Anticipating Semilinear SPDEs [a]

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$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad + Bv(t)\circ dW(t),\ t>0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

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admit a solution with a random initial condition $Y : \Omega \longrightarrow H$ in a Hilbert space H?

Answer:

YES! (provided Y is sufficiently regular).

Strategy

- Replace Y in see (1) by a deterministic initial condition x in H and get the corresponding (equivalent) Itô see:

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{2}$$

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with F a suitably modified non-linear drift.

- View the solution of the see (2) as a function (cocycle) $U(t,x,\omega)$ of three variables (t,x,ω) with Fréchet and Malliavin regularity in x and ω (resp.)

Strategy-Contd

- Consider the Stratonovich version of the Itô see (2):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F_0\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\circ dW(t), \quad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2')$$

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- *In the above semilinear see, is it justified to replace the deterministic initial condition x by an arbitrary random variable Y* (substitution theorem)?

Strategy-Contd

- Then get back the anticipating Stratonovich see (1) again:

$$\left.\begin{aligned} dU(t,Y) &= -AU(t,Y)\,dt + F_0\big(U(t,Y)\big)\,dt \\ &\qquad + BU(t,Y)\circ dW(t), \quad t>0 \\ U(0,Y) &= Y \end{aligned}\right\} \tag{1}$$

by taking $v(t) := U(t,Y),\ t \geq 0.$

Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde's via substitution theorems ([Nu.1-2], [M-S.2]).

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- Existing substitution theorems work under restrictive finite-dimensional or (σ-)compactness constraints ([G-Nu-S], [A-I]).

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- Failure of Sobolev inequalities in infinite dimensions.

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- Construct Fréchet differentiable stochastic semiflow for the semilinear see (2) using a chaos-type expansion technique ([M-Z-Z]).
- Develop global estimates on the semiflow generated by the spde.
- Use ideas and techniques of the Malliavin calculus: Assume Malliavin regularity of the initial condition -rather than imposing finite-dimensional or compactness restrictions on the values of the initial random condition.
- Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are infinite-dimensional.

Motivation

Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic/anticipating stationary states. (*Expect hyperbolicty to be a generic property rather than ergodicity of the invariant measure!*)

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Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Expect results in this talk to lead to regularity in distribution of the invariant manifolds for semilinear spde's and sfde's.

The Set-up

- $(\Omega, \mathcal{F}, P) :=$ Wiener space of all continuous paths $\omega : \mathbf{R} \to E, \omega(0) = 0$, where E is a real separable Hilbert space.

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- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

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- $H :=$ real (separable) Hilbert space, norm $|\cdot|_H$.
- $\mathcal{B}(H) :=$ Borel σ-algebra of H.
- $L(H) :=$ Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$.

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.

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  $\{f_k : k \geq 1\} :=$ complete orthonormal basis of K; $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4).

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- (W, θ) is a *helix*:
  $$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$$

Set-up-contd

- $L_2(K, H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2 \right]^{1/2}$$

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- $F_0 : H \to H$ is C_b^1.
- $F := F_0 + \frac{1}{2}\sum_{k=1}^{\infty} B_k^2$, where $B_k \in L(H)$ are given by

$B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$; and $\sum_{k=1}^{\infty} \|B_k\|^2$ converges.

Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad + Bu(t,x)\,dW(t), \qquad t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2)$$

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$A : D(A) \subset H \to H$ is a closed linear operator on H.

Assume A has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding positive eigenvalues $\{\mu_n, n \geq 1\}$; i.e., $Ae_n = \mu_n e_n,\ n \geq 1$.

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Suppose $B : H \to L_2(K, H)$ is a bounded linear operator. The stochastic integral in the see (2) is defined in the sense of ([D-Z.1], Chapter 4):

Standing Hypotheses

- *Hypothesis* (A_1): $\displaystyle\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|^2_{L_2(K,H)} < \infty.$

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- *Hypothesis (B)*: $B : H \to L_2(K, H)$ extends to a bounded linear operator $B \in L(H, L(E, H))$; $\sum_{k=1}^{\infty} \|B_k\|^2 < \infty$, where $B_k \in L(H)$ is defined by

$$B_k(x) := B(x)(f_k), x \in H, k \geq 1.$$

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- Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.
- No restriction on $\dim M$ under (A_1) for spdes.

Mild Solutions

A mild solution of the semilinear see (2) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \ x \in H$, satisfying the following stochastic integral equation:

$$
\begin{aligned}
u(t,x,\cdot) = T_t x + \int_0^t T_{t-s} F(u(s,x,\cdot))\, ds \\
+ \int_0^t T_{t-s} B u(s,x,\cdot)\, dW(s), \quad t \geq 0,
\end{aligned}
\tag{2'}
$$

([D-Z.1-2]).

