D}_u v_n(t,f_n) - Y_u(t,f)|_H^2 \\
= & -2\nu \int_0^t ||\mathcal{D}_u v_n(s,f_n) - Y_u(s,f)||_V^2 ds \\
& -2\int_0^t \mathcal{D}_u Q(s) b(v_n(s,f_n) + Z(s), v_n(s,f_n) - v(s,f), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f))ds \\
& -2\int_0^t \mathcal{D}_u Q(s) b(v_n(s,f_n) - v(s,f), v(s,f) + Z(s), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f))ds \\
& -2\int_0^t Q(s) b(\mathcal{D}_u v_n(s,f_n) + \mathcal{D}_u Z(s), v_n(s,f_n) - v(s,f), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f))ds \\
& -2\int_0^t Q(s) b(\mathcal{D}_u v_n(s,f_n) - Y_u(s,f), v(s,f) + Z(s), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f))ds \\
& -2\int_0^t Q(s) b(v_n(s,f_n) - v(s,f), \mathcal{D}_u v_n(s,f_n) + \mathcal{D}_u Z(s), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f))ds \\
& -2\int_0^t Q(s) b(v(s,f) + Z(s), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f), \mathcal{D}_u v_n(s,f_n) - Y_u(s,f))ds \\
:= & \; I_1^n + I_2^n + I_3^n + I_4^n + I_5^n + I_6^n + I_7^n \quad t \in [0,T]. \qquad (3.26)
\end{aligned}$$

Set

$$C_1^n(\omega) := \sup_{0\le s\le T} (|v_n(s,f_n)|_H + |Z(s)|_H), \quad C_2^n(\omega) := \sup_{0\le s\le T} (||v_n(s,f_n)||_V + \|Z(s)\|_V)$$

$$C_1(\omega) := \sup_{0\le s\le T} (|v(s,f)|_H + |Z(s)|_H), \quad C_2(\omega) := \sup_{0\le s\le T} (||v(s,f)||_V + \|Z(s)\|_V)$$

$$M_1^n(u,\omega) := \sup_{0\le s\le T}(|\mathcal{D}_u v_n(s,f_n)|_H + |\mathcal{D}_u Z(s)|_H),\, M_2^n(u,\omega) := \sup_{0\le s\le T}(||\mathcal{D}_u v_n(s,f_n)||_V + \|\mathcal{D}_u Z(s)\|_V$$

$$M_1(u,\omega) := \sup_{0\le s\le T}|Y_u(s,f)|_H,\, M_2(u,\omega) := \sup_{0\le s\le T}||Y_u(s,f)||_V$$

Now we estimate the terms on the right of (3.26). We start with

$$\begin{aligned} I_2^n &\le c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|\int_0^t |v_n(s,f_n)+Z(s)|_H^{\frac{1}{2}}||v_n(s,f_n)+Z(s)||_V^{\frac{1}{2}}|v_n(s,f_n)-v(s,f)|_H^{\frac{1}{2}} \\ &\qquad \times||v_n(s,f_n)-v(s,f)||_V^{\frac{1}{2}}||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V ds \\ &\le c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|(C_1^n(\omega)+C_1(\omega))(C_2^n(\omega)+C_2(\omega))(\int_0^T |v_n(s,f_n)-v(s,f)|_H^2 ds)^{\frac{1}{4}} \\ &\qquad \times(\int_0^T ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V^{\frac{4}{3}} ds)^{\frac{3}{4}} \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} I_3^n &\le c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|(C_1^n(\omega)+C_1(\omega))(C_2^n(\omega)+C_2(\omega))(\int_0^T |v_n(s,f_n)-v(s,f)|_H^2 ds)^{\frac{1}{4}} \\ &\qquad \times(\int_0^T ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V^{\frac{4}{3}} ds)^{\frac{3}{4}} \end{aligned} \tag{3.28}$$

For I_4^n, we have

$$\begin{aligned} I_4^n &\le c\sup_{0\le s\le T}|Q(s)|\int_0^t |\mathcal{D}_u v_n(s,f_n)+\mathcal{D}_u Z(s)|_H^{\frac{1}{2}}||\mathcal{D}_u v_n(s,f_n)+\mathcal{D}_u Z(s)||_V^{\frac{1}{2}}|v_n(s,f_n)-v(s,f)|_H^{\frac{1}{2}} \\ &\qquad ||v_n(s,f_n)-v(s,f)||_V^{\frac{1}{2}}||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V ds \\ &\le c||Q||_\infty[M_1^n(u,\omega)M_2^n(u,\omega)(C_2^n(\omega)+C_2(\omega))]^{\frac{1}{2}}(\int_0^T |v_n(s,f_n)-v(s,f)|_H^2 ds)^{\frac{1}{4}} \\ &\qquad \times(\int_0^T ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V^{\frac{4}{3}} ds)^{\frac{3}{4}} \end{aligned} \tag{3.29}$$

The term I_5^n can be bounded as follows:

$$\begin{aligned} I_5^n &\le c\sup_{0\le s\le T}|Q(s)|\int_0^t ||v(s,f)+Z(s)||_V||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_H ds \\ &\le +\frac{\nu}{2}\int_0^t ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V^2 ds \\ &\quad +c_\nu||Q||_\infty^2\int_0^t ||v(s,f)+Z(s)||_V^2|\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))|_H^2 ds \end{aligned} \tag{3.30}$$

Now,

$$\begin{aligned}
I_6^n &\leq c \sup_{0\leq s\leq T}|Q(s)|\int_0^t ||\mathcal{D}_u v_n(s,f_n)+\mathcal{D}_u Z(s)||_V |v_n(s,f_n)-v(s,f)|_H^{\frac{1}{2}}||v_n(s,f_n)-v(s,f)||_V^{\frac{1}{2}} \\
&\qquad ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V^{\frac{1}{2}}|\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))|_H^{\frac{1}{2}}ds \\
&\leq c||Q||_\infty M_2^n(u,\omega)[(M_1^n(u,\omega)+M_1(u,\omega))(C_2^n(u,\omega)+C_2(\omega))]^{\frac{1}{2}}(\int_0^T |v_n(s,f_n)-v(s,f)|_H^2 ds)^{\frac{1}{4}} \\
&\qquad \times(\int_0^T ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f))||_V^{\frac{4}{3}}ds)^{\frac{3}{4}} \qquad (3.31)
\end{aligned}$$

Finally, we note that $I_7^n = 0$ because $b(u,v,v)=0$ for all $u,v\in V$.

