Anticipating Semilinear SPDEs

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Anticipating stochastic evolution equation (see):
\[ dv(t) = Av(t) \, dt + F_0 v(t) \, dt + Bv(t) \, dW(t); \ t > 0; \]
v(0) = Y.

Question: does the following equation admit a solution with a random initial condition Y?

Answer: YES! (provided Y is sufficiently regular.)
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Does the following anticipating stochastic evolution equation (see):

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\begin{align*}
    dv(t) &= -Av(t) \, dt + F_0(v(t)) \, dt \\
    &\quad + Bv(t) \circ dW(t), \quad t > 0, \\
    v(0) &= Y
\end{align*}
\]  

admit a solution with a random initial condition \( Y : \Omega \to H \) in a Hilbert space \( H \)?
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$$
\begin{align*}
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&\quad + Bv(t) \circ dW(t), \ t > 0, \\
\end{align*}
\right\} \\
v(0) &= Y
\end{align*}
$$

admit a solution with a random initial condition $Y : \Omega \to H$ in a Hilbert space $H$?

Answer:

YES! (provided $Y$ is sufficiently regular).
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\begin{align*}
dv(t) &= -Av(t) \, dt + F_0(v(t)) \, dt \\
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admit a solution with a random initial condition \( Y : \Omega \to H \) in a Hilbert space \( H \)?

**Answer:**
YES! (provided \( Y \) is sufficiently regular).
Replace $Y$ in see (1) by a **deterministic** initial condition $x$ in $H$ and get the corresponding (equivalent) Itô see:

$$
du(t, x) = -Au(t, x) \, dt + F(u(t, x)) \, dt $$
$$
+ Bu(t, x) \, dW(t), \quad t > 0 $$

$$
u(0, x) = x \in H $$

with $F$ a suitably modified non-linear drift.

(2)
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$$
u(0, x) = x \in H
$$

with $F$ a suitably modified non-linear drift.

View the solution of the see (2) as a function (cocycle) $U(t, x, \omega)$ of three variables $(t, x, \omega)$ with Fréchet and Malliavin regularity in $x$ and $\omega$ (resp.)
Consider the Stratonovich version of the Itô see (2):

\[
du(t, x) = -Au(t, x) \, dt + F_0(u(t, x)) \, dt + Bu(t, x) \, dW(t), \quad t > 0
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(2')
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\[ du(t, x) = -Au(t, x) \, dt + F_0(u(t, x)) \, dt \]
\[ + Bu(t, x) \circ dW(t), \quad t > 0 \]
\[ u(0, x) = x \in H \]

\[ (2') \]

In the above semilinear see, is it justified to replace the deterministic initial condition \( x \) by an arbitrary random variable \( Y \) (substitution theorem)?
Then get back the anticipating Stratonovich see (1) again:

\[
\begin{align*}
    dU(t, Y) &= -AU(t, Y) \, dt + F_0(U(t, Y)) \, dt \\
    &\quad + BU(t, Y) \circ dW(t), \quad t > 0 \\
    U(0, Y) &= Y
\end{align*}
\]

by taking \( v(t) := U(t, Y), \quad t \geq 0. \)
Affirmative answer for the above question is known for a wide class of finite-dimensional sde’s via substitution theorems ([Nu.1-2], [M-S.2]).

Existing substitution theorems work under restrictive finite-dimensional or ( — ) compactness constraints ([G-Nu-S], [A-I]).
Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde’s via substitution theorems ([Nu.1-2], [M-S.2]).

- Known substitution theorems require a level of regularity of the cocycle $U(t, x, \omega)$ in $t$ that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).
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Difficulties-Contd

- Failure of Kolmogorov’s continuity theorem in infinite dimensions ([Mo.1], [Sk]).
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Failure of Sobolev inequalities in infinite dimensions.
Approach

- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).

