Anticipating Semilinear SPDEs [a]

Salah Mohammed [b]

http://sfde.math.siu.edu/

Kaiserslautern: April 4, 2007
Germany

[a] Results to appear in JFA [M-Z]
[b] Department of Mathematics, SIU-C, Carbondale, Illinois, USA

Acknowledgment

- Joint work with T.S. Zhang (Manchester, UK).

Acknowledgment

- Joint work with T.S. Zhang (Manchester, UK).
- Research supported by NSF: DMS-0203368.

Objective

Question:

Objective

Question:

Does the following anticipating stochastic evolution equation (see):

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad + Bv(t)\circ dW(t),\ t>0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

admit a solution with a random initial condition $Y:\Omega \longrightarrow H$ in a Hilbert space H?

Objective

Question:

Does the following anticipating stochastic evolution equation (see):

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad + Bv(t)\circ dW(t),\ t>0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

admit a solution with a random initial condition $Y : \Omega \longrightarrow H$ in a Hilbert space H?

Answer:

Objective

Question:

Does the following anticipating stochastic evolution equation (see):

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad\qquad + Bv(t)\circ dW(t),\ t>0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

admit a solution with a random initial condition $Y:\Omega \longrightarrow H$ in a Hilbert space H?

Answer:

YES! (provided Y is sufficiently regular).

Strategy

- Replace Y in see (1) by a deterministic initial condition x in H and get the corresponding (equivalent) Itô see:

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2}$$

with F a suitably modified non-linear drift.

Strategy

- Replace Y in see (1) by a deterministic initial condition x in H and get the corresponding (equivalent) Itô see:

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2}$$

  with F a suitably modified non-linear drift.

- View the solution of the see (2) as a function (cocycle) $U(t,x,\omega)$ of three variables (t,x,ω) with Fréchet and Malliavin regularity in x and ω (resp.)

Strategy-Contd

- Consider the Stratonovich version of the Itô see (2):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F_0\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\circ dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2'}$$

Strategy-Contd

- Consider the Stratonovich version of the Itô see (2):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F_0\big(u(t,x)\big)\,dt \\ &\qquad\qquad + Bu(t,x)\circ dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2'}$$

- *In the above semilinear see, is it justified to replace the deterministic initial condition* x *by an arbitrary random variable* Y (substitution theorem)?

Strategy-Contd

- Then get back the anticipating Stratonovich see (1) again:

$$\left.\begin{aligned} dU(t,Y) &= -AU(t,Y)\,dt + F_0\big(U(t,Y)\big)\,dt \\ &\qquad + BU(t,Y)\circ dW(t), \quad t>0 \\ U(0,Y) &= Y \end{aligned}\right\} \tag{1}$$

by taking $v(t) := U(t,Y),\ t \geq 0.$

Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde's via substitution theorems ([Nu.1-2], [M-S.2]).

Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde's via substitution theorems ([Nu.1-2], [M-S.2]).
- Known substitution theorems require a level of regularity of the cocycle $U(t, x, \omega)$ in t that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).

Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde's via substitution theorems ([Nu.1-2], [M-S.2]).
- Known substitution theorems require a level of regularity of the cocycle $U(t, x, \omega)$ in t that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).
- Existing substitution theorems work under restrictive finite-dimensional or compactness constraints ([G-Nu-M]).

Difficulties-Contd

- Failure of Kolmogorov's continuity theorem in infinite dimensions ([Mo.1], [Sk]).

Difficulties-Contd

- Failure of Kolmogorov's continuity theorem in infinite dimensions ([Mo.1], [Sk]).
- Failure of Sobolev inequalities in infinite dimensions.

Approach

- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).

Approach

- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).
- Develop global estimates on the semiflow generated by the spde.

Approach

- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).
- Develop global estimates on the semiflow generated by the spde.
- Use ideas and techniques of the Malliavin calculus: Assume Malliavin regularity of the initial condition -rather than imposing finite-dimensional or compactness restrictions on the values of the initial random condition.

Approach

- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).
- Develop global estimates on the semiflow generated by the spde.
- Use ideas and techniques of the Malliavin calculus: Assume Malliavin regularity of the initial condition -rather than imposing finite-dimensional or compactness restrictions on the values of the initial random condition.
- Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are infinite-dimensional.

Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by sfde's.

Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by sfde's.

Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by sfde's.
Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Expect results in this talk to lead to regularity in distribution of the invariant manifolds for semilinear spde's and sfde's.

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E, \omega(0) = 0$, where E is a real separable Hilbert space.

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E, \omega(0) = 0$, where E is a real separable Hilbert space.
- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R},\ \omega \in \Omega.$$

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E$, $\omega(0) = 0$, where E is a real separable Hilbert space.
- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

- $H :=$ real (separable) Hilbert space, norm $|\cdot|_H$.

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E$, $\omega(0) = 0$, where E is a real separable Hilbert space.
- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R},\ \omega \in \Omega.$$

- $H :=$ real (separable) Hilbert space, norm $|\cdot|_H$.
- $\mathcal{B}(H) :=$ Borel σ-algebra of H.

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E$, $\omega(0) = 0$, where E is a real separable Hilbert space.
- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \ \omega \in \Omega.$$

- $H :=$ real (separable) Hilbert space, norm $|\cdot|_H$.
- $\mathcal{B}(H) :=$ Borel σ-algebra of H.
- $L(H) :=$ Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$.

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.

- $$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbf{R};$$
  $\{f_k : k \geq 1\} :=$ complete orthonormal basis of K; $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4).

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.
- $$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbf{R};$$
  $\{f_k : k \geq 1\} :=$ complete orthonormal basis of K; $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4). Series converges absolutely in E but not necessarily in K.

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.
- $$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbf{R};$$
  $\{f_k : k \geq 1\} :=$ complete orthonormal basis of K; $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4). Series converges absolutely in E but not necessarily in K.
- (W, θ) is a *helix*:
  $$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$$

Set-up-contd

- $L_2(K, H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2\right]^{1/2}$$

Set-up-contd

- $L_2(K, H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2\right]^{1/2}$$

- $F_0 : H \to H$ is C_b^1.

