

# *Anticipating Semilinear SPDEs*<sup>a</sup>

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<sup>a</sup>Results to appear in JFA [M-Z]

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# Acknowledgment

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Does the following anticipating stochastic evolution equation (see):

$$\left. \begin{aligned} dv(t) &= -Av(t) dt + F_0(v(t)) dt \\ &\quad + Bv(t) \circ dW(t), t > 0, \\ v(0) &= Y \end{aligned} \right\} \quad (1)$$

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admit a solution with a random initial condition  $Y : \Omega \rightarrow H$  in a Hilbert space  $H$ ?

*Answer:*

**YES!** (provided  $Y$  is sufficiently **regular**).

# Strategy

- Replace  $Y$  in see (1) by a **deterministic** initial condition  $x$  in  $H$  and get the corresponding (equivalent) Itô see:

$$\left. \begin{aligned} du(t, x) &= -Au(t, x) dt + F(u(t, x)) dt \\ &\quad + Bu(t, x) dW(t), \quad t > 0 \\ u(0, x) &= x \in H \end{aligned} \right\} \quad (2)$$

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with  $F$  a suitably modified non-linear drift.

- View the solution of the see (2) as a function (**cocycle**)  $U(t, x, \omega)$  of three variables  $(t, x, \omega)$  with Fréchet and Malliavin regularity in  $x$  and  $\omega$  (resp.)

# Strategy-Contd

- Consider the Stratonovich version of the Itô see (2):

$$\left. \begin{aligned} du(t, x) &= -Au(t, x) dt + F_0(u(t, x)) dt \\ &\quad + Bu(t, x) \circ dW(t), \quad t > 0 \\ u(0, x) &= x \in H \end{aligned} \right\} (2')$$

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- *In the above semilinear see, is it justified to replace the deterministic initial condition  $x$  by an arbitrary random variable  $Y$  (substitution theorem)?*

# Strategy-Contd

- If **YES**, then get back the anticipating Stratonovich see (1) again:

$$\left. \begin{aligned} dU(t, Y) &= -AU(t, Y) dt + F_0(U(t, Y)) dt \\ &\quad + BU(t, Y) \circ dW(t), \quad t > 0 \\ U(0, Y) &= Y \end{aligned} \right\} (1)$$

by taking  $v(t) := U(t, Y)$ ,  $t \geq 0$ .

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- Existing substitution theorems work under restrictive finite-dimensional or compactness constraints ([G-Nu-M]).

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- Failure of Kolmogorov's continuity theorem in infinite dimensions ([Mo.1], [Sk]).
- Failure of Sobolev inequalities in infinite dimensions.

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- Develop global spatial estimates on the semiflow, its Malliavin and Fréchet derivatives.
- Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are **infinite-dimensional**.

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Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

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Substitution theorem provides a dynamic characterization of stable/unstable manifolds for semilinear see's near hyperbolic stationary states.

Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by sfde's.

Global moment estimates on the cocycle and its derivatives are interesting in their own right.

Expect results in this talk to lead to **regularity in distribution** of the invariant manifolds for semilinear spde's and sfde's.

# The Set-up

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- $(\Omega, \mathcal{F}, P) :=$  **Wiener space** of all continuous paths  $\omega : \mathbf{R} \rightarrow E$ ,  $\omega(0) = 0$ , where  $E$  is a real separable Hilbert space.

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- **Wiener shifts**  $\theta : \mathbf{R} \times \Omega \rightarrow \Omega$ : Group of  $P$ -preserving ergodic transformations on  $(\Omega, \mathcal{F}, P)$ :  
$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

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- $H :=$  real (separable) Hilbert space, norm  $|\cdot|_H$ .
- $\mathcal{B}(H) :=$  Borel  $\sigma$ -algebra of  $H$ .
- $L(H) :=$  Banach space of all bounded linear operators  $H \rightarrow H$  given the uniform operator norm  $\|\cdot\|_{L(H)}$ .

# Set-up: Brownian Motion

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- $$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbf{R};$$

$\{f_k : k \geq 1\} :=$  complete orthonormal basis of  $K$ ;  
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- $(W, \theta)$  is a **helix**:

$$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$$

# Set-up-contd

- $L_2(K, H) :=$  **Hilbert space** of all Hilbert-Schmidt operators  $S : K \rightarrow H$ , with norm

$$\|S\|_2 := \left[ \sum_{k=1}^{\infty} |S(f_k)|_H^2 \right]^{1/2}$$

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- $F := F_0 + \frac{1}{2} \sum_{k=1}^{\infty} B_k^2$ , where  $B_k \in L(H)$  are given by

$$B_k(x) := B(x)(f_k), x \in H, k \geq 1; \text{ and } \sum_{k=1}^{\infty} \|B_k\|^2$$

converges.