Develop global estimates on the semiflow generated by the spde. Use ideas and techniques of the Malliavin calculus: Assume Malliavin regularity of the initial condition—rather than imposing finite-dimensional or compactness restrictions on the values of the initial random condition. Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are infinite-dimensional.
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Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see’s near hyperbolic/anticipating stationary states. (Expect hyperbolicity to be a generic property rather than ergodicity of the invariant measure!)
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Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Expect results in this talk to lead to regularity in distribution of the invariant manifolds for semilinear spde’s and sfde’s.
The Set-up

- \((\Omega, \mathcal{F}, P) := \text{Wiener space}\) of all continuous paths \(\omega : \mathbb{R} \to E, \omega(0) = 0\), where \(E\) is a real separable Hilbert space.
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- **Wiener shifts** \(\theta : \mathbb{R} \times \Omega \to \Omega\): Group of \(P\)-preserving ergodic transformations on \((\Omega, \mathcal{F}, P)\):

\[
\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega.
\]
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- \(L(H) := \text{Banach space of all bounded linear operators } H \rightarrow H\) given the uniform operator norm \(\| \cdot \|_{L(H)}\).
Set-up: Brownian Motion

- \( W := E \)-valued Brownian motion \( W : \mathbb{R} \times \Omega \rightarrow E \) with separable covariance Hilbert space \( K \subset E \), Hilbert-Schmidt embedding.
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- $W(t) = \sum_{k=1}^{\infty} W^k(t) f_k$, $t \in \mathbb{R}$;

- $\{f_k : k \geq 1\} :=$ complete orthonormal basis of $K$;
- $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4).
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- $(W, \theta)$ is a helix:

$$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$$
Set-up-contd

- $L_2(K, H) := \text{Hilbert space of all Hilbert-Schmidt operators } S : K \to H$, with norm

$$\| S \|_2 := \left[ \sum_{k=1}^{\infty} |S(f_k)|_H^2 \right]^{1/2}$$
Set-up-contd

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- \( F_0 : H \to H \text{ is } C^1_b. \)
Set-up-contd

- $L_2(K, H) := \text{Hilbert space}$ of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[ \sum_{k=1}^{\infty} |S(f_k)|^2_H \right]^{1/2}$$

- $F_0 : H \to H$ is $C^1_b$.

- $F := F_0 + \frac{1}{2} \sum_{k=1}^{\infty} B_k^2$, where $B_k \in L(H)$ are given by

$$B_k(x) := B(x)(f_k), \quad x \in H, \quad k \geq 1; \quad \text{and} \quad \sum_{k=1}^{\infty} \|B_k\|^2$$

converges.
Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

\[
\begin{aligned}
    du(t, x) &= -Au(t, x) \, dt + F(u(t, x)) \, dt \\
    &\quad + Bu(t, x) \, dW(t), \quad t > 0 \\
    u(0, x) &= x \in H
\end{aligned}
\] (2)

in \( H \).
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\]

in \( H \).

\( A : D(A) \subset H \rightarrow H \) is a closed linear operator on \( H \).
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\[u(0, x) = x \in H\]

in \(H\).

\[A : D(A) \subset H \to H\] is a closed linear operator on \(H\). Assume \(A\) has a complete orthonormal system of eigenvectors \(\{e_n : n \geq 1\}\) with corresponding positive eigenvalues \(\{\mu_n, n \geq 1\}\); i.e., \(Ae_n = \mu_n e_n, \quad n \geq 1\).
Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geq 0$. 
The Set-up-contd

Suppose \(-A\) generates a strongly continuous semigroup of bounded linear operators \(T_t : H \rightarrow H, ~t \geq 0\).

\(F : H \rightarrow H\) is (Fréchet) \(C^1_b\): \(F\) has a globally bounded Fréchet derivative \(F : H \rightarrow L(H)\).
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$F : H \rightarrow H$ is (Fréchet) $C^1_b$: $F$ has a globally bounded Fréchet derivative $F : H \rightarrow L(H)$.