Using the estimates (3.27)–(3.31) in (3.26) and applying Gronwall's inequality we obtain

$$\begin{aligned}
&\sup_{u\leq t\leq T}|\mathcal{D}_u v_n(t,f_n)-Y_u(t,f)|_H^2+\nu\int_0^T||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f)||_V^2 ds \\
&\qquad \leq L_n(\omega)\exp\left(c_\nu||Q||_\infty^2\int_0^T||v(s,f)+Z(s)||_V^2 ds\right) \\
&\qquad \leq L_n(\omega)\exp\left(c_\nu||Q||_\infty^2\{(|f|_H^2+D_1)D_2+\int_0^T\|Z(s)\|_V^2 ds\}\right) \qquad (3.32)
\end{aligned}$$

where $L_n(\omega)$ is the sum of the right sides of (3.27), (3.28), (3.29), (3.31). By the dominated convergence theorem, we deduce that

$$\begin{aligned}
&E[\sup_{u\leq t\leq T}|\mathcal{D}_u v_n(t,f_n)-Y_u(t,f)|_H^2]+\nu E[\int_0^T ||\mathcal{D}_u v_n(s,f_n)-Y_u(s,f)||_V^2 ds] \\
&\to 0, \quad \text{as} \quad n\to\infty, \qquad (3.33)
\end{aligned}$$

where Proposition 3.3 has been used. □

Propostion 3.5. *For any $f\in V$, the Malliavin derivative $\mathcal{D}_u v(t,f)$ of $v(t,f)$ satisfies the following estimate:*

$$\begin{aligned}
&\sup_{u\leq t\leq T}|\mathcal{D}_u v(t,f)|_H^2]+\nu E[\int_0^T ||\mathcal{D}_u v(s,f)||_V^2 ds] \\
\leq\ &\{c\|Q\|_\infty^2 D_7 D_6 D_4^2(|f|_H^2+D_1)D_2+c\|\mathcal{D}_u Q\|_\infty^2(|f|_H^2+D_1)^2 D_2^2 \\
&+c\|Q\|_\infty^2 D_7^2 D_4(D_3^2+(|f|_H^2+D_1)D_2)\}\exp\{C||Q||_\infty^2(D_4^2+(|f|_H^2+D_1)D_2))\} (3.34)
\end{aligned}$$

Proof. By the chain rule,

$$\begin{aligned}
&|\mathcal{D}_u v(t,f)|_H^2 \\
=\ &-2\nu\int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds
\end{aligned}$$

$$
\begin{aligned}
& -2\int_0^t Q(s)b(\mathcal{D}_u v(s,f)+\mathcal{D}_u Z(s), v(s,f)+Z(s), \mathcal{D}_u v(s,f))ds \\
& -2\int_0^t Q(s)b(v(s,f)+Z(s), \mathcal{D}_u v(s,f)+\mathcal{D}_u Z(s), \mathcal{D}_u v(s,f))ds \\
& -2\int_0^t \mathcal{D}_u Q(s)b(v(s,f)+Z(s), v(s,f)+Z(s), \mathcal{D}_u v(s,f))ds \\
:= \ & K_1+K_2+K_3+K_4 \quad t\in[0,T]. \qquad (3.35)
\end{aligned}
$$

In view of (2.10), we have the following estimates for K_2 and K_4:

$$
\begin{aligned}
K_2 \ \leq \ & c||Q||_\infty \int_0^t ||\mathcal{D}_u v(s,f)||_V ||v(s,f)+Z(s)||_V |\mathcal{D}_u v(s,f)|_H ds \\
& +c||Q||_\infty \int_0^t ||\mathcal{D}_u Z(s)||_V^{\frac{1}{2}} |\mathcal{D}_u Z(s)|_H^{\frac{1}{2}} ||v(s,f)+Z(s)||_V ||\mathcal{D}_u v(s,f)||_V^{\frac{1}{2}} |\mathcal{D}_u v(s,f)|_H^{\frac{1}{2}} ds \\
\leq \ & \frac{\nu}{4}\int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds + c_\nu ||Q||_\infty^2 \int_0^t ||v(s,f)+Z(s)||_V^2 |\mathcal{D}_u v(s,f)|_H^2 ds \\
& +\frac{\nu}{4}\int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds + c\int_0^t |\mathcal{D}_u v(s,f)|_H^2 ds \\
& +c_\nu ||Q||_\infty^2 \int_0^T \|\mathcal{D}_u Z(s)\|_V ||v(s,f)+Z(s)||_V^2 |\mathcal{D}_u Z(s)|_H ds \qquad (3.36)
\end{aligned}
$$