Set-up-contd

- $L_2(K,H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2\right]^{1/2}$$

- $F_0 : H \to H$ is C_b^1.
- $F := F_0 + \frac{1}{2}\sum_{k=1}^{\infty} B_k^2$, where $B_k \in L(H)$ are given by

$B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$; and $\sum_{k=1}^{\infty} \|B_k\|^2$

converges.

Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2)$$

in H.

Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \quad (2)$$

in H.

$A : D(A) \subset H \to H$ is a closed linear operator on H.

Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2)$$

in H.

$A : D(A) \subset H \to H$ is a closed linear operator on H.

Assume A has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding positive eigenvalues $\{\mu_n, n \geq 1\}$; i.e., $Ae_n = \mu_n e_n,\ n \geq 1$.

The Set-up-contd

Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geq 0$.

The Set-up-contd

Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \rightarrow H$, $t \geq 0$.

$F : H \rightarrow H$ is (Fréchet) C_b^1: F has a globally bounded Fréchet derivative $F : H \rightarrow L(H)$.

The Set-up-contd

Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geq 0$.

$F : H \to H$ is (Fréchet) C_b^1: F has a globally bounded Fréchet derivative $F : H \to L(H)$.

Suppose $B : H \to L_2(K, H)$ is a bounded linear operator. The stochastic integral in the see (2) is defined in the following sense ([D-Z.1], Chapter 4):

Set-up: The Itô Integral

Let $\psi : [0, a] \times \Omega \to L_2(K, H)$ be jointly measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted and

$$\int_0^a E\|\psi(t)\|^2_{L_2(K,H)} \, dt < \infty.$$

Set-up: The Itô Integral

Let $\psi : [0, a] \times \Omega \to L_2(K, H)$ be jointly measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted and

$$\int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt < \infty.$$

Set

$$\int_0^a \psi(t)\, dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k)\, dW^k(t)$$

where the H-valued Itô integrals on the right hand side are with respect to the one-dimensional Wiener processes $W^k,\ k \geq 1$.

The Itô Integral-contd

Series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E\left| \int_0^a \psi(t)(f_k)\, dW^k(t) \right|^2 = \int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt$$
$$< \infty.$$

Standing Hypotheses

- *Hypothesis* (A_1): $\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|^2_{L_2(K,H)} < \infty.$

Standing Hypotheses

- *Hypothesis* (A_1): $\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|^2_{L_2(K,H)} < \infty.$

- *Hypothesis (B)*: $B : H \to L_2(K, H)$ extends to a bounded linear operator $B \in L(H, L(E, H))$; $\sum_{k=1}^{\infty} \|B_k\|^2 < \infty$, where $B_k \in L(H)$ is defined by

$$B_k(x) := B(x)(f_k), x \in H, k \geq 1.$$

Remarks

- Hypothesis (A_1) is implied by the following two requirements:

Remarks

- Hypothesis (A_1) is implied by the following two requirements:
  - (a) The operator $B : H \longrightarrow L_2(K, H)$ is Hilbert-Schmidt.

Remarks

- Hypothesis (A_1) is implied by the following two requirements:
  - (a) The operator $B : H \longrightarrow L_2(K, H)$ is Hilbert-Schmidt.
  - (b) $\liminf_{n\to\infty} \mu_n > 0$.

Remarks

- Hypothesis (A_1) is implied by the following two requirements:
  - (a) The operator $B : H \to L_2(K, H)$ is Hilbert-Schmidt.
  - (b) $\liminf_{n\to\infty} \mu_n > 0$.
- Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.

Remarks

- Hypothesis (A_1) is implied by the following two requirements:
  - (a) The operator $B : H \to L_2(K, H)$ is Hilbert-Schmidt.
  - (b) $\liminf_{n \to \infty} \mu_n > 0$.
- Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.
- No restriction on $\dim M$ under (A_1) for spdes.

Mild Solutions

A mild solution of the semilinear see (2) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \ x \in H$, satisfying the following stochastic integral equation:

$$\begin{aligned} u(t,x,\cdot) = T_t x &+ \int_0^t T_{t-s} F(u(s,x,\cdot))\, ds \\ &+ \int_0^t T_{t-s} B u(s,x,\cdot)\, dW(s), \quad t \geq 0, \end{aligned} \tag{$2'$}$$

([D-Z.1-2]).

Stratonovich Form

The Itô see (2) has the equivalent Stratonovich form

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,x)\,dt + Bu(t,x)\circ dW(t) \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{3}$$

where $B_k \in L(H)$ are given by $B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$.

The Cocycle

$k =$ non-negative integer. H real Hilbert.

The Cocycle

$k =$ non-negative integer. H real Hilbert.

A C^k perfect cocycle (U, θ) on H is a measurable random field $U : \mathbf{R}^+ \times H \times \Omega \to H$ such that:

The Cocycle

$k =$ non-negative integer. H real Hilbert.

A C^k perfect cocycle (U, θ) on H is a measurable random field $U : \mathbf{R}^+ \times H \times \Omega \to H$ such that:

- For each $\omega \in \Omega$, the map $U(t, x, \omega)$ is continuous in $(t, x) \in \mathbf{R}^+ \times H$; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, $U(t, x, \omega)$ is C^k in $x \in H$.

The Cocycle

$k =$ non-negative integer. H real Hilbert.