Suppose $B : H \rightarrow L_2(K, H)$ is a bounded linear operator. The stochastic integral in the see (2) is defined in the sense of ([D-Z.1], Chapter 4):
Standing Hypotheses

- **Hypothesis (A₁):** \( \sum_{n=1}^{\infty} \mu_n^{-1} \| B(e_n) \|_{L^2(K,H)}^2 < \infty. \)
Standing Hypotheses

- **Hypothesis (A<sub>1</sub>):** \[ \sum_{n=1}^{\infty} \mu_n^{-1} \| B(e_n) \|_{L_2(K,H)}^2 < \infty. \]

- **Hypothesis (B):** \[ B : H \rightarrow L_2(K, H) \text{ extends to a bounded linear operator } B \in L(H, L(E, H)); \]
  \[ \sum_{k=1}^{\infty} \| B_k \|^2 < \infty, \text{ where } B_k \in L(H) \text{ is defined by} \]
  \[ B_k(x) := B(x)(f_k), x \in H, k \geq 1. \]
Remarks

- Hypothesis \((A_1)\) is implied by the following two requirements:

  \(\text{(a)}\) The operator \(B_{H!L}^2(K;H)\) is Hilbert-Schmidt.

  \(\text{(b)}\) \(\lim\inf_{n \to 1} n > 0\).

Requirement (b) above is satisfied if \(A_1 = \Delta\), where \(\Delta\) is the Laplacian on a compact smooth \(d\)-dimensional Riemannian manifold \(M\) with boundary, under Dirichlet boundary conditions. No restriction on \(\dim M\) under \((A_1)\) for SPDEs.
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- Requirement (b) above is satisfied if \(A = -\Delta\), where \(\Delta\) is the Laplacian on a compact smooth \(d\)-dimensional Riemannian manifold \(M\) with boundary, under Dirichlet boundary conditions.
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No restriction on \(\dim M\) under \((A_1)\) for spdes.
A **mild solution** of the semilinear see (2) is a family of \((\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(H)))\)-measurable, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes \(u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H, \ x \in H\), satisfying the following stochastic integral equation:

\[
u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} F(u(s, x, \cdot)) \, ds \\
+ \int_0^t T_{t-s} B u(s, x, \cdot) \, dW(s), \quad t \geq 0,
\]

((D-Z.1-2)).
The Itô see (2) has the equivalent **Stratonovich** form

\[
\begin{align*}
    du(t, x) &= -Au(t, x) \, dt + F(u(t, x)) \, dt \\
    &\quad - \frac{1}{2} \sum_{k=1}^{\infty} B_k^2 u(t, x) \, dt + Bu(t, x) \circ dW(t)
\end{align*}
\]

\[u(0, x) = x \in H\]

where \(B_k \in L(H)\) are given by \(B_k(x) := B(x)(f_k),\)
\(x \in H, k \geq 1.\)
The Cocycle

**Theorem 1:**

Under Hypotheses (B) and (A₁), the see (2) (or (3)) admits a perfect jointly measurable $C^1$ cocycle $(U, \theta)$, $U : \mathbb{R}^+ \times H \times \Omega \to H$:

$$U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$$

for all $t_1, t_2 \in \mathbb{R}^+$, all $\omega \in \Omega$. 
The Cocycle

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for all $t_1, t_2 \in \mathbb{R}^+$, all $\omega \in \Omega$.

**Proof of Theorem 1:**

([M-Z-Z], Theorem 1.2.6); cf. [F.1-2].

□
The Cocycle Property

\[ U(t_1, \cdot, \omega) \quad U(t_2, \cdot, \theta(t_1, \omega)) \]

\[ U(t_1, x, \omega) \]

\[ \theta(t_1, \cdot) \quad \theta(t_2, \cdot) \]

\[ \Omega \]

\[ \omega \]

\[ t = 0 \quad t = t_1 \quad t = t_1 + t_2 \]

\[ H \]

\[ x \]
Malliavin Regularity

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are $p$-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).
Malliavin Regularity

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We now state the main substitution theorem in this talk.
**Theorem 2:** (The Substitution Theorem)

Assume Hypotheses (B) and (A\(_1\)). Let 
\( U : \mathbb{R}^+ \times H \times \Omega \to H \) be the \( C^1 \) cocycle generated by the see (2). Let \( Y \in \mathbb{D}^{1,4}(\Omega, H) \) be a random variable. Then \( v(t) := U(t, Y) \), \( t \geq 0 \), is a mild solution of the (anticipating) Stratonovich see
**Theorem 2:** (The Substitution Theorem)

