Anticipating Semilinear SPDEs

Salah Mohammed

http://sfde.math.siu.edu/

Kaiserslautern: April 4, 2007
Germany

\textsuperscript{a} Results to appear in JFA [M-Z]
\textsuperscript{b} Department of Mathematics, SIU-C, Carbondale, Illinois, USA
Acknowledgment

- Joint work with T.S. Zhang (Manchester, UK).
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- Research supported by NSF: DMS-0203368.
Objectives

**Question:**

Does the following anticipating stochastic evolution equation (see):

\[
\begin{align*}
\frac{dv(t)}{dt} &= Av(t) dt + F_0 v(t) dt + Bv(t) dW(t); \quad t > 0
\end{align*}
\]

admit a solution with a random initial condition \(v(0) = Y\)?

**Answer:** YES! (provided \(Y\) is sufficiently regular).
Question:

Does the following anticipating stochastic evolution equation (see):

\[ dv(t) = -Av(t) \, dt + F_0(v(t)) \, dt + Bv(t) \circ dW(t), \quad t > 0, \]

admit a solution with a random initial condition

\[ Y : \Omega \to H \text{ in a Hilbert space } H? \]
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Does the following anticipating stochastic evolution equation (see):

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Answer:

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+ Bv(t) \circ dW(t), \quad t > 0,
\]
\[v(0) = Y\]

admit a solution with a random initial condition \(Y : \Omega \rightarrow H\) in a Hilbert space \(H\)?

Answer:
YES! (provided \(Y\) is sufficiently regular).
Replace $Y$ in see (1) by a \textit{deterministic} initial condition $x$ in $H$ and get the corresponding (equivalent) Itô see:

\[
du(t, x) = -Au(t, x) \, dt + F(u(t, x)) \, dt + Bu(t, x) \, dW(t), \quad t > 0
\]

\[u(0, x) = x \in H\]

with $F$ a suitably modified non-linear drift.
Strategy

- Replace $Y$ in see (1) by a deterministic initial condition $x$ in $H$ and get the corresponding (equivalent) Itô see:

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$$

$u(0, x) = x \in H$

(2)

with $F$ a suitably modified non-linear drift.

- View the solution of the see (2) as a function (cocycle) $U(t, x, \omega)$ of three variables $(t, x, \omega)$ with Fréchet and Malliavin regularity in $x$ and $\omega$ (resp.)
Consider the Stratonovich version of the Itô see (2): 

\[ du(t, x) = -Au(t, x) \, dt + F_0(u(t, x)) \, dt \]

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\[(2')\]

In the above semilinear see, is it justified to replace the deterministic initial condition \(x\) by an arbitrary random variable \(Y\) (substitution theorem)?
Then get back the anticipating Stratonovich see (1) again:

\[
\begin{align*}
    dU(t, Y) &= -AU(t, Y) \, dt + F_0(U(t, Y)) \, dt \\
             &\quad + BU(t, Y) \circ dW(t), \quad t > 0
\end{align*}
\]

\[U(0, Y) = Y\]

by taking \(v(t) := U(t, Y), \quad t \geq 0.\)
Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde’s via substitution theorems ([Nu.1-2], [M-S.2]).

Known substitution theorems require a level of regularity of the cocycle $U(t;x;\omega)$ in $t$ that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).

Existing substitution theorems work under restrictive finite-dimensional or compactness constraints ([G-Nu-M]).
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- Failure of Sobolev inequalities in infinite dimensions.
Approach

- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).

Develop global estimates on the semiflow generated by the spde.

Use ideas and techniques of the Malliavin calculus:

Assume Malliavin regularity of the initial condition—rather than imposing finite-dimensional or compactness restrictions on the values of the initial random condition.

Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are infinite-dimensional.
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Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see’s near hyperbolic stationary states.
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Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see’s near hyperbolic stationary states.

Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by sfde’s.

Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Expect results in this talk to lead to regularity in distribution of the invariant manifolds for semilinear spde’s and sfde’s.
The Set-up

- $(\Omega, \mathcal{F}, P) := \textbf{Wiener space}$ of all continuous paths
  $\omega : \mathbb{R} \to E$, $\omega(0) = 0$, where $E$ is a real separable
  Hilbert space.
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- **Wiener shifts** \(\theta : \mathbb{R} \times \Omega \to \Omega\): Group of \(P\)-preserving ergodic transformations on \((\Omega, \mathcal{F}, P)\):

  \[\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega.\]
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- \(\mathcal{B}(H) := \text{Borel } \sigma\text{-algebra of } H\).
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- $L(H) := \text{Banach space of all bounded linear operators } H \rightarrow H \text{ given the uniform operator norm } \| \cdot \|_{L(H)}$. 

---

*Anticipating Semilinear SPDEs*
Set-up: Brownian Motion

- $W := E$-valued Brownian motion $W : \mathbb{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.

$$W(t) = \sum_{k=1}^{\infty} W_k(t) f_k; t \in \mathbb{R};$$

$W_k(t)$: complete orthonormal basis of $K$; standard independent one-dimensional Wiener processes (DEF.1), Chapter 4. Series converges absolutely in $E$ but not necessarily in $K$.