# Set-up: The Semilinear SEE

Consider the semilinear Itô stochastic evolution equation (see):

$$\left. \begin{aligned} du(t, x) &= -Au(t, x) dt + F(u(t, x)) dt \\ &\quad + Bu(t, x) dW(t), \quad t > 0 \\ u(0, x) &= x \in H \end{aligned} \right\} \quad (2)$$

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Assume  $A$  has a complete orthonormal system of eigenvectors  $\{e_n : n \geq 1\}$  with corresponding positive eigenvalues  $\{\mu_n, n \geq 1\}$ ; i.e.,  $Ae_n = \mu_n e_n, n \geq 1$ .



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Suppose  $B : H \rightarrow L_2(K, H)$  is a bounded linear operator. The Itô integral in the see (2) is defined in the following sense ([D-Z.1], Chapter 4):

# Set-up: The Itô Integral

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Let  $\psi : [0, a] \times \Omega \rightarrow L_2(K, H)$  be jointly measurable,  $(\mathcal{F}_t)_{t \geq 0}$ -adapted and

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Set

$$\int_0^a \psi(t) dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k) dW^k(t)$$

where the  $H$ -valued Itô integrals on the right hand side are with respect to the one-dimensional Wiener processes  $W^k$ ,  $k \geq 1$ .

# The Itô Integral-contd

Series converges in  $L^2(\Omega, H)$  because

$$\sum_{k=1}^{\infty} E \left| \int_0^a \psi(t)(f_k) dW^k(t) \right|^2 = \int_0^a E \|\psi(t)\|_{L_2(K,H)}^2 dt < \infty.$$

# Standing Hypotheses

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■ *Hypothesis (B)*:  $B : H \rightarrow L_2(K, H)$  extends to a bounded linear operator  $B \in L(H, L(E, H))$  ;

$$\sum_{k=1}^{\infty} \|B_k\|^2 < \infty, \text{ where } B_k \in L(H) \text{ is defined by}$$

$$B_k(x) := B(x)(f_k), x \in H, k \geq 1.$$



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- Requirement (b) above is satisfied if  $A = -\Delta$ , where  $\Delta$  is the Laplacian on a compact smooth  $d$ -dimensional Riemannian manifold  $M$  with boundary, under Dirichlet boundary conditions.

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- Requirement (b) above is satisfied if  $A = -\Delta$ , where  $\Delta$  is the Laplacian on a compact smooth  $d$ -dimensional Riemannian manifold  $M$  with boundary, under Dirichlet boundary conditions.
- No restriction on  $\dim M$  under  $(A_1)$  for spdes.

# Mild Solutions

A **mild solution** of the semilinear see (2) is a family of  $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$ -measurable,  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes  $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \rightarrow H$ ,  $x \in H$ , satisfying the following stochastic integral equation:

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} F(u(s, x, \cdot)) ds + \int_0^t T_{t-s} B u(s, x, \cdot) dW(s), \quad t \geq 0, \quad (3)$$

([D-Z.1-2]).

# Stratonovich Form

The Itô see (2) has the equivalent **Stratonovich** form

$$\left. \begin{aligned} du(t, x) &= -Au(t, x) dt + F(u(t, x)) dt \\ &\quad - \frac{1}{2} \sum_{k=1}^{\infty} B_k^2 u(t, x) dt + Bu(t, x) \circ dW(t) \\ u(0, x) &= x \in H \end{aligned} \right\} (2')$$

where  $B_k \in L(H)$  are given by  $B_k(x) := B(x)(f_k)$ ,  
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- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$  for all  $t_1, t_2 \in \mathbf{R}^+$ , all  $\omega \in \Omega$ .