A C^k perfect cocycle (U, θ) on H is a measurable random field $U : \mathbf{R}^+ \times H \times \Omega \to H$ such that:

- For each $\omega \in \Omega$, the map $U(t, x, \omega)$ is continuous in $(t, x) \in \mathbf{R}^+ \times H$; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, $U(t, x, \omega)$ is C^k in $x \in H$.
- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

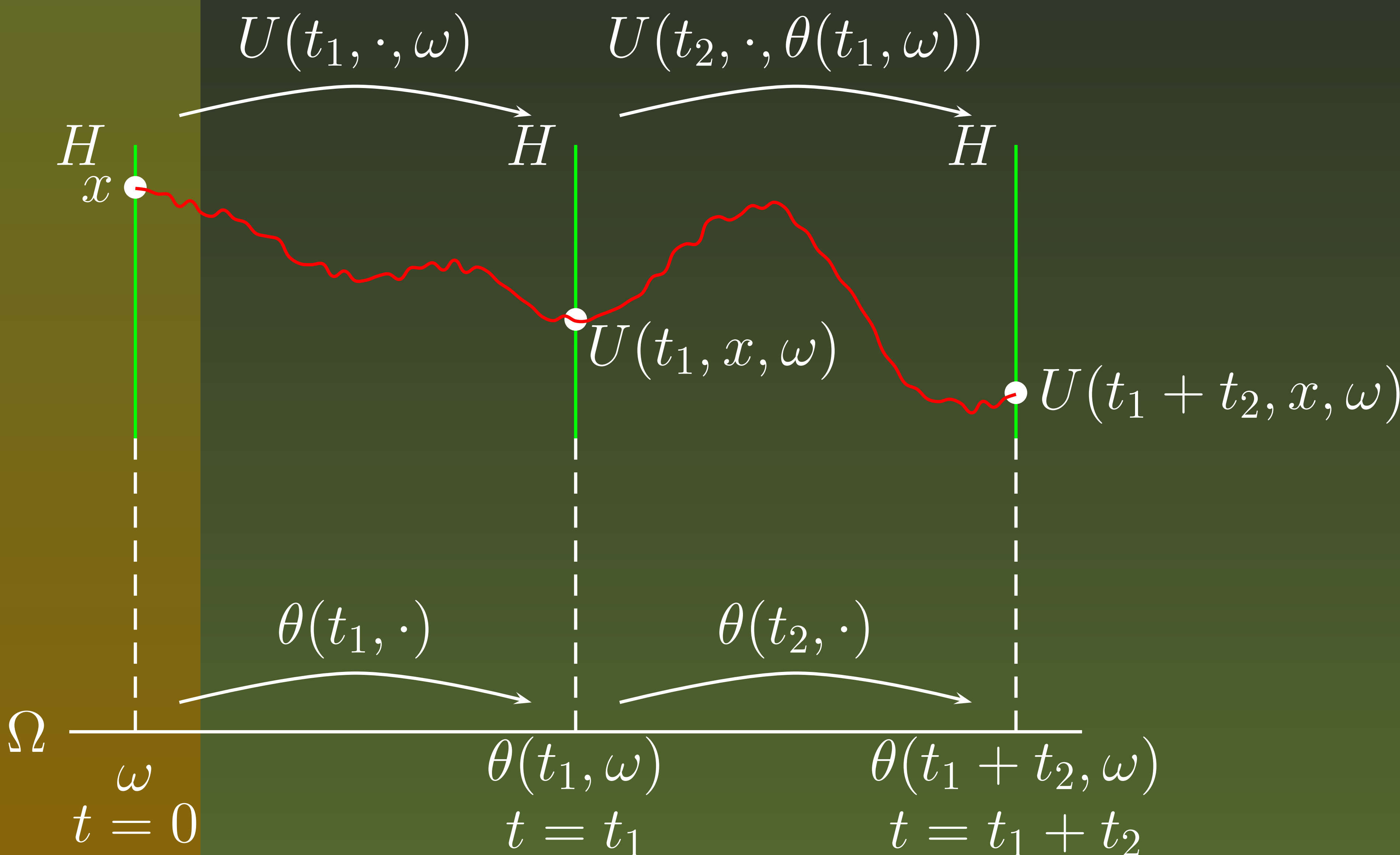
The Cocycle

$k =$ non-negative integer. H real Hilbert.

A C^k perfect cocycle (U, θ) on H is a measurable random field $U : \mathbf{R}^+ \times H \times \Omega \to H$ such that:

- For each $\omega \in \Omega$, the map $U(t, x, \omega)$ is continuous in $(t, x) \in \mathbf{R}^+ \times H$; for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, $U(t, x, \omega)$ is C^k in $x \in H$.
- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.
- $U(0, x, \omega) = x$ for all $x \in H, \omega \in \Omega$.

The Cocycle Property



Existence of the Cocycle

Theorem 1:

Under Hypotheses (B) and (A_1), the see (2) (or (3)) admits a perfect jointly measurable C^1 cocycle (U, θ) where

$$U : \mathbf{R}^+ \times H \times \Omega \to H.$$

Existence of the Cocycle

Theorem 1:

Under Hypotheses (B) and (A_1), the see (2) (or (3)) admits a perfect jointly measurable C^1 cocycle (U, θ) where

$$U : \mathbf{R}^+ \times H \times \Omega \to H.$$

Proof of Theorem 1:

([M-Z-Z], Theorem 1.2.6); cf. [F.1-2]. $\square$

Stationary Points

An $\mathcal{F}$-measurable random variable $Y : \Omega \to H$ is said be a stationary point for the cocycle (U, θ) if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $(t, \omega) \in \mathbf{R}^+ \times \Omega$.

Stationary Points

An $\mathcal{F}$-measurable random variable $Y : \Omega \to H$ is said be a stationary point for the cocycle (U, θ) if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $(t, \omega) \in \mathbf{R}^{+} \times \Omega$.

A stationary point of the see (2) corresponds to a stationary solution to the anticipating Stratonovich see (1).

Malliavin Regularity

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are p-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).

Malliavin Regularity

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are p-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).

We now state the main substitution theorem in this talk.

Substitution

Theorem 2: (The Substitution Theorem)

Assume Hypotheses (B) and (A_1). Let $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the C^1 cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y)$, $t \geq 0$, is a mild solution of the (anticipating) Stratonovich see

Substitution

Theorem 2: (The Substitution Theorem)

Assume Hypotheses (B) and (A_1). Let $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the C^1 cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y)$, $t \geq 0$, is a mild solution of the (anticipating) Stratonovich see

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad + Bv(t) \circ dW(t),\ t > 0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

where $F_0 = F - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2$.