$(W; t)$ is a helix:

$$W(t_1 + t_2) = W(t_2)(t_1).$$

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\( \{ f_k : k \geq 1 \} := \) complete orthonormal basis of \( K \);
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- $(W, \theta)$ is a helix:
  $W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$
Set-up-contd

- \( L_2(K, H) := \text{Hilbert space of all Hilbert-Schmidt operators } S : K \to H, \text{ with norm} \)

\[
\| S \|_2 := \left[ \sum_{k=1}^{\infty} |S(f_k)|_H^2 \right]^{1/2}
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- $F_0 : H \to H$ is $C^1_b$.

- $F := F_0 + \frac{1}{2} \sum_{k=1}^{\infty} B_k^2$, where $B_k \in L(H)$ are given by

$$
B_k(x) := B(x)(f_k), \ x \in H, \ k \geq 1; \text{ and } \sum_{k=1}^{\infty} \|B_k\|^2 \text{ converges.}
$$
Consider the semilinear Itô stochastic evolution equation (see):

\[
\begin{align*}
    du(t, x) &= -Au(t, x) \, dt + F(u(t, x)) \, dt \\
    &\quad + Bu(t, x) \, dW(t), \quad t > 0 \\
    u(0, x) &= x \in H
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in $H$. 

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\( A : D(A) \subset H \to H \) is a closed linear operator on \( H \).

Assume \( A \) has a complete orthonormal system of eigenvectors \( \{e_n : n \geq 1\} \) with corresponding positive eigenvalues \( \{\mu_n, n \geq 1\} \); i.e., \( Ae_n = \mu_n e_n, \ n \geq 1 \).
Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geq 0$. 
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\(F : H \to H\) is (Fréchet) \(C^1_b\): \(F\) has a globally bounded Fréchet derivative \(F : H \to L(H)\).
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Suppose $B : H \to L_2(K, H)$ is a bounded linear operator. The stochastic integral in the see (2) is defined in the following sense ([D-Z.1], Chapter 4):
Set-up: The Itô Integral

Let \( \psi : [0, a] \times \Omega \rightarrow L_2(K, H) \) be jointly measurable, \((\mathcal{F}_t)_{t \geq 0}\)-adapted and

\[
\int_0^a E \| \psi(t) \|_{L_2(K,H)}^2 \, dt < \infty.
\]
Set-up: The Itô Integral

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$$\int_0^a E\|\psi(t)\|^2_{L_2(K,H)} \, dt < \infty.$$ 

Set

$$\int_0^a \psi(t) \, dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k) \, dW^k(t)$$

where the $H$-valued Itô integrals on the right hand side are with respect to the one-dimensional Wiener processes $W^k$, $k \geq 1$. 
The Itô Integral- contd

Series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E \left| \int_{0}^{a} \psi(t)(f_k)\,dW^k(t) \right|^2 = \int_{0}^{a} E \left\| \psi(t) \right\|_{L^2(K,H)}^2 \,dt < \infty.$$
Standing Hypotheses

- Hypothesis (A₁): \[
\sum_{n=1}^{\infty} \mu_n^{-1} \| B(e_n) \|_{L^2(K,H)}^2 < \infty.
\]
Standing Hypotheses

- **Hypothesis (A1):** \[ \sum_{n=1}^{\infty} \mu_n^{-1} \| B(e_n) \|_{L_2(K,H)}^2 < \infty. \]

- **Hypothesis (B):** \( B : H \rightarrow L_2(K, H) \) extends to a bounded linear operator \( B \in L(H, L(E, H)) \); \[ \sum_{k=1}^{\infty} \| B_k \|^2 < \infty, \] where \( B_k \in L(H) \) is defined by
  \[ B_k(x) := B(x)(f_k), \quad x \in H, k \geq 1. \]
Remarks

- Hypothesis $(A_1)$ is implied by the following two requirements:

  (a) The operator $B^!_{H!L_2}(K;H!)$ is Hilbert-Schmidt.

  (b) $\lim \inf n!1 n! > 0$.

Requirement (b) above is satisfied if $A = A$, where $A$ is the Laplacian on a compact smooth $d$-dimensional Riemannian manifold $M$ with boundary, under Dirichlet boundary conditions. No restriction on $\text{dim } M$ under $(A_1)$ for SPDEs.
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- Requirement (b) above is satisfied if \(A = -\Delta\), where \(\Delta\) is the Laplacian on a compact smooth \(d\)-dimensional Riemannian manifold \(M\) with boundary, under Dirichlet boundary conditions.
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- Requirement (b) above is satisfied if \(A = -\Delta\), where \(\Delta\) is the Laplacian on a compact smooth \(d\)-dimensional Riemannian manifold \(M\) with boundary, under Dirichlet boundary conditions.