$$
\begin{aligned}
K_4 \ \leq \ & c||\mathcal{D}_u Q||_\infty \int_0^t ||\mathcal{D}_u v(s,f)||_V ||v(s,f)+Z(s)||_V |v(s,f)+Z(s)|_H ds \\
\leq \ & \frac{\nu}{4}\int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds + c_\nu ||\mathcal{D}_u Q||_\infty^2 \int_0^T ||v(s,f)+Z(s)||_V^2 |v(s,f)+Z(s)|_H^2 ds \\
\leq \ & \frac{\nu}{4}\int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds + c_\nu ||\mathcal{D}_u Q||_\infty^2 [(|f|_H^2+D_1)D_2]^2, \qquad (3.37)
\end{aligned}
$$

where (3.7), (3.8) were used. By (2.6), it follows that

$$
\begin{aligned}
K_3 \ = \ & -2\int_0^t Q(s)b(v(s,f)+Z(s), \mathcal{D}_u Z(s), \mathcal{D}_u v(s,f))ds \\
\leq \ & c||Q||_\infty \int_0^t ||\mathcal{D}_u v(s,f)||_V^{\frac{1}{2}} |\mathcal{D}_u v(s,f)|_H^{\frac{1}{2}} ||v(s,f)+Z(s)||_V^{\frac{1}{2}} |v(s,f)+Z(s)|_H^{\frac{1}{2}} \|D_u Z(s)\|_V ds \\
\leq \ & \frac{\nu}{4}\int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds + c\int_0^t |\mathcal{D}_u v(s,f)|_H^2 ds \\
& +c_\nu ||Q||_\infty^2 \int_0^T \|\mathcal{D}_u Z(s)\|_V^2 ||v(s,f)+Z(s)||_V |v(s,f)+Z(s)|_H ds \qquad (3.38)
\end{aligned}
$$

Substitute (3.36), (3.37), (3.38) into (3.35) to get

$$
\begin{aligned}
& |\mathcal{D}_u v(t,f)|_H^2 + \nu \int_0^t ||\mathcal{D}_u v(s,f)||_V^2 ds \\
\leq \ & c_\nu ||Q||_\infty^2 \int_0^t ||v(s,f)+Z(s)||_V^2 |\mathcal{D}_u v(s,f)|_H^2 ds + c \int_0^t |\mathcal{D}_u v(s,f)|_H^2 ds \\
& + c_\nu ||Q||_\infty^2 \int_0^T \|\mathcal{D}_u Z(s)\|_V ||v(s,f)+Z(s)||_V^2 |\mathcal{D}_u Z(s)|_H ds + c_\nu ||\mathcal{D}_u Q||_\infty^2 [(|f|_H^2 + D_1) D_2]^2 \\
& + c_\nu ||Q||_\infty^2 \int_0^T \|\mathcal{D}_u Z(s)\|_V^2 ||v(s,f)+Z(s)||_V |v(s,f)+Z(s)|_H ds \qquad (3.39)
\end{aligned}
$$

Applying Gronwall's inequality we obtain

$$
\begin{aligned}
& \sup_{0\leq t\leq T} [|\mathcal{D}_u v(t,f)|_H^2] + \nu \int_0^T ||\mathcal{D}_u v(s,f)||_V^2 ds \\
\leq \ & \{c_\nu ||Q||_\infty^2 \int_0^T \|\mathcal{D}_u Z(s)\|_V ||v(s,f)+Z(s)||_V^2 |\mathcal{D}_u Z(s)|_H ds + c_\nu ||\mathcal{D}_u Q||_\infty^2 [(|f|_H^2 + D_1) D_2]^2 \\
& + c_\nu ||Q||_\infty^2 \int_0^T \|\mathcal{D}_u Z(s)\|_V^2 ||v(s,f)+Z(s)||_V |v(s,f)+Z(s)|_H ds\} \\
& \times exp\{c||Q||_\infty^2 \int_0^T ||v(s,f)+Z(s)||_V^2 ds + cT\} \qquad (3.40)
\end{aligned}
$$

Now (3.36) follows from Proposition 3.3. □

Theorem 3.1. *For each $f \in H$, the solution map*

$$\Omega \ni \omega \to v(t,f,\omega) \in H$$

of the random NSE (3.6) is Malliavin differentiable. Its Malliavin derivative $\mathcal{D}_u v(t,f)$ solves the following random evolution equation:

$$
\begin{aligned}
& \mathcal{D}_u v(t,f) \\
= \ & -\int_0^t A\mathcal{D}_u v(s,f) ds - \int_0^t Q(s)(\mathcal{D}_u v(s,f) + \mathcal{D}_u Z(s)) \cdot \nabla)(v(s,f)+Z(s)) ds \\
& -\int_0^t Q(s)(v(s,f)+Z(s)) \cdot \nabla)(\mathcal{D}_u v(s,f) + \mathcal{D}_u Z(s)) ds \\
& -\int_0^t \mathcal{D}_u Q(s)(v(s,f)+Z(s)) \cdot \nabla)(v(s,f)+Z(s)) ds,
\end{aligned}
\qquad (3.41)
$$

for all $t \in [0,T]$, a.s..