Substitution Theorem-contd

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (2) (or (3)), then* $U(t, Y) = Y\big(\theta(t)\big),\ t \geq 0$, *is a stationary solution of the (anticipating) Stratonovich see (1):*

Substitution Theorem-contd

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (2) (or (3)), then* $U(t, Y) = Y\big(\theta(t)\big),\ t \geq 0$, *is a stationary solution of the (anticipating) Stratonovich see (1):*

$$\left.\begin{aligned} dY(\theta(t)) &= -AY(\theta(t))\,dt + F_0\big(Y(\theta(t))\big)\,dt \\ &\qquad + BY(\theta(t)) \circ dW(t), t > 0, \\ Y(\theta(0)) &= Y. \end{aligned}\right\} \qquad (4)$$

Substitution Theorem-contd

Furthermore, assume that F_0 *is* C_b^2*. Then the linearized cocycle* $DU(t, Y)$ *is a mild solution of the linearized anticipating see*

Substitution Theorem-contd

Furthermore, assume that F_0 is C_b^2. Then the linearized cocycle $DU(t,Y)$ is a mild solution of the linearized anticipating see

$$\left.\begin{aligned} dDU(t,Y) &= -ADU(t,Y)\,dt \\ &\quad + DF_0\big(U(t,Y)\big)DU(t,Y)\,dt \\ &\quad + \{B \circ DU(t,Y)\} \circ dW(t),\ t > 0, \\ DU(0,Y) &= \mathrm{id}_{L(H)}. \end{aligned}\right\} \tag{5}$$

Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):

Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
  - Lift linear see to the Hilbert space $L_2(H)$.

Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
  - Lift linear see to the Hilbert space $L_2(H)$.
  - Use chaos-type expansion in $L_2(H)$

Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
  - Lift linear see to the Hilbert space $L_2(H)$.
  - Use chaos-type expansion in $L_2(H)$
  - Prove convergence of the expansion in $L^{2p}(\Omega, L_2(H))$ via repeated application of moment estimates of the Itô integral

Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
  - Lift linear see to the Hilbert space $L_2(H)$.
  - Use chaos-type expansion in $L_2(H)$
  - Prove convergence of the expansion in $L^{2p}(\Omega, L_2(H))$ via repeated application of moment estimates of the Itô integral
- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.

Outline of Proof

- Construct a linear cocycle for the linear Itô see (with $F \equiv 0$):
  - Lift linear see to the Hilbert space $L_2(H)$.
  - Use chaos-type expansion in $L_2(H)$
  - Prove convergence of the expansion in $L^{2p}(\Omega, L_2(H))$ via repeated application of moment estimates of the Itô integral
- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.
- Derive moment estimates for the nonlinear cocycle, its Fréchet and Malliavin derivatives.

Outline of Proof-Contd

- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.

Outline of Proof-Contd

- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.
- Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.

Outline of Proof-Contd

- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.
- Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.
- Take n to ∞ via the moment estimates on the cocycle, its Fréchet and Malliavin derivatives and Dominated Convergence.

Linear SEE

Existence of semiflows for mild solutions of linear see:

$$
\begin{aligned}
du(t,x,\cdot) &= -Au(t,x,\cdot)\,dt \\
&\quad + Bu(t,x,\cdot)\,dW(t), \quad t > 0 \\
u(0,x,\omega) &= x \in H.
\end{aligned}
$$

Linear SEE

Existence of semiflows for mild solutions of linear see:

$$\begin{aligned} du(t,x,\cdot) &= -Au(t,x,\cdot)\,dt \\ &\quad + Bu(t,x,\cdot)\,dW(t), \quad t>0 \\ u(0,x,\omega) &= x \in H. \end{aligned}$$

$A : D(A) \subset H \to H$ closed linear operator on a separable real Hilbert space H.

Linear SEE

Existence of semiflows for mild solutions of linear see:

$$du(t,x,\cdot) = -Au(t,x,\cdot)\,dt + Bu(t,x,\cdot)\,dW(t), \quad t > 0$$

$$u(0,x,\omega) = x \in H.$$

$A : D(A) \subset H \to H$ closed linear operator on a separable real Hilbert space H.

e.g. $A = -\Delta$ on compact smooth Riemannian manifold.

Mild Solutions: Linear Case

A *mild solution* of the linear see is a family of jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes

$$u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \; x \in H$$

such that

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot)\, dW(s), \quad t \geq 0.$$

Integral equation holds x-almost surely, $x \in H$.

Mild Solutions: Linear Case

A *mild solution* of the linear see is a family of jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes

$$u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \; x \in H$$

such that

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot) \, dW(s), \quad t \geq 0.$$

Integral equation holds x-almost surely, $x \in H$.

Is $u(t, x, \cdot)$ pathwise continuous linear in x?

Kolmogorov Fails!

Kolmogorov's continuity theorem fails for random field $I : L^2([0,1], \mathbf{R}) \to L^2(\Omega, \mathbf{R})$

$$I(x) := \int_0^1 x(t)\, dW(t), \quad x \in L^2([0,1], \mathbf{R}).$$

Kolmogorov Fails!

Kolmogorov's continuity theorem fails for random field $I : L^2([0,1],\mathbf{R}) \to L^2(\Omega,\mathbf{R})$

$$I(x) := \int_0^1 x(t)\, dW(t), \quad x \in L^2([0,1],\mathbf{R}).$$

No continuous (or even measurable linear!) selection

$$L^2([0,1],\mathbf{R}) \times \Omega \to \mathbf{R}$$

$$(x,\omega) \mapsto I(x,\omega)$$

of I ([Mo.1], pp. 144-148).

Lifting

- Lift semigroup $T_t, t \geq 0$, to a strongly continuous semigroup of bounded linear operators $\tilde{T}_t : L_2(K,H) \to L_2(K,H), t \geq 0$, via composition $\tilde{T}_t(C) := T_t \circ C,\ C \in L_2(K,H), t \geq 0$.