- No restriction on \(\dim M\) under \((A_1)\) for spdes.
A **mild solution** of the semilinear see (2) is a family of $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H$, $x \in H$, satisfying the following stochastic integral equation:

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} F(u(s, x, \cdot)) \, ds$$

$$+ \int_0^t T_{t-s} B u(s, x, \cdot) \, dW(s), \quad t \geq 0,$$

$$([D-Z.1-2]).$$
The Itô see (2) has the equivalent Stratonovich form

\[
du(t, x) = -Au(t, x) \, dt + F(u(t, x)) \, dt
- \frac{1}{2} \sum_{k=1}^{\infty} B_k^2 u(t, x) \, dt + Bu(t, x) \circ dW(t)
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\[u(0, x) = x \in H\]

where \(B_k \in L(H)\) are given by \(B_k(x) := B(x)(f_k)\), \(x \in H, k \geq 1\).
The Cocycle

\[ k = \text{non-negative integer.} \quad H \text{ real Hilbert.} \]
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A $C^k$ perfect cocycle $(U, \theta)$ on $H$ is a measurable random field $U : \mathbb{R}^+ \times H \times \Omega \to H$ such that:

- For each $\omega \in \Omega$, the map $U(t; x; \omega)$ is continuous in $(t; x)$ in $\mathbb{R}^+ \times H$; for fixed $(t; \omega)$, $U(t; x; \omega)$ is $C^k$ in $x \in H$.
- $U(t_1 + t_2; x; \omega) = U(t_2; U(t_1; x; \omega); \omega)$ for all $t_1, t_2 \in \mathbb{R}^+$, all $\omega \in \Omega$.
- $U(0; x; \omega) = x$ for all $x \in H; \omega \in \Omega$. 
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- \( U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega) \)
for all \( t_1, t_2 \in \mathbb{R}^+ \), all \( \omega \in \Omega. \)
The Cocycle

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- For each \( \omega \in \Omega \), the map \( U(t, x, \omega) \) is continuous in \( (t, x) \in \mathbb{R}^+ \times H \); for fixed \( (t, \omega) \in \mathbb{R}^+ \times \Omega \), \( U(t, x, \omega) \) is \( C^k \) in \( x \in H \).

- \( U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega) \)
  for all \( t_1, t_2 \in \mathbb{R}^+ \), all \( \omega \in \Omega \).

- \( U(0, x, \omega) = x \) for all \( x \in H, \omega \in \Omega \).
The Cocycle Property

\[ U(t_1, \cdot, \omega) \rightarrow \theta(t_1, \cdot) \rightarrow U(t_1, x, \omega) \]

\[ U(t_2, \cdot, \theta(t_1, \omega)) \rightarrow \theta(t_2, \cdot) \rightarrow U(t_1 + t_2, x, \omega) \]
Existence of the Cocycle

Theorem 1:

Under Hypotheses (B) and (A_1), the see (2) (or (3)) admits a perfect jointly measurable $C^1$ cocycle $(U, \theta)$ where

$$U : \mathbb{R}^+ \times H \times \Omega \rightarrow H.$$
Existence of the Cocycle

**Theorem 1:**

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U : \mathbb{R}^+ \times H \times \Omega \rightarrow H.
\]

**Proof of Theorem 1:**

([M-Z-Z], Theorem 1.2.6); cf. [F.1-2].
An $\mathcal{F}$-measurable random variable $Y : \Omega \rightarrow H$ is said to be a **stationary point** for the cocycle $(U, \theta)$ if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$. 
Stationary Points

An $\mathcal{F}$-measurable random variable $Y : \Omega \rightarrow H$ is said be a stationary point for the cocycle $(U, \theta)$ if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$.

A stationary point of the see (2) corresponds to a stationary solution to the anticipating Stratonovich see (1).
Malliavin Regularity

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are $p$-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).
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We now state the main substitution theorem in this talk.
**Theorem 2:** (The Substitution Theorem)

Assume Hypotheses (B) and ($A_1$). Let $U : \mathbb{R}^+ \times H \times \Omega \rightarrow H$ be the $C^1$ cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y), \ t \geq 0$, is a mild solution of the (anticipating) Stratonovich see
Theorem 2: (The Substitution Theorem)

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$$
\begin{aligned}
\frac{dv(t)}{dt} &= -Av(t)\,dt + F_0(v(t))\,dt \\
&\quad + Bv(t) \circ dW(t), \ t > 0,
\end{aligned}
$$

(1)

where $F_0 = F - \frac{1}{2} \sum_{k=1}^{\infty} B_k^2$. 

Substitution
In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is a stationary point of the see (2) (or (3)), then $U(t, Y) = Y(\theta(t))$, $t \geq 0$, is a stationary solution of the (anticipating) Stratonovich see (1):
In particular, if $Y \in D^{1,4}(\Omega, H)$ is a stationary point of the see (2) (or (3)), then $U(t, Y) = Y(\theta(t)), \ t \geq 0$, is a stationary solution of the (anticipating) Stratonovich see (1):

$$
dY(\theta(t)) = -AY(\theta(t)) \ dt + F_0(Y(\theta(t))) \ dt + BY(\theta(t)) \circ dW(t), \ t > 0,
$$

$$
Y(\theta(0)) = Y.
$$

(4)
Substitution Theorem-contd

Furthermore, assume that \( F_0 \) is \( C^2_b \). Then the linearized cocycle \( DU(t, Y) \) is a mild solution of the linearized anticipating see

\[
\frac{d DU(t, Y)}{dt} = ADU(t, Y) dt + DF_0(U(t, Y)) DU(t, Y) dt + B DU(t, Y) dW(t), \quad t > 0; \\
DU(0, Y) = id L(H).
\]
Furthermore, assume that $F_0$ is $C^2_b$. Then the linearized cocycle $DU(t, Y)$ is a mild solution of the linearized anticipating see

\[
dDU(t, Y) = -ADU(t, Y) \, dt \\
+ DF_0(U(t, Y)) \, DU(t, Y) \, dt \\
+ \{ B \circ DU(t, Y) \} \circ dW(t), \; t > 0,
\]

\[DU(0, Y) = \text{id}_{L(H)}.\]

(5)
Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
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- Lift linear see to the Hilbert space $L_2(H)$. 
Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
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  - Use chaos-type expansion in $L_2(H)$.
Outline of Proof

Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):

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- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.
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- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.