Proof. Again, as in the proof of Proposition 3.4, we will implicitly assume $Q = Q_N$. Take $f_n \in V, n \geq 1$ such that $f_n \to f$ in H as $n \to \infty$. By Proposition 3.4 , we know that $v(t, f_n, \omega) \in H$ is Malliavin differentiable. The Malliavin derivative $\mathcal{D}_u v(t, f_n)$ solves the following random evolution equation:

$$\begin{aligned} & \mathcal{D}_u v(t, f_n) \\ = & -\int_0^t A\mathcal{D}_u v(s, f_n)ds - \int_0^t Q(s)(\mathcal{D}_u v(s, f_n) + \mathcal{D}_u Z(s)) \cdot \nabla)(v(s, f_n) + Z(s))ds \\ & -\int_0^t Q(s)(v(s, f_n) + Z(s)) \cdot \nabla)(\mathcal{D}_u v(s, f_n) + \mathcal{D}_u Z(s))ds \\ & -\int_0^t \mathcal{D}_u Q(s)(v(s, f_n) + Z(s)) \cdot \nabla)(v(s, f_n) + Z(s))ds, \end{aligned} \tag{3.42}$$

for all $t \in [0, T]$. On the other hand, as Proposition 3.1 we can show that

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} |v(t, f_n) - v(t, f)|_H = 0$$

$$\lim_{n\to\infty} E[\sup_{0\leq t\leq T} |v(t, f_n) - v(t, f)|_H^p] = 0$$

$$\lim_{n\to\infty} E\bigg(\int_0^T ||v(t, f_n) - v(t, f)||_V^2 dt\bigg)^p = 0$$

for any $p > 0$. Thus it suffices to show that the Malliavin derivatives $\mathcal{D}_u v(t, f_n), n \geq 1$, converge. Let $Z_u(t, f)$ be the solution of the following random evolution equation:

$$\begin{aligned} Z_u(t, f) = & -\int_0^t AZ_u(s, f)ds - \int_0^t Q(s)(Z_u(s, f) + \mathcal{D}_u Z(s)) \cdot \nabla)(v(s, f) + Z(s))ds \\ & -\int_0^t Q(s)(v(s, f) + Z(s)) \cdot \nabla)(Z_u(s, f) + \mathcal{D}_u Z(s))ds \\ & -\int_0^t \mathcal{D}_u Q(s)(v(s, f) + Z(s)) \cdot \nabla)(v(s, f) + Z(s))ds, \end{aligned} \tag{3.43}$$

for all $t \in [0, T]$. Since the Malliavin derivative operator $\mathcal{D}$ is closed, to prove the theorem it suffices to show that

$$\lim_{n\to\infty} \sup_{0\leq u\leq t} E[|\mathcal{D}_u v_n(t, f_n) - Z_u(t, f)|_H^2] = 0. \tag{3.44}$$

To prove (3.44), consider the following:

$$
\begin{aligned}
&|\mathcal{D}_u v(t,f_n)-Z_u(t,f)|_H^2\\
= & -2\nu\int_0^t ||\mathcal{D}_u v(s,f_n)-Z_u(s,f)||_V^2 ds\\
& -2\int_0^t \mathcal{D}_u Q(s) b(v(s,f_n)+Z(s), v(s,f_n)-v(s,f), \mathcal{D}_u v(s,f_n)-Z_u(s,f))ds\\
& -2\int_0^t \mathcal{D}_u Q(s) b(v(s,f_n)-v(s,f), v(s,f)+Z(s), \mathcal{D}_u v(s,f_n)-Z_u(s,f))ds\\
& -2\int_0^t Q(s) b(\mathcal{D}_u v(s,f_n)+\mathcal{D}_u Z(s), v(s,f_n)-v(s,f), \mathcal{D}_u v(s,f_n)-Z_u(s,f))ds\\
& -2\int_0^t Q(s) b(\mathcal{D}_u v(s,f_n)-Z_u(s,f), v(s,f)+Z(s), \mathcal{D}_u v(s,f_n)-Z_u(s,f))ds\\
& -2\int_0^t Q(s) b(v(s,f_n)-v(s,f), \mathcal{D}_u v(s,f_n)+\mathcal{D}_u Z(s), \mathcal{D}_u v(s,f_n)-Z_u(s,f))ds\\
& -2\int_0^t Q(s) b(v(s,f)+Z(s), \mathcal{D}_u v(s,f_n)-Z_u(s,f), \mathcal{D}_u v(s,f_n)-Z_u(s,f))ds\\
:= & J_1^n+J_2^n+J_3^n+J_4^n+J_5^n+J_6^n+J_7^n \qquad (3.45)
\end{aligned}
$$

for all $t\in[0,T]$. Note first that $J_7^n=0$ because the trilinear form $b(\cdot,\cdot,\cdot)$ is anti-symmetric with respect to the last two arguments. To estimate J_2^n and J_3^n, note that

$$
\begin{aligned}
J_2^n \le\ & c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|\int_0^t |v(s,f_n)+Z(s)|_H^{\frac12}||v(s,f_n)+Z(s)||_V^{\frac12}|v(s,f_n)-v(s,f)|_H^{\frac12}\\
&\qquad \times||v(s,f_n)-v(s,f)||_V^{\frac12}||\mathcal{D}_u v(s,f_n)-Z_u(s,f))||_V ds\\
\le\ & c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|\sup_{0\le s\le T}(|v(s,f_n)-v(s,f)|_H^{\frac12})\sup_{0\le s\le T}(|v(s,f_n)+Z(s)|_H^{\frac12})\\
&\times\int_0^t ||v(s,f_n)+Z(s)||_V^{\frac12}||v(s,f_n)-v(s,f)||_V^{\frac12}||\mathcal{D}_u v(s,f_n)-Z_u(s,f))||_V ds\\
&\longrightarrow 0, \quad \text{as} \quad n\to\infty. \qquad (3.46)
\end{aligned}
$$

and

$$
\begin{aligned}
J_3^n \le\ & c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|\int_0^t |v(s,f)+Z(s)|_H^{\frac12}||v(s,f)+Z(s)||_V^{\frac12}|v(s,f_n)-v(s,f)|_H^{\frac12}\\
&\qquad \times||v(s,f_n)-v(s,f)||_V^{\frac12}||\mathcal{D}_u v(s,f_n)-Z_u(s,f))||_V ds\\
\le\ & c\sup_{0\le s\le T}|\mathcal{D}_u Q(s)|\sup_{0\le s\le T}(|v(s,f_n)-v(s,f)|_H^{\frac12})\sup_{0\le s\le T}(|v(s,f)+Z(s)|_H^{\frac12})\\
&\times\int_0^t ||v(s,f)+Z(s)||_V^{\frac12}||v(s,f_n)-v(s,f)||_V^{\frac12}||\mathcal{D}_u v(s,f_n)-Z_u(s,f))||_V ds
\end{aligned}
$$