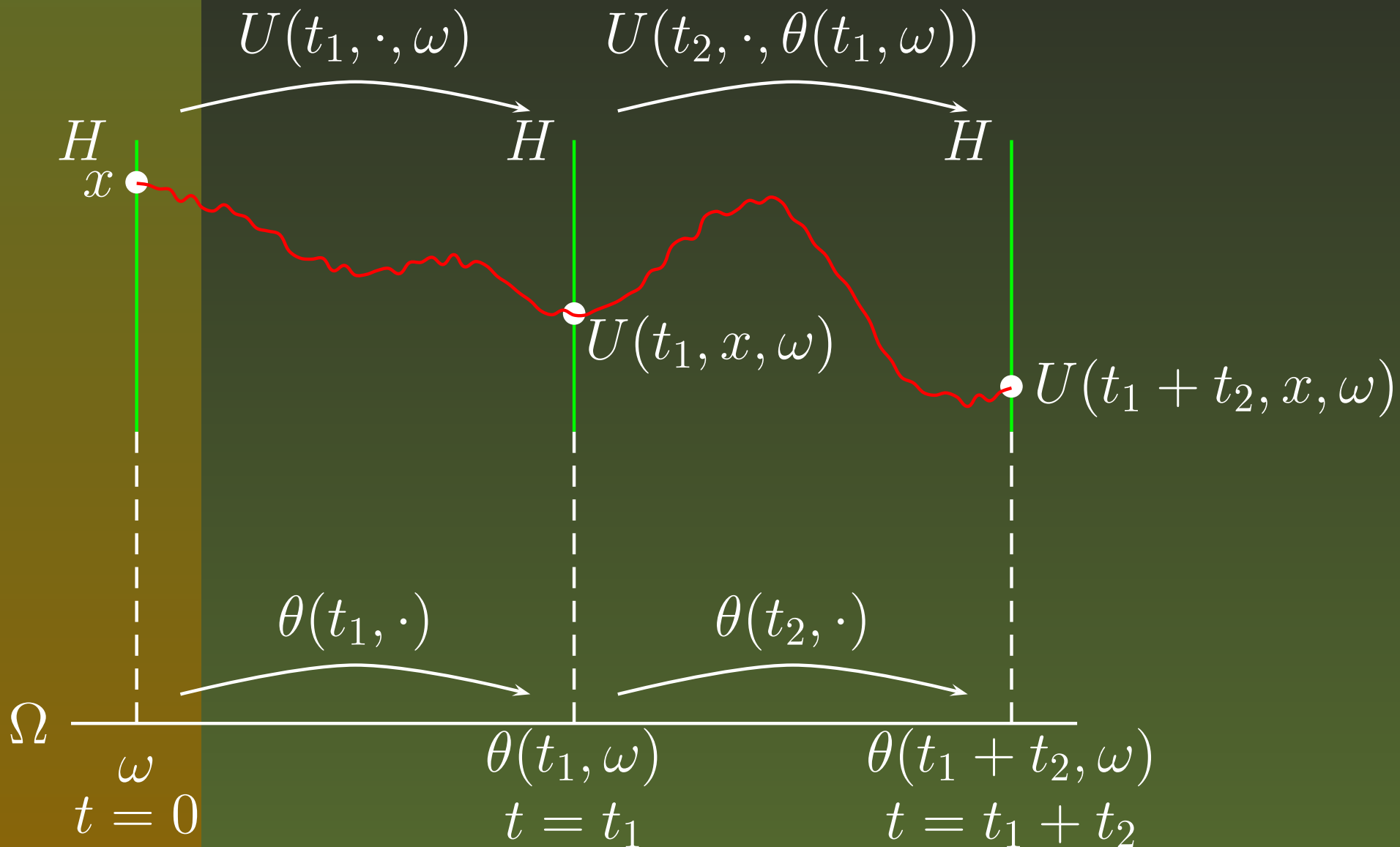
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- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$  for all  $t_1, t_2 \in \mathbf{R}^+$ , all  $\omega \in \Omega$ .
- $U(0, x, \omega) = x$  for all  $x \in H, \omega \in \Omega$ .

# The Cocycle Property



# Existence of the Cocycle

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## *Theorem 1:*

*Under Hypotheses (B) and  $(A_1)$ , the Itô see (2) (or its Stratonovich version (2')) admits a perfect jointly measurable  $C^1$  cocycle  $(U, \theta)$  where*

$$U : \mathbf{R}^+ \times H \times \Omega \rightarrow H.$$

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## *Proof of Theorem 1:*

([M-Z-Z], Theorem 1.2.6). □

# Stationary Points

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An  $\mathcal{F}$ -measurable random variable  $Y : \Omega \rightarrow H$  is said to be a **stationary point** for the cocycle  $(U, \theta)$  if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

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A stationary point of the see (2) corresponds to a **stationary solution** to the anticipating Stratonovich see (1).

# Malliavin Regularity

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For any integer  $p \geq 2$ , denote by  $\mathbb{D}^{1,p}(\Omega, H)$  the Sobolev space of all  $\mathcal{F}$ -measurable random variables  $Y : \Omega \rightarrow H$  which are  $p$ -integrable together with their Malliavin derivatives  $\mathcal{D}Y$  ([Nu.1-2]).

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We now state the main substitution theorem in this talk.