- Derive moment estimates for the nonlinear cocycle, its Fréchet and Malliavin derivatives.
Outline of Proof-Contd

- Prove the substitution theorem when $Y$ is replaced by its finite-dimensional projections $Y_n$: Use finite-dimensional projections to smooth out the semigroup $T_t$ in $t$, and apply finite-dimensional substitution techniques.
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Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.
Outline of Proof-Contd

- Prove the substitution theorem when $Y$ is replaced by its finite-dimensional projections $Y_n$: Use finite-dimensional projections to smooth out the semigroup $T_t$ in $t$, and apply finite-dimensional substitution techniques.

- Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.

- Take $n$ to $\infty$ via the moment estimates on the cocycle, its Fréchet and Malliavin derivatives and Dominated Convergence.
Existence of semiflows for mild solutions of linear see:

\[
\begin{align*}
du(t, x, \cdot) &= -Au(t, x, \cdot) \, dt \\
&\quad + Bu(t, x, \cdot) \, dW(t), \quad t > 0 \\
u(0, x, \omega) &= x \in H.
\end{align*}
\]
Existence of semiflows for mild solutions of linear see:

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\[ u(0, x, \omega) = x \in H. \]

\[ A : D(A) \subset H \to H \] closed linear operator on a separable real Hilbert space \( H \).
Linear SEE

Existence of semiflows for mild solutions of linear see:

\[ du(t, x, \cdot) = -Au(t, x, \cdot) \, dt \]
\[ + Bu(t, x, \cdot) \, dW(t), \quad t > 0 \]
\[ u(0, x, \omega) = x \in H. \]

\[ A : D(A) \subset H \rightarrow H \] closed linear operator on a separable real Hilbert space \( H \).

e.g. \( A = -\Delta \) on compact smooth Riemannian manifold.
Mild Solutions: Linear Case

A mild solution of the linear see is a family of jointly measurable, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes

\[ u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \rightarrow H, \ x \in H \]

such that

\[ u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} Bu(s, x, \cdot) \, dW(s), \quad t \geq 0. \]

Integral equation holds \( x\text{-almost surely}, \ x \in H. \)
A mild solution of the linear see is a family of jointly measurable, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes

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\[ u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot) \, dW(s), \quad t \geq 0. \]

Integral equation holds \( x\)-almost surely, \( x \in H \).

Is \( u(t, x, \cdot) \) pathwise continuous linear in \( x \)?
Kolmogorov Fails!

Kolmogorov’s continuity theorem fails for random field $I : L^2([0, 1], \mathbb{R}) \to L^2(\Omega, \mathbb{R})$

$$I(x) := \int_0^1 x(t) \, dW(t), \quad x \in L^2([0, 1], \mathbb{R}).$$
Kolmogorov Fails!

Kolmogorov’s continuity theorem fails for random field $I : L^2([0, 1], \mathbb{R}) \to L^2(\Omega, \mathbb{R})$

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No continuous (or even measurable linear!) selection

$$L^2([0, 1], \mathbb{R}) \times \Omega \to \mathbb{R}$$

$$(x, \omega) \mapsto I(x, \omega)$$

of $I$ ([Mo.1], pp. 144-148).
Lifting semigroup $T_t$, $t \geq 0$, to a strongly continuous semigroup of bounded linear operators

$$\tilde{T}_t : L_2(K, H) \to L_2(K, H), t \geq 0,$$

via composition

$$\tilde{T}_t(C) := T_t \circ C, \ C \in L_2(K, H), t \geq 0.$$
Lifting

- Lift semigroup $T_t$, $t \geq 0$, to a strongly continuous semigroup of bounded linear operators
  \[ \tilde{T}_t : L_2(K, H) \rightarrow L_2(K, H), t \geq 0, \] via composition
  \[ \tilde{T}_t(C) := T_t \circ C, \quad C \in L_2(K, H), t \geq 0. \]

- Lift stochastic integral

\[
\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\}) \, dW(s), \quad x \in H, \quad t \geq 0,
\]

(to $L_2(H)$ for adapted square-integrable $v : \mathbb{R}^+ \times \Omega \rightarrow L_2(H)$). Denote lifting by

\[
\int_0^t T_{t-s}Bv(s) \, dW(s) \in L_2(H).
\]
That is:

\[
\left[ \int_0^t T_{t-s} Bv(s) \, dW(s) \right](x) = \\
\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\}) \, dW(s)
\]

for all \( t \geq 0, \ x-a.s. \)
The Linear Flow

**Theorem 3:**
Assume hypothesis (B) and (A₁). Then the mild solution of the linear see has a Borel (strongly) measurable $(F_t)_{t\geq 0}$-adapted version $\Phi : \mathbb{R}^+ \times \Omega \rightarrow L(H)$ with the following properties:
The Linear Flow