$$\longrightarrow 0, \quad \text{as} \quad n \to \infty. \tag{3.47}$$

For J_4^n, we have

$$\begin{aligned} J_4^n &\le c \sup_{0\le s\le T} |Q(s)| \int_0^t |\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)|_H^{\frac{1}{2}} ||\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)||_V^{\frac{1}{2}} |v_n(s,f_n) - v(s,f)|_H^{\frac{1}{2}} \\ &\qquad ||v_n(s,f_n) - v(s,f)||_V^{\frac{1}{2}} ||\mathcal{D}_u v_n(s,f_n) - Z_u(s,f))||_V ds \\ &\le c||Q||_\infty \sup_{0\le s\le T} (|v(s,f_n) - v(s,f)|_H^{\frac{1}{2}}) \sup_{0\le s\le T} (|\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)|_H^{\frac{1}{2}}) \\ &\qquad \times (\int_0^T ||\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)||_V^{\frac{1}{2}} ||v_n(s,f_n) - v(s,f)||_V^{\frac{1}{2}} ||\mathcal{D}_u v_n(s,f_n) - Z_u(s,f))||_V ds \\ &\longrightarrow 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.48}$$

The term J_5^n can be estimated as follows:

$$\begin{aligned} J_5^n &\le c \sup_{0\le s\le T} |Q(s)| \int_0^t ||v(s,f) + Z(s)||_V ||\mathcal{D}_u v(s,f_n) - Z_u(s,f))||_V |\mathcal{D}_u v(s,f_n) - Z_u(s,f))|_H ds \\ &\le +\frac{\nu}{2} \int_0^t ||\mathcal{D}_u v_n(s,f_n) - Y_u(s,f))||_V^2 ds \\ &\quad + c_\nu ||Q||_\infty^2 \int_0^t ||v(s,f) + Z(s)||_V^2 |\mathcal{D}_u v_n(s,f_n) - Y_u(s,f))|_H^2 \, ds. \end{aligned} \tag{3.49}$$

Furthermore,

$$\begin{aligned} J_6^n &\le c \sup_{0\le s\le T} |Q(s)| \int_0^t ||\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)||_V |v(s,f_n) - v(s,f)|_H^{\frac{1}{2}} ||v(s,f_n) - v(s,f)||_V^{\frac{1}{2}} \\ &\qquad ||\mathcal{D}_u v_n(s,f_n) - Y_u(s,f))||_V |\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)|_H^{\frac{1}{2}} ds \\ &\le c||Q||_\infty \sup_{0\le s\le T} (|v(s,f_n) - v(s,f)|_H^{\frac{1}{2}} \sup_{0\le s\le T} (|\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)|_H^{\frac{1}{2}} \\ &\quad \times \int_0^t ||\mathcal{D}_u v(s,f_n) + \mathcal{D}_u Z(s)||_V^{\frac{1}{2}} ||v(s,f_n) - v(s,f)||_V^{\frac{1}{2}} ||\mathcal{D}_u v_n(s,f_n) - Y_u(s,f))||_V ds \\ &\quad \longrightarrow 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.50}$$

Substituting (3.46)–(3.50) into (3.45) and applying Gronwall's inequality, we obtain

$$\begin{aligned} \sup_{u\le t\le T} |\mathcal{D}_u v(t,f_n) - Z_u(t,f)|_H^2 \qquad & + \nu \int_0^T ||\mathcal{D}_u v(s,f_n) - Z_u(s,f)||_V^2 ds \\ &\le \tilde{L}_n(\omega) \exp\left(c_\nu ||Q||_\infty^2 \int_0^T ||v(s,f) + Z(s)||_V^2 ds \right) \end{aligned} \tag{3.51}$$

where $\tilde{L}_n(\omega)$ is the sum of the right-hand sides of (3.46), (3.47), (3.48), (3.50) and is such that $\tilde{L}_n \to 0$ as $n \to \infty$. Finally, (3.44) follows from the dominated convergence theorem. $\square$

4 The Anticipating SNSE

We are now ready to state and prove our main result whereby we replace the deterministic initial function f in the the SNSE (3.1) by an anticipating random field $Y \in \mathcal{D}^{1,4}(H)$:

Theorem 4.1. *Suppose $Y \in \mathcal{D}^{1,4}(H)$. Then $u(t,Y), t \geq 0$, is the unique solution of the following anticipating Stratonovich SNSE:*

$$u(t,Y) = Y - \int_0^t Au(s,Y)ds - \int_0^t B(u(s,Y))ds + \int_0^t u(s,Y) \circ dW(s) + \sigma_0 W_0(t), \quad t \geq 0, \tag{4.1}$$

satisfying the following conditions: $u(\cdot, Y) \in C([0,T], H)$ and $u(t,Y) \in \mathcal{D}^{1,4}_{loc}(H)$ for all $t \in [0,T]$ and any $T \in (0,\infty)$.

Proof. The proof is in two steps.

Step 1: Existence.

Note that $u(t,Y)$ takes values in H for all $t \geq 0$. But $Au(s,Y)$ and $B(u(s,Y))$ belong to V'. Because of this special infinite-dimensional setting, the existing product rules in the literature could not be applied. We will therefore give a direct proof of the product rule within this context.