Lifting

- Lift semigroup $T_t, t \geq 0$, to a strongly continuous semigroup of bounded linear operators $\tilde{T}_t : L_2(K,H) \to L_2(K,H), t \geq 0$, via composition $\tilde{T}_t(C) := T_t \circ C, \; C \in L_2(K,H), t \geq 0$.
- Lift stochastic integral

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s), \; x \in H, \, t \geq 0,$$

to $L_2(H)$ for adapted square-integrable $v : \mathbf{R}^+ \times \Omega \to L_2(H)$. Denote lifting by

$$\int_0^t T_{t-s} B v(s)\, dW(s) \in L_2(H).$$

Lifting-contd

That is:

$$\left[\int_0^t T_{t-s} B v(s)\, dW(s)\right](x) =$$

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$$

for all $t \geq 0$, x-a.s..

The Linear Flow

Theorem 3:
Assume hypothesis (B) and (A_1). Then the mild solution of the linear see has a Borel (strongly) measurable $(\mathcal{F}_t)_{t\geq 0}$-adapted version $\Phi : \mathbf{R}^+ \times \Omega \to L(H)$ with the following properties:

The Linear Flow

Theorem 3:

Assume hypothesis (B) and (A_1). Then the mild solution of the linear see has a Borel (strongly) measurable $(\mathcal{F}_t)_{t\geq 0}$-adapted version $\Phi : \mathbf{R}^+ \times \Omega \to L(H)$ with the following properties:

- $E \sup_{0\leq t\leq a} \|\Phi(t,\cdot)\|^{2p}_{L(H)} < \infty$, *whenever* $p \geq 1$.

The Linear Flow

Theorem 3:
Assume hypothesis (B) and (A_1). Then the mild solution of the linear see has a Borel (strongly) measurable $(\mathcal{F}_t)_{t\geq 0}$-adapted version $\Phi : \mathbf{R}^+ \times \Omega \to L(H)$ with the following properties:

- $E \sup\limits_{0\leq t\leq a} \|\Phi(t,\cdot)\|_{L(H)}^{2p} < \infty$, *whenever* $p \geq 1$.
- (Φ, θ) *is a perfect* $L(H)$*-valued cocycle:*

$$\Phi(t+s,\omega) = \Phi(t,\theta(s,\omega)) \circ \Phi(s,\omega)$$

for all $s,t \geq 0$ *and all* $\omega \in \Omega$;

The Linear Flow

Theorem 3:
Assume hypothesis (B) and (A_1)*. Then the mild solution of the linear see has a Borel (strongly) measurable* $(\mathcal{F}_t)_{t\geq 0}$*-adapted version* $\Phi : \mathbf{R}^+ \times \Omega \to L(H)$ *with the following properties:*

- $E \sup_{0\leq t\leq a} \|\Phi(t,\cdot)\|_{L(H)}^{2p} < \infty$, *whenever* $p \geq 1$.
- (Φ, θ) *is a perfect* $L(H)$*-valued cocycle:*

$$\Phi(t+s,\omega) = \Phi(t,\theta(s,\omega)) \circ \Phi(s,\omega)$$

  for all $s, t \geq 0$ *and all* $\omega \in \Omega$*;*
- $\sup_{0\leq s\leq t\leq a} \|\Phi(t-s,\theta(s,\omega))\|_{L(H)} < \infty$, *for all* $\omega \in \Omega$.

Linear Flow-Contd: “Chaos"!

- *For each* $t > 0$ *and almost all* $\omega \in \Omega$, $\Phi(t,\omega) \in L_2(H)$ *has “chaos-type” representation*

$$\Phi(t,\cdot) = T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n}\, dW(s_n) \cdots dW(s_2)\, dW(s_1).$$

Iterated Itô stochastic integrals are lifted integrals in $L_2(H)$, *and series converges absolutely in* $L_2(H)$.

Semilinear SEE

Consider the semilinear Itô see:

$$\left.\begin{aligned} du(t) &= -Au(t)dt + F(u(t))dt \\ &\qquad\qquad + Bu(t)\,dW(t), \quad t > 0, \\ u(0) &= x \in H \end{aligned}\right\} \qquad (2)$$

Semilinear SEE

Consider the semilinear Itô see:

$$\left.\begin{aligned} du(t) &= -Au(t)dt + F(u(t))dt \\ &\qquad + Bu(t)\,dW(t), \quad t > 0, \\ u(0) &= x \in H \end{aligned}\right\} \tag{2}$$

Operators A, B satisfy hypothesis (B) and (A_1).
$F : H \to H$ is (Fréchet) C_b^1, with linear growth:

$$|F(v)| \leq C(1 + |v|), \quad v \in H$$

for some positive constant C.

Mild Solution: Semilinear SEE

Define a *mild solution* of semilinear Itô see (2) as a family of jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \;\; x \in H$, satisfying:

$$u(t,x,\cdot) = T_t(x) + \int_0^t T_{t-s}(F(u(s,x,\cdot)))\,ds + \int_0^t T_{t-s}Bu(s,x,\cdot)\,dW(s),$$

for all $t \geq 0$, x-a.s. ([D–Z], Chapter 7, p. 182).

Random Integral Equation

Obtain a C^k perfect cocycle (U, θ) for mild solutions of the semilinear see, via the random integral equation on H:

$$U(t,x,\omega) = \Phi(t,\omega)(x) + \int_0^t \Phi(t-s,\theta(s,\omega))(F(U(s,x,\omega)))\,ds$$

for each $\omega \in \Omega$, $t \geq 0$, $x \in H$.

Estimates of the Cocycle

Get new global estimates on the non-linear cocycle $U : \mathbf{R}^{+} \times H \times \Omega \to H$, its spatial Fréchet derivative $DU(t, x, \cdot)$ and its Malliavin derivatives $\mathcal{D}_u U(t, x, \cdot)$ for $u, t \in [0, a]$ and $x \in H$.