**Theorem 3:**
Assume hypothesis (B) and \((A_1)\). Then the mild solution of the linear see has a Borel (strongly) measurable \((\mathcal{F}_t)_{t \geq 0}\)-adapted version \(\Phi : \mathbb{R}^+ \times \Omega \rightarrow L(H)\) with the following properties:

\[ E \sup_{0 \leq t \leq a} \| \Phi(t, \cdot) \|_{L(H)}^{2p} < \infty, \text{ whenever } p \geq 1. \]
The Linear Flow

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- \(E \sup_{0 \leq t \leq a} \|\Phi(t, \cdot)\|_{L(H)}^{2p} < \infty\), whenever \(p \geq 1\).

- \((\Phi, \theta)\) is a perfect \(L(H)\)-valued cocycle:

\[
\Phi(t + s, \omega) = \Phi(t, \theta(s, \omega)) \circ \Phi(s, \omega)
\]

for all \(s, t \geq 0\) and all \(\omega \in \Omega\);
The Linear Flow

**Theorem 3:**
Assume hypothesis (B) and \((A_1)\). Then the mild solution of the linear see has a Borel (strongly) measurable \((\mathcal{F}_t)_{t \geq 0}\)-adapted version \(\Phi : \mathbb{R}^+ \times \Omega \to L(H)\) with the following properties:

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\Phi(t + s, \omega) = \Phi(t, \theta(s, \omega)) \circ \Phi(s, \omega)
\]

for all \(s, t \geq 0\) and all \(\omega \in \Omega\);
- \(\sup_{0 \leq s \leq t \leq a} \| \Phi(t - s, \theta(s, \omega)) \|_{L(H)} < \infty\), for all \(\omega \in \Omega\).
For each $t > 0$ and almost all $\omega \in \Omega$, 
$\Phi(t, \omega) \in L_2(H)$ has "chaos-type" representation

$$
\Phi(t, \cdot) = T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n} \, dW(s_n)
$$

$$
\cdots \cdots dW(s_2) \, dW(s_1).
$$

Iterated Itô stochastic integrals are lifted integrals in $L_2(H)$, and series converges absolutely in $L_2(H)$.
Consider the semilinear Itô see:

\[ du(t) = -Au(t)dt + F(u(t))dt \]
\[ + Bu(t)\,dW(t), \quad t > 0, \]
\[ u(0) = x \in H \]

(2)
Consider the semilinear Itô see:

\[
du(t) = -Au(t)dt + F(u(t))dt + Bu(t)\,dW(t), \quad t > 0,
\]

\[
u(0) = x \in H
\]

Operators \(A, B\) satisfy hypothesis (B) and (\(A_1\)).

\(F : H \rightarrow H\) is (Fréchet) \(C_b^1\), with linear growth:

\[|F(v)| \leq C(1 + |v|), \quad v \in H\]

for some positive constant \(C\).
Define a *mild solution* of semilinear Itô see (2) as a family of jointly measurable, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes \(u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H, \ x \in H\), satisfying:

\[
u(t, x, \cdot) = T_t(x) + \int_0^t T_{t-s}(F(u(s, x, \cdot))) \, ds \\
+ \int_0^t T_{t-s}Bu(s, x, \cdot) \, dW(s),
\]

for all \(t \geq 0, x\text{-a.s.} \ ([D-Z], \text{Chapter 7, p. 182}).\)
Random Integral Equation

Obtain a $C^k$ perfect cocycle $(U, \theta)$ for mild solutions of the semilinear see, via the random integral equation on $H$:

$$U(t, x, \omega) = \Phi(t, \omega)(x) + \int_0^t \Phi(t - s, \theta(s, \omega))(F(U(s, x, \omega))) \, ds$$

for each $\omega \in \Omega$, $t \geq 0$, $x \in H$. 

Anticipating Semilinear SPDEs – p.41/77
Estimates of the Cocycle

Get new global estimates on the non-linear cocycle $U : \mathbb{R}^+ \times H \times \Omega \to H$, its spatial Fréchet derivative $DU(t, x, \cdot)$ and its Malliavin derivatives $D_u U(t, x, \cdot)$ for $u, t \in [0, a]$ and $x \in H$. Derivations are based on results in [M.Z.Z], Gronwall’s lemma and the fact that $W$ has stationary independent increments.
Estimates of the Cocycle

Get new global estimates on the non-linear cocycle $U : \mathbb{R}^+ \times H \times \Omega \rightarrow H$, its spatial Fréchet derivative $DU(t, x, \cdot)$ and its Malliavin derivatives $\mathcal{D}_u U(t, x, \cdot)$ for $u, t \in [0, a]$ and $x \in H$.