Fix $t > 0$ and let $\{0 = t_0^n < t_1^n < ... < t_{k_n}^n = t\}, n \geq 1$ be a sequence of partitions of the interval $[0,t]$ such that $\tau^n = max_i(t_{i+1}^n - t_i^n) \to 0$ as $n \to \infty$. Write

$$\begin{aligned}
& u(t,Y) - Y - Q(t)Z(t) \\
= \ & v(t,Y)Q(t) - Y \\
= \ & \sum_{i=0}^{k_n-1} [v(t_{i+1},Y)Q(t_{i+1}) - v(t_i,Y)Q(t_i)] \\
= \ & \sum_{i=0}^{k_n-1} Q(t_{i+1})(v(t_{i+1},Y) - v(t_i,Y)) + \sum_{i=0}^{k_n-1} v(t_i,Y)(Q(t_{i+1}) - Q(t_i)) \\
= \ & -\sum_{i=0}^{k_n-1} \int_{t_i}^{t_{i+1}} Q(t_{i+1})Av(s,Y)ds - \sum_{i=0}^{k_n-1} \int_{t_i}^{t_{i+1}} Q(t_{i+1})Q(s)B(v(s,Y) + Z(s))ds \\
+ \ & \sum_{i=0}^{k_n-1} v(t_i,Y) \int_{t_i}^{t_{i+1}} Q(s)dW(s) + \frac{1}{2} \sum_{i=0}^{k_n-1} v(t_i,Y) \int_{t_i}^{t_{i+1}} Q(s)\, ds \\
:= \ & T_1^n + T_2^n + T_3^n + T_4^n.
\end{aligned} \tag{4.2}$$

As $v(s,Y), Q(s)$ are continuous in s, then

$$\lim_{n\to\infty} T_1^n = -\int_0^t Au(s,Y)ds + \int_0^t Q(s)AZ(s)ds, \tag{4.3}$$

$$\lim_{n\to\infty} T_2^n = -\int_0^t Q(s)^2 B(v(s,Y)+Z(s))ds = -\int_0^t B(u(s,Y))ds \tag{4.4}$$

and

$$\lim_{n\to\infty} T_4^n = \frac{1}{2}\int_0^t Q(s)v(s,Y)ds. \tag{4.5}$$

By the property of the Skorohod integral ([N], p. 40), we have

$$\begin{aligned} T_3^n &= \sum_{i=0}^{k_n-1}\int_{t_i}^{t_{i+1}} v(t_i,Y)Q(s)dW(s) + \sum_{i=0}^{k_n-1}\int_{t_i}^{t_{i+1}} \mathcal{D}_s(v(t_i,Y))Q(s)ds \\ &= \int_0^t \sum_{i=0}^{k_n-1} v(t_i,Y)Q(s)\chi_{(t_i,t_{i+1}]}(s)dW(s) + \int_0^t \sum_{i=0}^{k_n-1} \mathcal{D}_s(v(t_i,Y))\chi_{(t_i,t_{i+1}]}(s)Q(s)ds \\ &= \int_0^t F^n(s)dW(s) + \int_0^t G^n(s)ds, \end{aligned} \tag{4.6}$$

where

$$F^n(s) := \sum_{i=0}^{k_n-1} v(t_i,Y)Q(s)\chi_{(t_i,t_{i+1}]}(s),$$

$$G^n(s) := \sum_{i=0}^{k_n-1} \mathcal{D}_s(v(t_i,Y))\chi_{(t_i,t_{i+1}]}(s)Q(s), \quad 0\le s\le t.$$

Recall that $\mathbb{L}^{1,2}(H)$ (see [N]) is the class of H-valued processes u such that $u(t)\in\mathcal{D}^{1,2}(H)$ for almost all t, and there exists a measurable version of the two parameter process $D_s u(t)$ satisfying $E[\int_0^T\int_0^T |D_s u(t)|_H^2 dsdt] < \infty$. We say that $u\in\mathbb{L}^{1,2}_{loc}(H)$ if there exists a sequence $\{(\Omega_n,u^n), n\ge 1\}\subset \mathcal{F}\times\mathbb{L}^{1,2}(H)$ such that Ω_n increases to Ω a.s. and $u=u^n$ a.e on $[0,T]\times\Omega_n$. We first show that $F^n(\cdot)\to v(\cdot,Y)Q(\cdot)$ in $\mathbb{L}^{1,2}_{loc}(H)$ as $n\to\infty$. To this end, we may assume without loss of generality that $|Y|_H\le M$ for some constant M and $Q=Q_N$. This is because, otherwise, we can replace Y by $Y\phi(|Y|_N)$ where $\phi\in C_0^\infty(R)$ is a smooth bump function satisfying $\phi(x)=1$ whenever $|x|\le M$ and $\phi(x)=0$ when $|x|>M+1$. Note that $v(s,Y)$ is continuous in s. It is clear that $F^n(s)\to v(s,Y)Q(s)$ for every $s\ge 0$. Moreover,

$$\begin{aligned} \sup_{0\le s\le t}|F^n(s)|_H &\le (\sup_{0\le s\le t}|v(s,Y)|_H)(\sup_{0\le s\le t}Q(s)) \\ &\le c(|Y|_H + D_1^{\frac{1}{2}})D_2^{\frac{1}{2}}(\sup_{0\le s\le t}Q(s)) \end{aligned} \tag{4.7}$$