Estimates of the Cocycle

Get new global estimates on the non-linear cocycle $U : \mathbf{R}^{+} \times H \times \Omega \to H$, its spatial Fréchet derivative $DU(t, x, \cdot)$ and its Malliavin derivatives $\mathcal{D}_u U(t, x, \cdot)$ for $u, t \in [0, a]$ and $x \in H$.

Derivations are based on results in [M.Z.Z], Gronwall's lemma and the fact that W has stationary independent increments.

Estimates of Cocycle-Contd

Theorem 4:

Assume Hypotheses (B), (A_1) and let F_0 be C_b^1. Let $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the cocycle generated by the mild solutions of the see (2). Fix any $a \in (0, \infty)$. Then:

$$E \sup_{\substack{0 \leq t \leq a \\ x \in H}} \frac{|U(t,x,\cdot)|^{2p}}{(1 + |x|_H^{2p})} < \infty, \quad p \geq 1$$

$$E \sup_{\substack{0 \leq t \leq a \\ x \in H}} \|DU(t,x,\cdot)\|^{2p} < \infty, \quad p \geq 1$$

$DU :=$ Fréchet derivative of U in the spatial variable x.

More Estimates

Theorem 4′:

In the see (2), assume Hypotheses (B) and (A_1).

More Estimates

Theorem 4′:

In the see (2), assume Hypotheses (B) and (A_1).

(i) *Let* $u, t \in [0, a]$. *Define*

$$V(t,\cdot) := \Phi(t,\cdot) - T_t, \quad t \in [0,a].$$

Then $V(t,\cdot) \in \mathbb{D}^{1,2p}(\Omega, L_2(H))$ *and*

$$E\left[\sup_{u\le t\le a} \|\mathcal{D}_u V(t,\cdot)\|^{2p}_{L_2(H)}\right] < \infty.$$

for all $p \geq 1$.

More Estimates-contd

(ii) *Suppose* F *is* C_b^1. *Then*

$$E\left[\sup_{\substack{0\le t\le a\\ x\in H}} \frac{|\mathcal{D}U(t,x,\cdot)|_H^{2p}}{(1+|x|_H^{2p})}\right] < \infty,$$

for all $p \geq 1$. $\mathcal{D}$:= *Malliavin derivative.*

More Estimates-contd

(ii) *Suppose* F *is* C_b^1. *Then*

$$E\left[\sup_{\substack{0\le t\le a\\ x\in H}} \frac{|\mathcal{D}U(t,x,\cdot)|_H^{2p}}{(1+|x|_H^{2p})}\right] < \infty,$$

for all $p \ge 1$. $\mathcal{D} :=$ *Malliavin derivative.*

(iii) *Let* F *be* C_b^2. *Then*

$$E\left[\sup_{\substack{0\le u,t\le a\\ x\in H}} \frac{\|\mathcal{D}_u DU(t,x,\cdot)\|^{2p}}{(1+|x|_H^{2p})}\right] < \infty$$

for all $p \ge 1$.

Finite-dimensional Projections

Objective:

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on H.

Finite-dimensional Projections

Objective:

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on H.

$\{e_n : n \geq 1\} :=$ complete orthonormal system of eigenvectors of A.

Finite-dimensional Projections

Objective:

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on H.

$\{e_n : n \geq 1\} :=$ complete orthonormal system of eigenvectors of A.

$H_n := L\{e_i : 1 \leq i \leq n\}$, the n-dimensional linear subspace of H spanned by $\{e_i : 1 \leq i \leq n\}$, for each $n \geq 1$.

Projections-contd

Define the projections $P_n : H \to H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$

Projections-contd

Define the projections $P_n : H \to H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$

Define $Y_n : \Omega \to H_n$ by

$$Y_n := P_n \circ Y, \quad n \geq 1.$$

Then $Y_n \to Y$ as $n \to \infty$ a.s.

Finite-dimensional Substitution

Theorem 5:

Assume (B) and (A_1) and suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$. Then

$$\left.\begin{aligned} dU(t, Y_n) &= -AU(t, Y_n)\,dt + F_0\big(U(t, Y_n)\big)\,dt \\ &\qquad + BU(t, Y_n) \circ dW(t), t > 0, \\ U(0, Y_n) &= Y_n. \end{aligned}\right\} \qquad (6)$$

for each $n \geq 1$.

Proof of Theorem 5

Proof still requires Malliavin calculus techniques, largely due to the underlying strongly continuous semi-group dynamics in $\{T_t\}_{t\geq 0}$.

Proof of Theorem 5

Proof still requires Malliavin calculus techniques, largely due to the underlying strongly continuous semi-group dynamics in $\{T_t\}_{t\geq 0}$.

Rewrite see in mild Stratonovich form:

$$U(t,x) = T_t(x) + \int_0^t T_{t-s}F_0\big(U(s,x)\big)\,ds + \int_0^t T_{t-s}BU(s,x)\circ dW(s), \quad t>0. \tag{3'}$$

Proof of Theorem 5-contd

Sufficient to show that x in $(3')$ can be replaced by Y_n:

$$\left.\begin{aligned} U(t, Y_n) &= T_t(Y_n) + \int_0^t T_{t-s} F_0\big(U(s, Y_n)\big)\, ds \\ &\quad + \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s), \\ &\qquad t > 0,\ n \geq 1. \end{aligned}\right\} \tag{7}$$

Proof of Theorem 5-contd

To prove (7), we show that the random field

$$\int_0^t T_{t-s} BU(s,x) \circ dW(s), \qquad x \in H_n,$$

has a version satisfying

Proof of Theorem 5-contd

To prove (7), we show that the random field

$$\int_0^t T_{t-s} BU(s,x) \circ dW(s), \quad x \in H_n,$$

has a version satisfying

$$\int_0^t T_{t-s} BU(s,x) \circ dW(s)\Big|_{x=Y_n} = \int_0^t T_{t-s} BU(s,Y_n) \circ dW(s) \quad (8)$$

a.s. for fixed $t > 0$.