Derivations are based on results in [M.Z.Z], Gronwall’s lemma and the fact that $W$ has stationary independent increments.
Estimates of Cocycle-Contd

**Theorem 4:**

Assume Hypotheses (B), \((A_{1})\) and let \(F_0\) be \(C^1_b\). Let \(U : \mathbb{R}^+ \times H \times \Omega \to H\) be the cocycle generated by the mild solutions of the see (2). Fix any \(a \in (0, \infty)\). Then:

\[
\mathbb{E} \sup_{0 \leq t \leq a} \frac{|U(t, x, \cdot)|^{2p}}{(1 + |x|^{2p}_{H})} < \infty, \quad p \geq 1
\]

\[
\mathbb{E} \sup_{0 \leq t \leq a} \|DU(t, x, \cdot)\|^{2p} < \infty, \quad p \geq 1
\]

\(DU :=\) Fréchet derivative of \(U\) in the spatial variable \(x\).
More Estimates

**Theorem 4**:

*In the see (2), assume Hypotheses (B) and \((A_1)\).*
More Estimates

Theorem 4':

In the see (2), assume Hypotheses (B) and \((A_1)\).

(i) Let \(u, t \in [0, a]\). Define

\[
V(t, \cdot) := \Phi(t, \cdot) - T_t, \quad t \in [0, a].
\]

Then \(V(t, \cdot) \in \mathbb{D}^{1,2p}(\Omega, L_2(H))\) and

\[
E \left[ \sup_{u \leq t \leq a} \|D_u V(t, \cdot)\|_{L_2(H)}^{2p} \right] < \infty.
\]

for all \(p \geq 1\).
More Estimates-contd

(ii) Suppose $F$ is $C^1_b$. Then

$$E \left[ \sup_{0 \leq t \leq a} \frac{\| DU(t, x, \cdot) \|_{H}^{2p}}{(1 + |x|_H^{2p})} \right] < \infty,$$

for all $p \geq 1$. $D :=$ Malliavin derivative.
More Estimates-contd

(ii) Suppose $F$ is $C^{1}_{b}$. Then

$$E \left[ \sup_{0 \leq t \leq a} \frac{\|DU(t, x, \cdot)\|_{H}^{2p}}{(1 + |x|_{H}^{2p})} \right] < \infty,$$

for all $p \geq 1$. $D :=$ Malliavin derivative.

(iii) Let $F$ be $C^{2}_{b}$. Then

$$E \left[ \sup_{0 \leq u, t \leq a} \frac{\|DU(t, x, \cdot)\|_{H}^{2p}}{(1 + |x|_{H}^{2p})} \right] < \infty$$

for all $p \geq 1$. 
Finite-dimensional Projections

**Objective:**

To prove the substitution theorem when the random variable $Y \in D^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on $H$. 

A complete orthonormal system of eigenvectors of $A$ is denoted as $f_i$, and $H_n$ is the $n$-dimensional linear subspace of $H$ spanned by $f_1, \ldots, f_n$, for each $n$. 

Anticipating Semilinear SPDEs
Finite-dimensional Projections

**Objective:**

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on $H$.

$\{e_n : n \geq 1\} :=$ complete orthonormal system of eigenvectors of $A$. 
Finite-dimensional Projections

**Objective:**

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on $H$.

\[ \{e_n : n \geq 1\} := \text{complete orthonormal system of eigenvectors of } A. \]

\[ H_n := L\{e_i : 1 \leq i \leq n\}, \text{ the } n\text{-dimensional linear subspace of } H \text{ spanned by } \{e_i : 1 \leq i \leq n\}, \text{ for each } n \geq 1. \]
Define the projections $P_n : H \to H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$
Define the projections $P_n : H \rightarrow H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$ 

Define $Y_n : \Omega \rightarrow H_n$ by

$$Y_n := P_n \circ Y, \quad n \geq 1.$$ 

Then $Y_n \rightarrow Y$ as $n \rightarrow \infty$ a.s.
**Theorem 5:**

Assume (B) and (A₁) and suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$. Then

$$
\begin{align*}
    dU(t, Y_n) &= -AU(t, Y_n) \, dt + F_0(U(t, Y_n)) \, dt \\
    &\quad + BU(t, Y_n) \circ dW(t), \ t > 0,
\end{align*}
$$

(6)

$$
U(0, Y_n) = Y_n.
$$

for each $n \geq 1$. 
Proof of Theorem 5

Proof still requires Malliavin calculus techniques, largely due to the underlying strongly continuous semi-group dynamics in \( \{T_t\}_{t \geq 0} \).
Proof of Theorem 5

Proof still requires Malliavin calculus techniques, largely due to the underlying strongly continuous semi-group dynamics in \( \{T_t\}_{t \geq 0} \).