The dominated convergence theorem yields that

$$\lim_{n\to\infty} E[\int_0^t |F^n(s)-v(s,Y)Q(s)|_H^2 ds] = 0 \tag{4.8}$$

The Malliavin derivative of F^n is given by

$$\begin{aligned}\mathcal{D}_u F^n(s) &= \sum_{i=0}^{k_n-1} \mathcal{D}_u[v(t_i,Y)Q(s)]\chi_{(t_i,t_{i+1}]}(s)\\ &= \sum_{i=0}^{k_n-1} [\mathcal{D}_u(v(t_i,Y))Q(s) + v(t_i,Y)\mathcal{D}_u Q(s)]\chi_{(t_i,t_{i+1}]}(s)\\ &= \sum_{i=0}^{k_n-1} v(t_i,Y)\mathcal{D}_u Q(s)\chi_{(t_i,t_{i+1}]}(s)\\ &+ \sum_{i=0}^{k_n-1} [\mathcal{D}_u v(t_i,Y) + Dv(t_i,Y)(\mathcal{D}_u Y)]Q(s)\chi_{(t_i,t_{i+1}]}(s), \qquad (4.9)\end{aligned}$$

where $Dv(s,f)$ denotes the Fréchet derivative at $f \in H$ for the mapping $v(s,\cdot)$, and $\mathcal{D}_u v(t_i,Y) := \mathcal{D}_u v(t_i,f)\Big|_{f=Y}$. Since $v(s,Y)$, $\mathcal{D}_u v(s,Y)$ and $Dv(s,Y)$ are continuous in s, it is easily seen that $\mathcal{D}_u F^n(s) \to \mathcal{D}_u(v(s,Y)Q(s))$ for every $s \geq 0$. In view of Proposition 3.1, Proposition 3.2, Proposition 3.5, it follows from (4.9) that

$$\begin{aligned}|\mathcal{D}_u F^n(s)|_H \;\leq\; & c(|Y|_H + D_1^{\frac{1}{2}})D_2^{\frac{1}{2}} \sup_{0\leq s\leq t} |\mathcal{D}_u Q(s)|\chi_{[0,t]}(u)\\ & +\{c\|Q\|_\infty^2 D_7 D_6 D_4^2(|Y|_H^2 + D_1)D_2 + c\|\mathcal{D}_u Q\|_\infty^2(|Y|_H^2 + D_1)^2 D_2^2\\ & +c\|Q\|_\infty^2 D_7^2 D_4(D_3^2 + (|Y|_H^2 + D_1)D_2)\}\exp\{C||Q||_\infty^2(D_4^2 + (|Y|_H^2 + D_1)D_2))\}\\ & +|\mathcal{D}_u Y|_H \exp(c||Q||_\infty^2|Y|_H^2)||Q||_\infty \qquad (4.10)\end{aligned}$$

Thus from the dominated convergence theorem it follows that

$$\lim_{n\to\infty} E\left[\int_0^t \int_0^\infty |\mathcal{D}_u F^n(s) - \mathcal{D}_u(v(s,Y)Q(s))|_H^2\, du\, ds\right] = 0 \qquad (4.11)$$

The relations (4.8) and (4.11) imply that $F^n(\cdot) \to v(\cdot,Y)Q(\cdot)$ in $\mathbb{L}^{1,2}_{loc}(H)$ as $n \to \infty$. Consequently, we have

$$\lim_{n\to\infty} \int_0^t F^n(s)dW(s) = \int_0^t v(s,Y)Q(s)dW(s), \qquad (4.12)$$

where the integrals in the above relation are Skorohod integrals. To compute $\lim\limits_{n\to\infty} G^n(s)$, consider

$$\begin{aligned}G^n(s) &= Q(s)\sum_{i=0}^{k_n-1} [\mathcal{D}_s v(t_i,Y) + Dv(t_i,Y)(\mathcal{D}_s Y)]\chi_{(t_i,t_{i+1}]}(s)\\ &= Q(s)\sum_{i=0}^{k_n-1} Dv(t_i,Y)(\mathcal{D}_s Y)\chi_{(t_i,t_{i+1}]}(s), \qquad (4.13)\end{aligned}$$

where we have used the fact that $\mathcal{D}_s v(t_i, f) = 0$ for $s > t_i$. Since $Dv(s, Y)$ is continuous in s, we see that

$$\lim_{n\to\infty} \int_0^t G^n(s)ds = \int_0^t Q(s)Dv(s,Y)(\mathcal{D}_s Y)ds \tag{4.14}$$

for all $t \geq 0$.

Putting (4.2)–(4.14) together and letting $n \to \infty$, we arrive at

$$\begin{aligned} & u(t,Y) - Y - Q(t)Z(t) \\ = \;& v(t,Y)Q(t) - Y \\ = \;& -\int_0^t Au(s,Y)ds - \int_0^t B(u(s,Y))ds + \int_0^t v(s,Y)Q(s)dW(s) \\ & + \frac{1}{2}\int_0^t v(s,Y)Q(s)ds + \int_0^t Q(s)Dv(s,Y)(\mathcal{D}_s Y)ds + \int_0^t Q(s)AZ(s)ds \end{aligned} \tag{4.15}$$

for all $t \geq 0$.

To proceed with the proof of (4.1), we first show that

$$\begin{aligned} & \int_0^t v(s,Y)Q(s) \circ dW(s) \\ = \;& \int_0^t v(s,Y)Q(s)dW(s) + \frac{1}{2}\int_0^t v(s,Y)Q(s)ds + \int_0^t Q(s)Dv(s,Y)(\mathcal{D}_s Y)\,ds \end{aligned} \tag{4.16}$$

for all $t \geq 0$. Define

$$\mathcal{D}_s^+[v(s,Y)Q(s)] := \lim_{\varepsilon\to 0+} \mathcal{D}_s[v(s+\varepsilon,Y)Q(s+\varepsilon)],$$

$$D_s^-[v(s,Y)Q(s)] := \lim_{\varepsilon\to 0+} \mathcal{D}_s[v(s-\varepsilon,Y)Q(s-\varepsilon)],$$

$$(\nabla[v(\cdot,Y)Q(\cdot)])(s) := \frac{1}{2}[\mathcal{D}_s^+[v(s,Y)Q(s)] + D_s^-[v(s,Y)Q(s)]]$$

for $s > 0$. Then, by Theorem 3.1.1 in [N], we know that

$$\begin{aligned} & \int_0^t v(s,Y)Q(s) \circ dW(s) \\ = \;& \int_0^t v(s,Y)Q(s)dW(s) + \frac{1}{2}\int_0^t (\nabla[v(\cdot,Y)Q(\cdot)])_s ds, \quad t \geq 0. \end{aligned} \tag{4.17}$$