Proof of Theorem 5-contd

To prove (8), fix $m \geq 1$: H_m is invariant under T_t. Therefore, $T_{t-s}P_m$ is smooth in s. Hence by finite-dimensional substitutions ([Nu.1-2]):

$$\int_0^t T_{t-s}P_m BU(s,x) \circ dW(s) \Big|_{x=Y_n} = \int_0^t T_{t-s}P_m BU(s,Y_n) \circ dW(s) \tag{9}$$

a.s. for all $m, n \geq 1$.

Proof of Theorem 5-contd

Use global estimates on U to represent the Stratonovich integrals (in (8) and (9)) in terms of Skorohod integrals. Then pass to the limit as $m \to \infty$ in (9), using finite-dimensional substitutions, global estimates on U and dominated convergence. $\square$

Proof of Substitution Theorem 2

Step 1:

Suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$, and assume Hypothesis (B) and (A_1).

Proof of Substitution Theorem 2

Step 1:

Suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$, and assume Hypothesis (B) and (A_1).

Sufficient to show

$$\left.\begin{aligned} U(t,Y) = T_t(Y) + \int_0^t T_{t-s}F_0\big(U(s,Y)\big)\,ds \\ + \int_0^t T_{t-s}BU(s,Y)\circ dW(s). \end{aligned}\right\} \quad (10)$$

Proof of Theorem 2-contd

Step 2:

Pass to the limit as $n \to \infty$ in the finite-dimensional result:

$$\left.\begin{aligned} U(t, Y_n) &= T_t(Y_n) + \int_0^t T_{t-s} F_0\big(U(s, Y_n)\big)\, ds \\ &\quad + \int_0^t T_{t-s} B U(s, Y_n) \circ dW(s), \\ &\qquad t > 0,\ n \geq 1. \end{aligned}\right\} \qquad (7)$$

Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t] \times \Omega \to H$ such that $v \in L^2([0,t] \times \Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s,\cdot)\|_H^2 \, du \, ds] < \infty$.

Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t] \times \Omega \to H$ such that $v \in L^2([0,t] \times \Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s,\cdot)\|_H^2 \, du \, ds] < \infty$.

We say that v belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t] \times \Omega \to H$ such that $v \in L^2([0,t] \times \Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s,\cdot)\|_H^2 \, du \, ds] < \infty$.

We say that v belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \to \infty$,

Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t] \times \Omega \to H$ such that $v \in L^2([0,t] \times \Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s,\cdot)\|_H^2 \, du \, ds] < \infty$.

We say that v belongs to $\mathbb{L}_{loc}^{1,2}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \to \infty$,

(ii) $v = v^m$ on Ω_m.

Proof of Theorem 2

Step 3:

The Stratonovich integral

$$\int_0^t T_{t-s} BU(s,Y) \circ dW(s)$$

in (10) is well-defined:

Proof of Theorem 2

Step 3:

The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (10) is well-defined:
Sufficient to show that the process

$$v(s) := T_{t-s} BU(s, Y), s \leq t$$

is in $\mathbb{L}_{loc}^{1,2}$ ([Nu.2], Theorem 5.2.3).

Proof of Theorem 2

Localize v*:*

Proof of Theorem 2

Localize v*:*

$m \geq 1$ any integer. $\phi_m \in C_b^2(\mathbf{R}, \mathbf{R})$ a bump function such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. Define

$$v^m(s) := v(s)\phi_m(|Y|_H), \; s \leq t.$$

Proof of Theorem 2

Localize v:

$m \geq 1$ any integer. $\phi_m \in C_b^2(\mathbf{R}, \mathbf{R})$ a bump function such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. Define

$$v^m(s) := v(s)\phi_m(|Y|_H), \ s \leq t.$$

Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.

Proof of Theorem 2

Localize v:

$m \geq 1$ any integer. $\phi_m \in C_b^2(\mathbf{R}, \mathbf{R})$ a bump function such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. Define

$$v^m(s) := v(s)\phi_m(|Y|_H), \; s \leq t.$$

Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.
$v^m \in \mathbb{L}^{1,2}$ for every $m \geq 1$ because $Y \in \mathbb{D}^{1,4}(\Omega, H)$ and the global moment estimates on U and its Fréchet and Malliavin derivatives.

Proof of Theorem 2

Localize v:

$m \geq 1$ any integer. $\phi_m \in C_b^2(\mathbf{R}, \mathbf{R})$ a bump function such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. Define

$$v^m(s) := v(s)\phi_m(|Y|_H), \; s \leq t.$$

Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.
$v^m \in \mathbb{L}^{1,2}$ for every $m \geq 1$ because $Y \in \mathbb{D}^{1,4}(\Omega, H)$ and the global moment estimates on U and its Fréchet and Malliavin derivatives.
Hence v is Stratonovich integrable.

Easy Limits

Step 4:

Pass to the limit a.s. as $n \to \infty$ in (7). Get easy a.s. limits:

$$\lim_{n\to\infty} U(t, Y_n) = U(t, Y)$$

$$\lim_{n\to\infty} T_t(Y_n) = T_t(Y)$$

$$\lim_{n\to\infty} \int_0^t T_{t-s} F\big(U(s, Y_n)\big)\, ds$$

$$= \int_0^t T_{t-s} F\big(U(s, Y)\big)\, ds$$

Easy Limits-contd

and

$$\lim_{n\to\infty} \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2 U(s, Y_n)\, ds$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2 U(s, Y)\, ds.$$

A Not-So-Easy Limit

Step 5:

A Not-So-Easy Limit

Step 5:

But following limit is non-trivial:

$$\left.\begin{aligned} &\lim_{n\to\infty} \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s) \\ &\qquad = \int_0^t T_{t-s} BU(s, Y) \circ dW(s) \end{aligned}\right\} \qquad (11)$$

in probability.

Proof of Theorem 2-contd

Step 6:

Proof of Theorem 2-contd

Step 6:

To prove (11), use localization:

Proof of Theorem 2-contd

Step 6:

To prove (11), use localization:

$$\int_0^t T_{t-s} BU(s, Y_n) \circ dW(s)$$

$$= \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s),$$

on $\Omega_m := \{\omega : |Y(\omega)|_H \leq m\};$

Proof of Theorem 2-contd

and

$$\int_0^t T_{t-s} BU(s,Y) \circ dW(s)$$

$$= \int_0^t T_{t-s} BU(s, Y) \phi_m(|Y|_H) \circ dW(s)$$

on Ω_m for any fixed integer $m \geq 1$.