Rewrite see in mild Stratonovich form:

\[
U(t, x) = T_t(x) + \int_0^t T_{t-s} F_0(U(s, x)) \, ds \\
+ \int_0^t T_{t-s} B U(s, x) \circ dW(s), \quad t > 0.
\] (3')
Proof of Theorem 5-contd

Sufficient to show that $x$ in (3') can be replaced by $Y_n$:

\[
U(t, Y_n) = T_t(Y_n) + \int_0^t T_{t-s}F_0(U(s, Y_n)) \, ds \\
+ \int_0^t T_{t-s}BU(s, Y_n) \circ dW(s),
\]

(7)

$t > 0, \ n \geq 1.$
Proof of Theorem 5-contd

To prove (7), we show that the random field

$$\int_0^t T_{t-s}BU(s, x) \circ dW(s), \quad x \in H_n,$$

has a version satisfying

$$Z_t = \int_0^t T_{t-s}BU(s, Y_n) \circ dW(s) \quad \text{a.s. for fixed} \quad t > 0.$$
Proof of Theorem 5-contd

To prove (7), we show that the random field

\[ \int_0^t T_{t-s}BU(s, x) \circ dW(s), \quad x \in H_n, \]

has a version satisfying

\[ \int_0^t T_{t-s}BU(s, x) \circ dW(s) \bigg|_{x=Y_n} = \int_0^t T_{t-s}BU(s, Y_n) \circ dW(s) \quad (8) \]

a.s. for fixed \( t > 0 \).
Proof of Theorem 5-contd

To prove (8), fix $m \geq 1$: $H_m$ is invariant under $T_t$. Therefore, $T_{t-s}P_m$ is smooth in $s$. Hence by finite-dimensional substitutions ([Nu.1-2]):

$$
\left. \int_0^t T_{t-s}P_mBU(s, x) \circ dW(s) \right|_{x=Y_n} = \int_0^t T_{t-s}P_mBU(s, Y_n) \circ dW(s)
$$

(9)
a.s. for all $m, n \geq 1$. 

Anticipating Semilinear SPDEs
Proof of Theorem 5-contd

Use global estimates on $U$ to represent the Stratonovich integrals (in (8) and (9)) in terms of Skorohod integrals. Then pass to the limit as $m \to \infty$ in (9), using finite-dimensional substitutions, global estimates on $U$ and dominated convergence.
Proof of Substitution Theorem 2

Step 1:

Suppose \( Y \in \mathbb{D}^{1,4}(\Omega, H) \), and assume Hypothesis (B) and (\( A_1 \)).
Proof of Substitution Theorem 2

**Step 1:**

Suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$, and assume Hypothesis (B) and $(A_1)$.

Sufficient to show

$$U(t, Y) = T_t(Y) + \int_0^t T_{t-s} F_0(U(s, Y)) \, ds$$

$$+ \int_0^t T_{t-s} B U(s, Y) \circ dW(s).$$

(10)
Proof of Theorem 2-contd

**Step 2:**

Pass to the limit as \( n \to \infty \) in the finite-dimensional result:

\[
U(t, Y_n) = T_t(Y_n) + \int_0^t T_{t-s} F_0(U(s, Y_n)) \, ds \\
+ \int_0^t T_{t-s} B U(s, Y_n) \circ dW(s),
\]

\( t > 0, \ n \geq 1. \)
Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0, t] \times \Omega \to H$ such that $v \in L^2([0, t] \times \Omega, H)$, $v(s, \cdot) \in D^{1,2}(\Omega, H)$ for almost all $s \in [0, t]$ and $E[\int_0^t \int_0^t \|D_u v(s, \cdot)\|_H^2 \, du \, ds] < \infty$. 
Denote by $\mathbb{L}^{1,2}$ the class of all processes $\nu : [0, t] \times \Omega \to H$ such that $\nu \in L^2([0, t] \times \Omega, H)$, $\nu(s, \cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0, t]$ and

$$E[\int_0^t \int_0^t ||D_u \nu(s, \cdot)||^2_H du ds] < \infty.$$ 

We say that $\nu$ belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, \nu^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:
Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0, t] \times \Omega \rightarrow H$ such that $v \in L^2([0, t] \times \Omega, H)$, $v(s, \cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0, t]$ and

$$E[\int_0^t \int_0^t \|D_u v(s, \cdot)\|_H^2 \, du \, ds] < \infty.$$ 

We say that $v$ belongs to $\mathbb{L}^{1,2}_{\text{loc}}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \rightarrow \infty$, 

(ii) $v^m \rightarrow v$ in $\mathbb{L}^{1,2}_{\text{loc}}$.
Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0, t] \times \Omega \to H$ such that $v \in L^2([0, t] \times \Omega, H)$, $v(s, \cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0, t]$ and $E[\int_0^t \int_0^t \|D_u v(s, \cdot)\|^2_H du \, ds] < \infty$.

We say that $v$ belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \to \infty$,

(ii) $v = v^m$ on $\Omega_m$. 
Proof of Theorem 2

Step 3:

The Stratonovich integral

\[ \int_0^T T_{t-s} BU(s, Y) \circ dW(s) \]

in (10) is well-defined:
Proof of Theorem 2

Step 3:
The Stratonovich integral

\[ \int_0^t T_{t-s} BU(s, Y) \circ dW(s) \]

in (10) is well-defined:
Sufficient to show that the process

\[ v(s) := T_{t-s} BU(s, Y), s \leq t \]

is in \( \mathbb{L}^{1,2}_{loc} \) ([Nu.2], Theorem 5.2.3).
Proof of Theorem 2

Localyze $v$:
Proof of Theorem 2

Localize $v$:

$m \geq 1$ any integer. $\phi_m \in C^2_b(\mathbb{R}, \mathbb{R})$ a bump function such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. Define

$$v^m(s) := v(s)\phi_m(|Y|_H), \quad s \leq t.$$
Proof of Theorem 2

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Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.
Proof of Theorem 2

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$v^m \in L^{1,2}$ for every $m \geq 1$ because $Y \in D^{1,4}(\Omega, H)$ and the global moment estimates on $U$ and its Fréchet and Malliavin derivatives.
**Proof of Theorem 2**

**Localize v:**

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Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.