# Substitution

*Theorem 2:* (The Substitution Theorem)

*Assume Hypotheses (B) and (A<sub>1</sub>). Let  $U : \mathbf{R}^+ \times H \times \Omega \rightarrow H$  be the  $C^1$  cocycle generated by the see (2). Let  $Y \in \mathbb{D}^{1,4}(\Omega, H)$  be a random variable. Then  $v(t) := U(t, Y)$ ,  $t \geq 0$ , is a mild solution of the (anticipating) Stratonovich see*

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$$\left. \begin{aligned} dv(t) &= -Av(t) dt + F_0(v(t)) dt \\ &\quad + Bv(t) \circ dW(t), \quad t > 0, \\ v(0) &= Y. \end{aligned} \right\} \quad (1)$$

# Substitution Theorem-contd

---

*In particular, if  $Y \in \mathbb{D}^{1,4}(\Omega, H)$  is a stationary point of the see (2) (or (2')), then  $U(t, Y) = Y(\theta(t))$ ,  $t \geq 0$ , is a stationary solution of the (anticipating) Stratonovich see (1):*

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$$\left. \begin{aligned} dY(\theta(t)) &= -AY(\theta(t)) dt + F_0(Y(\theta(t))) dt \\ &\quad + BY(\theta(t)) \circ dW(t), t > 0, \\ Y(\theta(0)) &= Y. \end{aligned} \right\} \quad (4)$$

# Substitution Theorem-contd

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*Furthermore, assume that  $F_0$  is  $C_b^2$ . Then the linearized cocycle  $DU(t, Y)$  is a mild solution of the linearized anticipating see*



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$$\left. \begin{aligned} dDU(t, Y) &= -ADU(t, Y) dt \\ &\quad + DF_0(U(t, Y)) DU(t, Y) dt \\ &\quad + \{B \circ DU(t, Y)\} \circ dW(t), \quad t > 0, \\ DU(0, Y) &= \text{id}_{L(H)}. \end{aligned} \right\} (5)$$

# Outline of Proof

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- Use the linear cocycle to get a pathwise variational integral equation equivalent to the semilinear see.
- Derive moment estimates for the nonlinear cocycle, its Fréchet and Malliavin derivatives.

# Outline of Proof-Contd

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- Prove the substitution theorem when  $Y$  is replaced by its finite-dimensional projections  $Y_n$ : Use finite-dimensional projections to smooth out the semigroup  $T_t$  in  $t$ , and apply finite-dimensional substitution techniques.



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- Prove the substitution theorem when  $Y$  is replaced by its finite-dimensional projections  $Y_n$ : Use finite-dimensional projections to smooth out the semigroup  $T_t$  in  $t$ , and apply finite-dimensional substitution techniques.
- Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.
- Let  $n \rightarrow \infty$  using dominated convergence and the moment estimates on the cocycle, its Fréchet and Malliavin derivatives.

# Linear SEE

---

Existence of semiflows for mild solutions of linear see:

$$\begin{aligned} du(t, x, \cdot) = & -Au(t, x, \cdot) dt \\ & + Bu(t, x, \cdot) dW(t), \quad t > 0 \end{aligned}$$

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e.g.  $A = -\Delta$  on compact smooth Riemannian manifold.

# Mild Solutions: Linear Case

A *mild solution* of the linear see is a family of jointly measurable,  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes

$$u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \rightarrow H, \quad x \in H$$

such that

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot) dW(s), \quad t \geq 0.$$

Integral equation holds  *$x$ -almost surely*,  $x \in H$ .

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Is  $u(t, x, \cdot)$  pathwise continuous linear in  $x$ ?

# Kolmogorov Fails!

---

*Kolmogorov's continuity theorem fails* for random field

$$I : L^2([0, 1], \mathbf{R}) \rightarrow L^2(\Omega, \mathbf{R})$$

$$I(x) := \int_0^1 x(t) dW(t), \quad x \in L^2([0, 1], \mathbf{R}).$$



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No **continuous** (or even **measurable linear!**) selection

$$L^2([0, 1], \mathbf{R}) \times \Omega \rightarrow \mathbf{R}$$

$$(x, \omega) \mapsto I(x, \omega)$$

of  $I$  ([Mo.1], pp. 144-148).

# Lifting

---

- Lift semigroup  $T_t, t \geq 0$ , to a strongly continuous semigroup of bounded linear operators

$\tilde{T}_t : L_2(K, H) \rightarrow L_2(K, H), t \geq 0$ , via composition

$\tilde{T}_t(C) := T_t \circ C, C \in L_2(K, H), t \geq 0$ .

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$$\tilde{T}_t(C) := T_t \circ C, \quad C \in L_2(K, H), t \geq 0.$$

- Lift stochastic integral

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\}) dW(s), \quad x \in H, t \geq 0,$$

to  $L_2(H)$  for adapted square-integrable

$v : \mathbf{R}^+ \times \Omega \rightarrow L_2(H)$ . Denote lifting by

$$\int_0^t T_{t-s} B v(s) dW(s) \in L_2(H).$$

# Lifting-contd

That is:

$$\left[ \int_0^t T_{t-s} B v(s) dW(s) \right] (x) = \int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\}) dW(s)$$

for all  $t \geq 0$ ,  $x$ -a.s..