Proof of Theorem 2-contd

Step 7:

(11) will follow from

$$\lim_{n\to\infty}\int_0^t T_{t-s}BU(s,Y_n)\phi_m(|Y|_H)\circ dW(s) = \int_0^t T_{t-s}BU(s,Y)\phi_m(|Y|_H)\circ dW(s) \tag{12}$$

in probability for each $m \geq 1$.

Proof of Theorem 2-contd

To prove (12), fix $m \geq 1$ and let

$$g_n(s) := T_{t-s}BU(s, Y_n)\phi_m(|Y|_H),$$

$$g(s) := T_{t-s}BU(s, Y)\phi_m(|Y|_H)$$

for all $s \in [0, t]$. Then

$$\lim_{n\to\infty} E\left[\int_0^T \|g_n(s) - g(s)\|^2_{L_2(K,H)}\, ds\right] = 0 \qquad (13)$$

$$\lim_{n\to\infty} E\left[\int_0^T \int_0^T \|\mathcal{D}_u g_n(s) - \mathcal{D}_u g(s)\|^2_{L_2(K,H)}\, du\, ds\right] = 0. \qquad (14)$$

Proof of Theorem 2-contd

Compute:

$$(\mathcal{D}_+ g)_u := \lim_{s \to u+} \mathcal{D}_u g(s)$$

$$(\mathcal{D}_- g)_u := \lim_{s \to u-} \mathcal{D}_u g(s)$$

$$(\nabla g)_u := (\mathcal{D}_+ g)_u + (\mathcal{D}_- g)_u$$

Proof of Theorem 2-contd

Compute:

$$(\mathcal{D}_+ g)_u := \lim_{s\to u+} \mathcal{D}_u g(s)$$

$$(\mathcal{D}_- g)_u := \lim_{s\to u-} \mathcal{D}_u g(s)$$

$$(\nabla g)_u := (\mathcal{D}_+ g)_u + (\mathcal{D}_- g)_u$$

and use path continuity to get

$$\lim_{n\to\infty} (\nabla g_n)_u = (\nabla g)_u, \quad a.s.$$

Proof of Theorem 2-contd

Step 7:

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s \, ds, \tag{15}$$

for $n \geq 1$; and

Proof of Theorem 2-contd

Step 7:

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s \, ds, \tag{15}$$

for $n \geq 1$; and

$$\int_0^t g(s) \circ dW(s) = \int_0^t g(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g)_s \, ds \tag{16}$$

a.s.. Skorohod integrals on RHS.

Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: Lengthy computation.

Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: Lengthy computation.

Step 8:

Take $n \to \infty$ in RHS of (15). $\square$

REFERENCES

D-Z.1 Da Prato, G., and Zabczyk, J., *Stochastic Equations in Infi nite Dimensions*, Cambridge University Press (1992).

D-Z.2 Da Prato, G., and Zabczyk, J., *Ergodicity for Infi nite Dimensional Systems*, Cambridge University Press (1996).

G-Nu-S Grorud, A., Nualart, D., and Sanz-Solé, M., Hilbert-valued anticipating stochastic differential equations, *Annales de l'institut Henri Poincaré (B) Probabilits et Statistiques*, 30 no. 1 (1994), 133-161. (<–)

REFERENCES-contd

Ma Malliavin, P., Stochastic calculus of variations and hypoelliptic operators, *Proceedings of the International Conference on Stochastic Differential Equations, Kyoto*, Kinokuniya, 1976, 195-263.

Mo.1 Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Research Notes in Mathematics, no. 99, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).(<–)

REFERENCES-contd

Mo.2 Mohammed, S.-E. A., Non-Linear Flows for Linear Stochastic Delay Equations, *Stochastics,* Vol. l7 #3, (1987), 207–212.

M-S.1 Mohammed, S.-E. A., and Scheutzow, M. K. R., The Stable Manifold Theorem for Nonlinear Stochastic Systems with Memory, Part I: Existence of the Semiflow, *Journal of Functional Analysis*, 205, (2003), 271-305. Part II: The Local Stable Manifold Theorem, *Journal of Functional Analysis*, 206, (2004), 253-306.

REFERENCES-contd

M-S.2 Mohammed, S.-E. A., and Scheutzow, M. K. R., The stable manifold theorem for stochastic differential equations, *The Annals of Probability,* Vol. 27, No. 2, (1999), 615-652. (<–)

M-Z-Z Mohammed, S.-E. A., Zhang, T. S. and Zhao, H. Z., The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations, Part 1: The Stochastic semifbw, Part 2: Existence of stable and unstable manifolds, pp. 98 (2006), *Memoirs of the American Mathematical Society* (to appear).(<–)

REFERENCES-contd

M-Z Mohammed, S.-E. A. and Zhang, T. S., The substitution theorem for semilinear stochastic partial differential equations, *Journal of Functional Analysis* (to appear) (preprint, 2007) (<–)

Nu.1 Nualart, D., *The Malliavin Calculus and Related Topics*, Probability and its Applications, Springer-Verlag (1995).

Nu.2 Nualart, D., *Analysis on Wiener space and anticipating stochastic calculus*, Springer LNM, 1690, Ecole d'Et'e de Probabilit'es de Saint-Flour XXV-1995, ed: P. Bernard (1995).(<–)

REFERENCES-contd

N-P Nualart, D., and Pardoux, E., Stochastic calculus with anticipating integrands, Analysis on Wiener space and anticipating stochastic calculus, *Probab. Th. Rel. Fields* , 78 (1988), 535-581.

Sk Skorohod, A. V., *Random Linear Operators*, Riedel 1984. (<–)

HERZLICHEN GLÜCKWUNSCH

ZUM GEBURTSTAG, HEINRICH!

THE END!

THANK YOU!