Thus, we need to show that

$$\frac{1}{2}(\nabla[v(\cdot,Y)Q(\cdot)])_s = \frac{1}{2}v(s,Y)Q(s) + Q(s)Dv(s,Y)(\mathcal{D}_s Y) \tag{4.18}$$

The Malliavin derivative of $v(t,Y)Q(s)$ is given by

$$\begin{aligned} & \mathcal{D}_s(v(t,Y)Q(t)) \\ = & \mathcal{D}_s(v(t,Y))Q(t) + v(t,Y)\mathcal{D}_s Q(t) \\ = & \mathcal{D}_s v(t,Y)Q(t) + Dv(t,Y)(\mathcal{D}_s Y)Q(t) + v(t,Y)\mathcal{D}_s Q(t) \end{aligned} \tag{4.19}$$

for all $t \geq 0$. Replacing t by $s-\varepsilon$ in (4.19) we get

$$\mathcal{D}_s(v(s-\varepsilon,Y)Q(s-\varepsilon)) = Dv(s-\varepsilon,Y)(\mathcal{D}_s Y)Q(s-\varepsilon),$$

where we have used the fact that $\mathcal{D}_s v(s-\varepsilon,Y)=0$, $\mathcal{D}_s Q(s-\varepsilon)=0$. This yields

$$D_s^-([v(\cdot,Y)Q(\cdot)])(s) = Dv(s,Y)(\mathcal{D}_s Y)Q(s). \tag{4.20}$$

Next, we replace t by $s+\varepsilon$ in (4.19), let $\varepsilon \to 0$ and use the continuity of the functions involved to obtain

$$D_s^+([v(\cdot,Y)Q(\cdot)])(s) = Dv(s,Y)(\mathcal{D}_s Y)Q(s) + v(s,Y)Q(s), \tag{4.21}$$

where we have used the facts that $\lim_{\varepsilon\to 0}\mathcal{D}_s v(s+\varepsilon,Y)=0$ and $\mathcal{D}_s Q(s+\varepsilon)=Q(s+\varepsilon)$. Finally, (4.18) follows from (4.20) and (4.21). Thus, (4.15) becomes

$$\begin{aligned} & u(t,Y) - Y - Q(t)Z(t) \\ = & v(t,Y)Q(t) - Y \\ = & -\int_0^t Au(s,Y)ds - \int_0^t B(u(s,Y))ds + \int_0^t v(s,Y)Q(s)\circ dW(s) \\ & + \int_0^t Q(s)AZ(s)ds \end{aligned} \tag{4.22}$$

On the other hand, by Itô's formula

$$\begin{aligned} Q(t)Z(t) &= \int_0^t Z(s)\circ dQ(s) + \int_0^t Q(s)\circ dZ(s) \\ &= -\int_0^t Q(s)AZ(s)ds + \sigma_0 W_0(t) + \int_0^t Z(s)Q(s)\circ dW(s) \end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23) we obtain

$$\begin{aligned} u(t,Y) &= Y - \int_0^t Au(s,Y)ds - \int_0^t B(u(s,Y))ds + \int_0^t u(s,Y)\circ dW(s) \\ & +\sigma_0 W_0(t) \end{aligned} \tag{4.24}$$

This proves the existence.

Step 2: Uniqueness.

We finally prove uniqueness for the solution u of the anticipating SNSE (4.1) among the class satisfying the conditions $u(\cdot,\omega) \in C([0,T],H)$ for a.a. $\omega \in \Omega$, and $u(t,\cdot) \in \mathcal{D}^{1,4}_{loc}(H)$ for all $t \in [0,T]$ and any $T \in (0,\infty)$. By localization and for the rest of the argument, it is sufficient to look at the class of all anticipating solutions u of the SNSE (4.1) with initial condition $Y \in \mathcal{D}^{1,4}(H)$. Consider the process $Q^{-1}(t) := Q(t)^{-1}$, $t \geq 0$. Then clearly $\circ dQ^{-1}(t) = -\sigma Q^{-1}(t) \circ dW(t)$ for all $t \geq 0$. Define the process $v(t) := u(t)Q^{-1}(t) - Z(t)$ for all $t \geq 0$. Now by the product rule for Stratonovich differentials in H, we get:

$$dv(t) = u(t) \circ dQ^{-1}(t) + Q^{-1}(t) \circ du(t) - dZ(t), \quad t > 0.$$

Using the fact that $\circ dQ^{-1}(t) = -\sigma Q^{-1}(t) \circ dW(t), \quad t > 0$, it follows immediately from the above product rule that $v(\cdot,\omega)$ is of bounded variation for a.a. $\omega \in \Omega$ and satisfies the following random SNSE:

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt - Q(t)B\big(v(t) + Z(t)\big)\,dt, \quad t \geq 0, \\ v(0) &= Y \in \mathcal{D}^{1,4}(H)\,. \end{aligned}\right\} \tag{4.25}$$

Now uniqueness of the solution for the anticipating SNSE (4.1) follows easily from the uniqueness of the solution to the above random NSE. This completes the proof of Theorem 4.1. □

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