# The Linear Flow

---

## *Theorem 3:*

*Assume hypothesis (B) and  $(A_1)$ . Then the mild solution of the linear see has a Borel (strongly) measurable  $(\mathcal{F}_t)_{t \geq 0}$ -adapted version  $\Phi : \mathbf{R}^+ \times \Omega \rightarrow L(H)$  with the following properties:*

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- $E \sup_{0 \leq t \leq a} \|\Phi(t, \cdot)\|_{L(H)}^{2p} < \infty$ , whenever  $p \geq 1$ .

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- $E \sup_{0 \leq t \leq a} \|\Phi(t, \cdot)\|_{L(H)}^{2p} < \infty$ , whenever  $p \geq 1$ .
- $(\Phi, \theta)$  is a perfect  $L(H)$ -valued cocycle:

$$\Phi(t + s, \omega) = \Phi(t, \theta(s, \omega)) \circ \Phi(s, \omega)$$

*for all  $s, t \geq 0$  and all  $\omega \in \Omega$ ;*

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for all  $s, t \geq 0$  and all  $\omega \in \Omega$ ;

- $\sup_{0 \leq s \leq t \leq a} \|\Phi(t - s, \theta(s, \omega))\|_{L(H)} < \infty$ , for all  $\omega \in \Omega$ .



# Linear Flow-Contd: “Chaos”!

- For each  $t > 0$  and almost all  $\omega \in \Omega$ ,  $\Phi(t, \omega) \in L_2(H)$  has “chaos-type” representation

$$\begin{aligned} \Phi(t, \cdot) = & T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \\ & \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n} dW(s_n) \\ & \cdots dW(s_2) dW(s_1). \end{aligned}$$

*Iterated Itô stochastic integrals are lifted integrals in  $L_2(H)$ , and series converges absolutely in  $L_2(H)$ .*

# Semilinear SEE

Consider the semilinear Itô see:

$$\left. \begin{aligned} du(t) &= -Au(t)dt + F(u(t))dt \\ &\quad + Bu(t) dW(t), \quad t > 0, \\ u(0) &= x \in H \end{aligned} \right\} \quad (2)$$

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Operators  $A, B$  satisfy hypothesis (B) and  $(A_1)$ .  
 $F : H \rightarrow H$  is (Fréchet)  $C_b^1$ , with linear growth:

$$|F(v)| \leq C(1 + |v|), \quad v \in H$$

for some positive constant  $C$ .

# Mild Solution: Semilinear SEE

Recall a *mild solution* of semilinear Itô see (2) is a family of jointly measurable,  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes  $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \rightarrow H$ ,  $x \in H$ , satisfying:

$$u(t, x, \cdot) = T_t(x) + \int_0^t T_{t-s}(F(u(s, x, \cdot))) ds + \int_0^t T_{t-s} B u(s, x, \cdot) dW(s),$$

for all  $t \geq 0$ ,  $x$ -a.s. ([D–Z], Chapter 7, p. 182).

# Random Integral Equation

Obtain a  $C^k$  perfect cocycle  $(U, \theta)$  for mild solutions of the semilinear see, via the **random integral equation** on  $H$ :

$$U(t, x, \omega) = \Phi(t, \omega)(x) + \int_0^t \Phi(t-s, \theta(s, \omega))(F(U(s, x, \omega))) ds$$

for each  $\omega \in \Omega$ ,  $t \geq 0$ ,  $x \in H$ .

# Estimates of the Cocycle

---

Get new global estimates on the non-linear cocycle  $U : \mathbb{R}^+ \times H \times \Omega \rightarrow H$ , its spatial Fréchet derivative  $DU(t, x, \cdot)$  and its Malliavin derivatives  $\mathcal{D}_u U(t, x, \cdot)$  for  $u, t \in [0, a]$  and  $x \in H$ .

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Derivations are based on results in [M.Z.Z], Gronwall's lemma and the fact that  $W$  has stationary independent increments.

# Estimates of Cocycle-Contd

## Theorem 4:

Assume Hypotheses (B),  $(A_1)$  and let  $F$  be  $C_b^1$ . Let  $U : \mathbf{R}^+ \times H \times \Omega \rightarrow H$  be the cocycle generated by the mild solutions of the see (2). Fix any  $a \in (0, \infty)$ . Then:

$$E \sup_{\substack{0 \leq t \leq a \\ x \in H}} \frac{|U(t, x, \cdot)|^{2p}}{(1 + |x|_H^{2p})} < \infty, \quad p \geq 1$$

$$E \sup_{\substack{0 \leq t \leq a \\ x \in H}} \|DU(t, x, \cdot)\|^{2p} < \infty, \quad p \geq 1$$

$DU :=$  Fréchet derivative of  $U$  in the spatial variable  $x$ .