Assume Hypotheses (B) and (A$_1$). Let $U : \mathbb{R}^+ \times H \times \Omega \to H$ be the $C^1$ cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y)$, $t \geq 0$, is a mild solution of the (anticipating) Stratonovich see

\[
\begin{align*}
    dv(t) &= -Av(t) \, dt + F_0(v(t)) \, dt \\
    &\quad + Bv(t) \circ dW(t), \; t > 0,
\end{align*}
\]

where $F_0 = F - \frac{1}{2} \sum_{k=1}^{\infty} B_k^2$. 

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In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is a stationary point of the see (2) (or (3)), then $U(t, Y) = Y(\theta(t)), \quad t \geq 0,$ is a stationary solution of the (anticipating) Stratonovich see (1):
In particular, if \( Y \in \mathbb{D}^{1,4}(\Omega, H) \) is a stationary point of the see (2) (or (3)), then \( U(t, Y) = Y(\theta(t)), \ t \geq 0, \) is a stationary solution of the (anticipating) Stratonovich see (1):

\[
\begin{align*}
    dY(\theta(t)) &= -AY(\theta(t)) \, dt + F_0(Y(\theta(t))) \, dt \\
    &\quad + BY(\theta(t)) \circ dW(t), \ t > 0, \\
    Y(\theta(0)) &= Y.
\end{align*}
\]
Substitution Theorem-contd

Furthermore, assume that $F_0$ is $C^2_b$. Then the linearized cocycle $DU(t,Y)$ is a mild solution of the linearized anticipating see
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$$
dDU(t, Y) = -ADU(t, Y)\, dt$$

$$+ DF_0(U(t, Y))\, DU(t, Y)\, dt$$

$$+ \{ B \circ DU(t, Y) \} \circ dW(t), \ t > 0,$$

$$DU(0, Y) = \text{id}_{L(H)}.$$

(5)
Outline of Proof

- Construct a linear cocycle $(\Phi, \theta)$ for the linear Itô see (with $F \equiv 0$):
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  - Prove convergence of the expansion in \(L^{2p}(\Omega, L_2(H))\) via repeated application of moment estimates of the Itô integral.
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  - Prove convergence of the expansion in \(L^{2p}(\Omega, L_2(H))\) via repeated application of moment estimates of the Itô integral.

- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.
Construct a linear cocycle $(\Phi, \theta)$ for the linear Itô see (with $F \equiv 0$):

- Lift linear see to the Hilbert space $L_2(H)$.
- Use chaos-type expansion in $L_2(H)$.
- Prove convergence of the expansion in $L^{2p}(\Omega, L_2(H))$ via repeated application of moment estimates of the Itô integral.

Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.

Derive moment estimates for the nonlinear cocycle, its Fréchet and Malliavin derivatives.
Prove the substitution theorem when $Y$ is replaced by its finite-dimensional projections $Y_n$: Use finite-dimensional projections to smooth out the semigroup $T_t$ in $t$, and apply finite-dimensional substitution techniques.
Outline of Proof-Contd

- Prove the substitution theorem when $Y$ is replaced by its finite-dimensional projections $Y_n$: Use finite-dimensional projections to smooth out the semigroup $T_t$ in $t$, and apply finite-dimensional substitution techniques.

- Use moment estimates on the cocycle to rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term involving Malliavin derivatives of the cocycle.
Outline of Proof-Contd

- Prove the substitution theorem when $Y$ is replaced by its finite-dimensional projections $Y_n$: Use finite-dimensional projections to smooth out the semigroup $T_t$ in $t$, and apply finite-dimensional substitution techniques.

- Use moment estimates on the cocycle to rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term involving Malliavin derivatives of the cocycle.

- Take $n$ to $\infty$ via the moment estimates on the cocycle, its Fréchet and Malliavin derivatives and dominated convergence. □
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