Stratonovich Form

The Itô see (2) has the equivalent Stratonovich form

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ &\qquad - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,x)\,dt + Bu(t,x)\circ dW(t) \\ u(0,x) &= x \in H \end{aligned}\right\} \tag{3}$$

where $B_k \in L(H)$ are given by $B_k(x) := B(x)(f_k)$, $x \in H$, $k \geq 1$.

The Cocycle

Theorem 1:

Under Hypotheses (B) and (A_1), the see (2) (or (3)) admits a perfect jointly measurable C^1 cocycle (U, θ), $U : \mathbf{R}^+ \times H \times \Omega \to H$:

$$U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$$

for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

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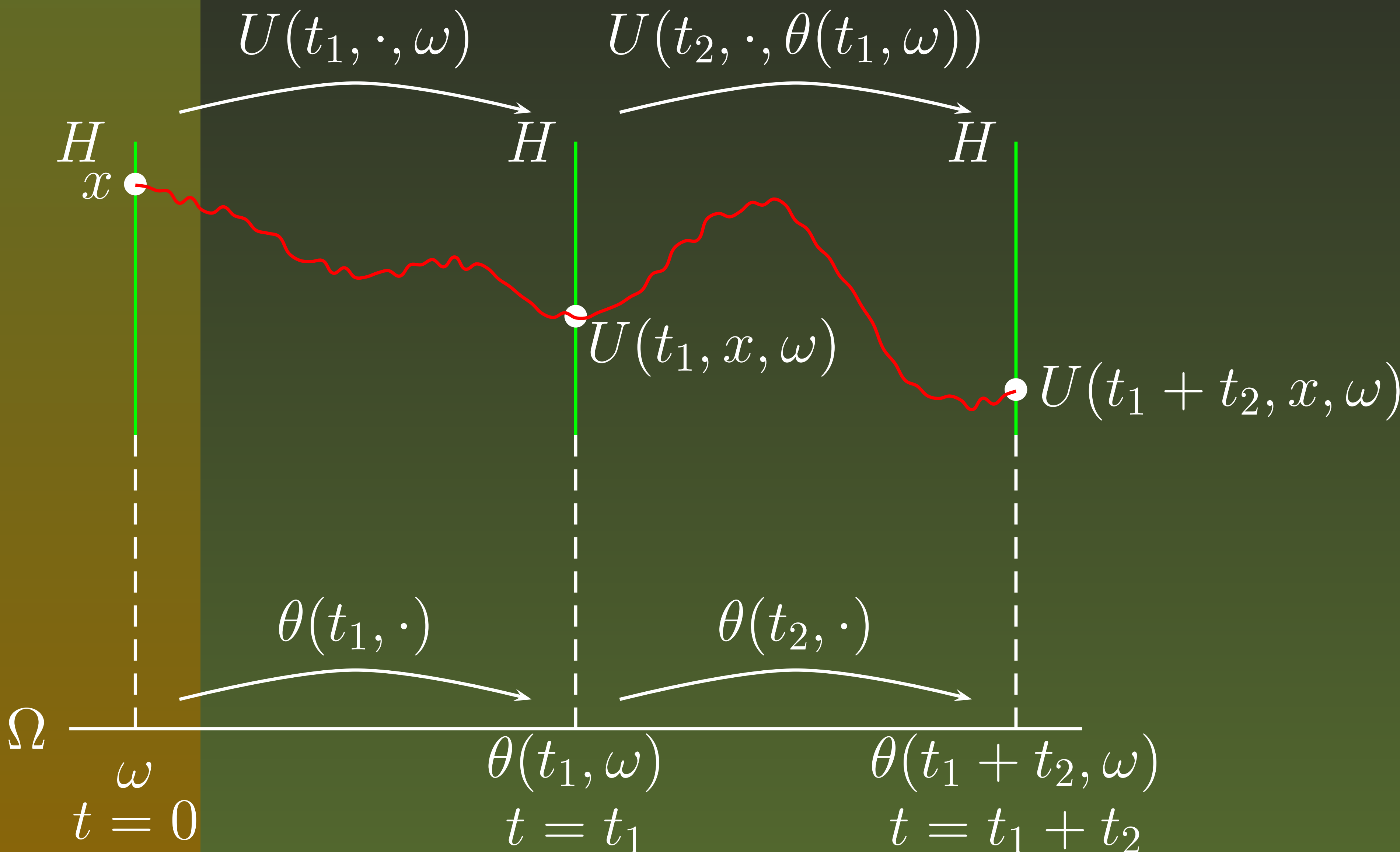
Proof of Theorem 1:

([M-Z-Z], Theorem 1.2.6); cf. [F.1-2]. □

The Cocycle Property



Malliavin Regularity

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are p-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).

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We now state the main substitution theorem in this talk.

Substitution

Theorem 2: (The Substitution Theorem)

Assume Hypotheses (B) and (A_1). Let $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the C^1 cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y)$, $t \geq 0$, is a mild solution of the (anticipating) Stratonovich see

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$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + F_0\big(v(t)\big)\,dt \\ &\qquad\qquad + Bv(t) \circ dW(t),\ t > 0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

where $F_0 = F - \frac{1}{2}\sum_{k=1}^{\infty} B_k^2$.