# More Estimates

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*Theorem 4':*

*In the see (2), assume Hypotheses (B) and  $(A_1)$ .*

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*In the see (2), assume Hypotheses (B) and  $(A_1)$ .*

*(i) Let  $u, t \in [0, a]$ . Define*

$$V(t, \cdot) := \Phi(t, \cdot) - T_t, \quad t \in [0, a].$$

*Then  $V(t, \cdot) \in \mathbb{D}^{1,2p}(\Omega, L_2(H))$  and*

$$E \left[ \sup_{u \leq t \leq a} \|\mathcal{D}_u V(t, \cdot)\|_{L_2(H)}^{2p} \right] < \infty$$

*for all  $p \geq 1$ .*

# More Estimates-contd

(ii) Suppose  $F$  is  $C_b^1$ . Then

$$E \left[ \sup_{\substack{0 \leq t \leq a \\ x \in H}} \frac{|\mathcal{D}U(t, x, \cdot)|_H^{2p}}{(1 + |x|_H^{2p})} \right] < \infty,$$

for all  $p \geq 1$ .  $\mathcal{D} :=$  Malliavin derivative.

# More Estimates-contd

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for all  $p \geq 1$ .  $\mathcal{D} :=$  Malliavin derivative.

(iii) Let  $F$  be  $C_b^2$ . Then

$$E \left[ \sup_{\substack{0 \leq u, t \leq a \\ x \in H}} \frac{\|\mathcal{D}_u \mathcal{D}U(t, x, \cdot)\|^{2p}}{(1 + |x|_H^{2p})} \right] < \infty$$

for all  $p \geq 1$ .

# Finite-dimensional Projections

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## *Objective:*

To prove the substitution theorem when the random variable  $Y \in \mathbb{D}^{1,4}(\Omega, H)$  is replaced by its finite-dimensional projections on  $H$ .

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$H_n := L\{e_i : 1 \leq i \leq n\}$ , the  $n$ -dimensional linear subspace of  $H$  spanned by  $\{e_i : 1 \leq i \leq n\}$ , for each  $n \geq 1$ .

# Projections-contd

---

Define the projections  $P_n : H \rightarrow H_n$ ,  $n \geq 1$ , by

$$P_n(x) := \sum_{k=1}^n \langle x, e_k, \rangle e_k, \quad x \in H.$$



# Projections-contd

Define the projections  $P_n : H \rightarrow H_n$ ,  $n \geq 1$ , by

$$P_n(x) := \sum_{k=1}^n \langle x, e_k, \rangle e_k, \quad x \in H.$$

Define  $Y_n : \Omega \rightarrow H_n$  by

$$Y_n := P_n \circ Y, \quad n \geq 1.$$

Then  $Y_n \rightarrow Y$  as  $n \rightarrow \infty$  a.s.

# Finite-dimensional Substitution

## *Theorem 5:*

*Assume (B) and (A<sub>1</sub>) and suppose  $Y \in \mathbb{D}^{1,4}(\Omega, H)$ . Then*

$$\left. \begin{aligned} dU(t, Y_n) &= -AU(t, Y_n) dt + F_0(U(t, Y_n)) dt \\ &\quad + BU(t, Y_n) \circ dW(t), t > 0, \\ U(0, Y_n) &= Y_n. \end{aligned} \right\} \quad (6)$$

*for each  $n \geq 1$ .*

# Proof of Theorem 5

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- Proof still requires Malliavin calculus techniques, largely due to the underlying **strongly continuous** semi-group dynamics in  $\{T_t\}_{t \geq 0}$ .



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- Use global estimates on  $U$  to represent the Stratonovich integrals in terms of Skorohod integrals.
- Project the semigroup  $\{T_t\}_{t \geq 0}$  onto  $H_m$  and use finite-dimensional substitutions.
- Then pass to the limit as  $m \rightarrow \infty$  using global estimates on  $U$  and dominated convergence.



# Proof of Substitution Theorem 2

*Step 1:*

Suppose  $Y \in \mathbb{D}^{1,4}(\Omega, H)$ , and assume Hypothesis (B) and  $(A_1)$ .

Sufficient to show

$$\left. \begin{aligned} U(t, Y) &= T_t(Y) + \int_0^t T_{t-s} F_0(U(s, Y)) ds \\ &+ \int_0^t T_{t-s} BU(s, Y) \circ dW(s). \end{aligned} \right\} \quad (10)$$

# Proof of Theorem 2-contd

*Step 2:*

Pass to the limit as  $n \rightarrow \infty$  in the finite-dimensional result:

$$\left. \begin{aligned} U(t, Y_n) &= T_t(Y_n) + \int_0^t T_{t-s} F_0(U(s, Y_n)) ds \\ &+ \int_0^t T_{t-s} B U(s, Y_n) \circ dW(s), \\ &t > 0, n \geq 1. \end{aligned} \right\} (7)$$



# Localization

Denote by  $\mathbb{L}^{1,2}$  the class of all processes  $v : [0, t] \times \Omega \rightarrow H$  such that  $v \in L^2([0, t] \times \Omega, H)$ ,  $v(s, \cdot) \in \mathbb{D}^{1,2}(\Omega, H)$  for almost all  $s \in [0, t]$  and  $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s, \cdot)\|_H^2 du ds] < \infty$ .