$v^m \in L^{1,2}$ for every $m \geq 1$ because $Y \in D^{1,4}(\Omega, H)$ and the global moment estimates on $U$ and its Fréchet and Malliavin derivatives. Hence $v$ is Stratonovich integrable.
Step 4: 

Pass to the limit a.s. as $n \to \infty$ in (7). Get easy a.s. limits:

\[
\lim_{n \to \infty} U(t, Y_n) = U(t, Y)
\]

\[
\lim_{n \to \infty} T_t(Y_n) = T_t(Y)
\]

\[
\lim_{n \to \infty} \int_0^t T_{t-s} F(U(s, Y_n)) \, ds = \int_0^t T_{t-s} F(U(s, Y)) \, ds
\]
and

\[
\lim_{n \to \infty} \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t T_{t-s} B_{k}^{2} U(s, Y_n) \, ds = \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t T_{t-s} B_{k}^{2} U(s, Y) \, ds.
\]
A Not-So-Easy Limit

Step 5:

\[
\lim_{t \to 0} Z_t = Z_{t_0} = Z_t \Rightarrow \Rightarrow \Rightarrow (11)
\]
in probability.

Anticipating Semilinear SPDEs
A Not-So-Easy Limit

**Step 5:**

But following limit is non-trivial:

\[
\lim_{n \to \infty} \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s)
= \int_0^t T_{t-s} BU(s, Y) \circ dW(s)
\]  

in probability.
Proof of Theorem 2-contd

Step 6:

To prove (11), use localization:

\[ Z_t = \int_{0}^{T} \mathbf{B}(s; \mathbf{Y}) m(H_j) dW(s) \]
Proof of Theorem 2-contd

Step 6:

To prove (11), use localization:
Proof of Theorem 2-contd

**Step 6:**

To prove (11), use localization:

\[
\int_0^t T_{t-s} BU(s, Y_n) \circ dW(s) = \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s),
\]

on \( \Omega_m := \{ \omega : |Y(\omega)|_H \leq m \} \);
Proof of Theorem 2-contd

and

$$\int_0^t T_{t-s} BU(s,Y) \circ dW(s)$$

$$= \int_0^t T_{t-s} BU(s,Y) \phi_m(|Y|_H) \circ dW(s)$$

on $\Omega_m$ for any fixed integer $m \geq 1$. 

Anticipating Semilinear SPDEs
Proof of Theorem 2-contd

Step 7:

(11) will follow from

\[
\lim_{n \to \infty} \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s)
\]

\[
= \int_0^t T_{t-s} BU(s, Y) \phi_m(|Y|_H) \circ dW(s)
\]

(12)
in probability for each \( m \geq 1 \).
Proof of Theorem 2-contd

To prove (12), fix $m \geq 1$ and let

$$g_n(s) := T_{t-s} BU(s, Y_n) \phi_m(\|Y\|_H),$$

$$g(s) := T_{t-s} BU(s, Y) \phi_m(\|Y\|_H)$$

for all $s \in [0, t]$. Then

$$\lim_{n \to \infty} E \left[ \int_0^T \|g_n(s) - g(s)\|_{L_2(K,H)}^2 \, ds \right] = 0 \quad (13)$$

$$\lim_{n \to \infty} E \left[ \int_0^T \int_0^T \|D_u g_n(s) - D_u g(s)\|_{L_2(K,H)}^2 \, du \, ds \right] = 0. \quad (14)$$
Proof of Theorem 2-contd

Compute:

\[(D + g)_u := \lim_{s \to u^+} D_u g(s)\]

\[(D - g)_u := \lim_{s \to u^-} D_u g(s)\]

\[(\nabla g)_u := (D + g)_u + (D - g)_u\]
Proof of Theorem 2- contd

Compute:

\[(\mathcal{D}+g)_u := \lim_{s \to u^+} \mathcal{D}_u g(s)\]

\[(\mathcal{D}-g)_u := \lim_{s \to u^-} \mathcal{D}_u g(s)\]

\[(\nabla g)_u := (\mathcal{D}+g)_u + (\mathcal{D}-g)_u\]

and use path continuity to get

\[\lim_{n \to \infty} (\nabla g_n)_u = (\nabla g)_u, \quad a.s.\]
Step 7:

Proof of substitution theorem will be complete if:

\[ \int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s \, ds, \]

for \( n \geq 1 \); and
Step 7:

Proof of substitution theorem will be complete if:

\[
\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s \, ds,
\]

for \( n \geq 1 \); and

\[
\int_0^t g(s) \circ dW(s) = \int_0^t g(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g)_s \, ds
\]

a.s. Skorohod integrals on RHS.
Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: *Lengthy computation.*
Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: Lengthy computation.

Step 8:
Take \( n \to \infty \) in RHS of (15).


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HERZLICHERN GLÜCKWUNSCH

ZUM GEBURTSTAG, HEINRICH!
THE END!
THANK YOU!