Substitution Theorem-contd

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (2) (or (3)), then* $U(t, Y) = Y\big(\theta(t)\big),\ t \geq 0$*, is a stationary solution of the (anticipating) Stratonovich see (1):*

Substitution Theorem-contd

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ *is a stationary point of the see (2) (or (3)), then* $U(t, Y) = Y\big(\theta(t)\big),\ t \geq 0$*, is a stationary solution of the (anticipating) Stratonovich see (1):*

$$\left.\begin{aligned} dY(\theta(t)) &= -AY(\theta(t))\,dt + F_0\big(Y(\theta(t))\big)\,dt \\ &\quad + BY(\theta(t)) \circ dW(t), t > 0, \\ Y(\theta(0)) &= Y. \end{aligned}\right\} \qquad (4)$$

Substitution Theorem-contd

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Substitution Theorem-contd

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$$\left.\begin{aligned} dDU(t,Y) &= -ADU(t,Y)\,dt \\ &\quad + DF_0\big(U(t,Y)\big)DU(t,Y)\,dt \\ &\quad + \big\{B \circ DU(t,Y)\big\} \circ dW(t),\ t>0, \\ DU(0,Y) &= \mathrm{id}_{L(H)}. \end{aligned}\right\} \tag{5}$$

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- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.(–>)

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- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.(–>)
- Derive moment estimates for the nonlinear cocycle, its Fréchet and Malliavin derivatives. (–>)

Outline of Proof-Contd

- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.

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- Use moment estimates on the cocycle to rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term involving Malliavin derivatives of the cocycle.
- Take n to ∞ via the moment estimates on the cocycle, its Fréchet and Malliavin derivatives and dominated convergence. $\square$

REFERENCES

A-I Arnold, L., and Imkeller, P., Stratonovich calculus with spatial parameters and anticipative problems in multiplicative ergodic theory, *Stochastic Processes and their Applications,* Vol. 62 (1996), 19–54.(<–)

D-Z.1 Da Prato, G., and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press (1992).

D-Z.2 Da Prato, G., and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press (1996).

REFERENCES-contd

G-Nu-S Grorud, A., Nualart, D., and Sanz-Solé, M., Hilbert-valued anticipating stochastic differential equations, *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, 30 no. 1 (1994), 133-161. (<–)

Ma Malliavin, P., Stochastic calculus of variations and hypoelliptic operators, *Proceedings of the International Conference on Stochastic Differential Equations, Kyoto*, Kinokuniya, 1976, 195-263.

Mo.1 Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Research Notes in Mathematics, no. 99, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).(<–)

REFERENCES-contd

Mo.2 Mohammed, S.-E. A., Non-Linear Flows for Linear Stochastic Delay Equations, *Stochastics,* Vol. l7 #3, (1987), 207–212.

M-S.1 Mohammed, S.-E. A., and Scheutzow, M. K. R., The Stable Manifold Theorem for Nonlinear Stochastic Systems with Memory, Part I: Existence of the Semiflow, *Journal of Functional Analysis*, 205, (2003), 271-305. Part II: The Local Stable Manifold Theorem, *Journal of Functional Analysis*, 206, (2004), 253-306.

REFERENCES-contd

M-S.2 Mohammed, S.-E. A., and Scheutzow, M. K. R., The stable manifold theorem for stochastic differential equations, *The Annals of Probability,* Vol. 27, No. 2, (1999), 615-652. (<–)

M-Z-Z Mohammed, S.-E. A., Zhang, T. S. and Zhao, H. Z., The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations, Part 1: The Stochastic semifbw, Part 2: Existence of stable and unstable manifolds, pp. 98 (2006), *Memoirs of the American Mathematical Society* (to appear).(<–)

REFERENCES-contd

M-Z Mohammed, S.-E. A. and Zhang, T. S., The substitution theorem for semilinear stochastic partial differential equations, *Journal of Functional Analysis* (to appear) (preprint, 2007) (<–)

Nu.1 Nualart, D., *The Malliavin Calculus and Related Topics*, Probability and its Applications, Springer-Verlag (1995).

Nu.2 Nualart, D., *Analysis on Wiener space and anticipating stochastic calculus*, Springer LNM, 1690, Ecole d'Et'e de Probabilit'es de Saint-Flour XXV-1995, ed: P. Bernard (1995).(<–)

REFERENCES-contd

N-P Nualart, D., and Pardoux, E., Stochastic calculus with anticipating integrands, Analysis on Wiener space and anticipating stochastic calculus, *Probab. Th. Rel. Fields* , 78 (1988), 535-581.

Sk Skorohod, A. V., *Random Linear Operators*, Riedel 1984.

(<–)