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We say that  $v$  belongs to  $\mathbb{L}_{loc}^{1,2}$  if there exists a sequence  $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$  with the following properties:

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Denote by  $\mathbb{L}^{1,2}$  the class of all processes  $v : [0, t] \times \Omega \rightarrow H$  such that  $v \in L^2([0, t] \times \Omega, H)$ ,  $v(s, \cdot) \in \mathbb{D}^{1,2}(\Omega, H)$  for almost all  $s \in [0, t]$  and  $E[\int_0^t \int_0^t \|\mathcal{D}_u v(s, \cdot)\|_H^2 du ds] < \infty$ .

We say that  $v$  belongs to  $\mathbb{L}_{loc}^{1,2}$  if there exists a sequence  $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$  with the following properties:

- (i)  $\Omega_m \uparrow \Omega$  as  $m \rightarrow \infty$ ,
- (ii)  $v = v^m$  on  $\Omega_m$ .

# Proof of Theorem 2

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*Step 3:*

The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (10) is well-defined:

# Proof of Theorem 2

*Step 3:*

The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (10) is well-defined:

Sufficient to show that the process

$$v(s) := T_{t-s} BU(s, Y), s \leq t$$

is in  $\mathbb{L}_{loc}^{1,2}$ : *Localize*  $v$  using a bump function

$\phi_m \in C_b^2(\mathbb{R}, \mathbb{R})$  such that  $\phi_m(z) = 1$  for  $|z| \leq m$  and  $\phi_m(z) = 0$  for  $|z| > m + 1$ . ([Nu.2], Theorem 5.2.3).

# Easy Limits

*Step 4:*

Pass to the limit a.s. as  $n \rightarrow \infty$  in (7). Get easy a.s. limits:

$$\lim_{n \rightarrow \infty} U(t, Y_n) = U(t, Y)$$

$$\lim_{n \rightarrow \infty} T_t(Y_n) = T_t(Y)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^t T_{t-s} F_0(U(s, Y_n)) ds \\ = \int_0^t T_{t-s} F_0(U(s, Y)) ds \end{aligned}$$

# A Not-So-Easy Limit

---

*Step 5:*



# A Not-So-Easy Limit

*Step 5:*

But following limit is non-trivial:

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s) \\ = \int_0^t T_{t-s} BU(s, Y) \circ dW(s) \end{aligned} \right\} \quad (11)$$

in probability.

# Proof of Theorem 2-contd

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*Step 6:*

# Proof of Theorem 2-contd

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*Step 6:*

To prove (11), use localization:

# Proof of Theorem 2-contd

*Step 6:*

To prove (11), use localization:

$$\begin{aligned} \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s) \\ = \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s), \end{aligned}$$

on  $\Omega_m := \{\omega : |Y(\omega)|_H \leq m\}$ ;

# Proof of Theorem 2-contd

---

and

$$\begin{aligned} \int_0^t T_{t-s} BU(s, Y) \circ dW(s) \\ = \int_0^t T_{t-s} BU(s, Y) \phi_m(|Y|_H) \circ dW(s) \end{aligned}$$

on  $\Omega_m$  for any fixed integer  $m \geq 1$ .

# Proof of Theorem 2-contd

Step 7:

(11) will follow from

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s) \\ = \int_0^t T_{t-s} BU(s, Y) \phi_m(|Y|_H) \circ dW(s) \end{aligned} \tag{12}$$

in probability for each  $m \geq 1$ .

# Proof of Theorem 2-contd

To prove (12), fix  $m \geq 1$  and let

$$g_n(s) := T_{t-s}BU(s, Y_n)\phi_m(|Y|_H),$$

$$g(s) := T_{t-s}BU(s, Y)\phi_m(|Y|_H)$$

for all  $s \in [0, t]$ . Then

$$\lim_{n \rightarrow \infty} E \left[ \int_0^T \|g_n(s) - g(s)\|_{L_2(K, H)}^2 ds \right] = 0 \quad (13)$$

$$\lim_{n \rightarrow \infty} E \left[ \int_0^T \int_0^T \|\mathcal{D}_u g_n(s) - \mathcal{D}_u g(s)\|_{L_2(K, H)}^2 du ds \right] = 0. \quad (14)$$

# Proof of Theorem 2-contd

---

Compute:

$$(\mathcal{D}_+g)_u := \lim_{s \rightarrow u+} \mathcal{D}_u g(s)$$

$$(\mathcal{D}_-g)_u := \lim_{s \rightarrow u-} \mathcal{D}_u g(s)$$

$$(\nabla g)_u := (\mathcal{D}_+g)_u + (\mathcal{D}_-g)_u$$



# Proof of Theorem 2-contd

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Compute:

$$(\mathcal{D}_+g)_u := \lim_{s \rightarrow u+} \mathcal{D}_u g(s)$$

$$(\mathcal{D}_-g)_u := \lim_{s \rightarrow u-} \mathcal{D}_u g(s)$$

$$(\nabla g)_u := (\mathcal{D}_+g)_u + (\mathcal{D}_-g)_u$$

and use path continuity to get

$$\lim_{n \rightarrow \infty} (\nabla g_n)_u = (\nabla g)_u, \quad a.s.$$

# Proof of Theorem 2-contd

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*Step 8:*

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s ds, \quad (15)$$

for  $n \geq 1$ ; and

# Proof of Theorem 2-contd

*Step 8:*

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s ds, \quad (15)$$

for  $n \geq 1$ ; and

$$\int_0^t g(s) \circ dW(s) = \int_0^t g(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g)_s ds \quad (16)$$

a.s.. Skorohod integrals on RHS.

# Proof of Theorem 2-contd

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Prove (15) and (16) from first principles, using approximations by Riemann sums: **Lengthy computation.**

# Proof of Theorem 2-contd

Prove (15) and (16) from first principles, using approximations by Riemann sums: **Lengthy computation.**

*Step 9:*

Take  $n \rightarrow \infty$  in RHS of (15) and get:

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^t T_{t-s} BU(s, Y_n) \phi_m(|Y|_H) \circ dW(s) \\ = \int_0^t T_{t-s} BU(s, Y) \phi_m(|Y|_H) \circ dW(s) \end{aligned} \tag{12}$$

in probability for each  $m \geq 1$ . □

# REFERENCES

---

- D-Z.1 Da Prato, G., and Zabczyk, J., *Stochastic Equations in Infinite Dimensions*, Cambridge University Press (1992).
- D-Z.2 Da Prato, G., and Zabczyk, J., *Ergodicity for Infinite Dimensional Systems*, Cambridge University Press (1996).
- G-Nu-S Grorud, A., Nualart, D., and Sanz-Solé, M., Hilbert-valued anticipating stochastic differential equations, *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques*, 30 no. 1 (1994), 133-161. (←)

# REFERENCES-contd

---

- Ma            Malliavin, P., Stochastic calculus of variations and hypoelliptic operators, *Proceedings of the International Conference on Stochastic Differential Equations, Kyoto*, Kinokuniya, 1976, 195-263.
- Mo.1        Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Research Notes in Mathematics, no. 99, Pitman Advanced Publishing Program, Boston-London-Melbourne (1984).(<-)

# REFERENCES-contd

---

- Mo.2 Mohammed, S.-E. A., Non-Linear Flows for Linear Stochastic Delay Equations, *Stochastics*, Vol. 17 #3, (1987), 207–212.
- M-S.1 Mohammed, S.-E. A., and Scheutzow, M. K. R., The Stable Manifold Theorem for Nonlinear Stochastic Systems with Memory, Part I: Existence of the Semiflow, *Journal of Functional Analysis*, 205, (2003), 271-305. Part II: The Local Stable Manifold Theorem, *Journal of Functional Analysis*, 206, (2004), 253-306.



# REFERENCES-contd

- M-S.2 Mohammed, S.-E. A., and Scheutzow, M. K. R., The stable manifold theorem for stochastic differential equations, *The Annals of Probability*, Vol. 27, No. 2, (1999), 615-652. (←)
- M-Z-Z Mohammed, S.-E. A., Zhang, T. S. and Zhao, H. Z., The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations, Part 1: The Stochastic semiflow, Part 2: Existence of stable and unstable manifolds, pp. 98 *Memoirs of the American Mathematical Society*.(←)  
(To appear)

# REFERENCES-contd

- M-Z Mohammed, S.-E. A. and Zhang, T. S., The substitution theorem for semilinear stochastic partial differential equations, *Journal of Functional Analysis* (to appear) (preprint, 2007) (<-)
- Nu.1 Nualart, D., *The Malliavin Calculus and Related Topics*, Probability and its Applications, Springer-Verlag (1995).
- Nu.2 Nualart, D., *Analysis on Wiener space and anticipating stochastic calculus*, Springer LNM, 1690, Ecole d'Et'e de Probabilit'es de Saint-Flour XXV-1995, ed: P. Bernard (1995).(<-)

# REFERENCES-contd

---

N-P

Nualart, D., and Pardoux, E., Stochastic calculus with anticipating integrands, Analysis on Wiener space and anticipating stochastic calculus, *Probab. Th. Rel. Fields* , 78 (1988), 535-581.

Sk

Skorohod, A. V., *Random Linear Operators*, Riedel 1984. (<-)

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