MARKOV BEHAVIOR
AND THE WEAK GENERATOR

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MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

\[
\begin{align*}
    dx(t) &= H(t, x_t) \, dt + G(t, x_t) \, dW(t), \quad t > 0 \\
    x_0 &= \eta \in C := C([-r, 0], \mathbb{R}^d)
\end{align*}
\]

(XIII)

with coefficients \( H : [0, T] \times C \to \mathbb{R}^d \), \( G : [0, T] \times C \to \mathbb{R}^{d \times m} \), \( m \)-dimensional Brownian motion \( W \) and trajectory field \( \{ x_t : t \geq 0, \eta \in C \} \).

1. Questions

(i) For the sfde (XIII) does the trajectory field \( x_t \) give a diffusion in \( C \) (or \( M_2 \))? 

(ii) How does the trajectory \( x_t \) transform under smooth non-linear functionals \( \phi : C \to \mathbb{R} \)?

(iii) What “diffusions” on \( C \) (or \( M_2 \)) correspond to sfde’s on \( \mathbb{R}^d \)?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.
Difficulties

(i) Although the current state $x(t)$ is a semimartingale, the trajectory $x_t$ does not seem to possess any martingale properties when viewed as $C$- (or $M_2$)-valued process: e.g. for Brownian motion $W \ (H \equiv 0, G \equiv 1)$:

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_t(0), \quad s \in [-r,0]$$

whenever $t_1 \leq t - r$.

(ii) Lack of strong continuity leads to the use of weak limits in $C$ which tend to live outside $C$.

(iii) We will show that $x_t$ is a Markov process in $C$. However almost all tame functions lie outside the domain of the (weak) generator.

(iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses $(M)$

(i) $\mathcal{F}_t :=$ completion of $\sigma\{W(u) : 0 \leq u \leq t\}, \quad t \geq 0$.

(ii) $H, G$ are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t, \eta_1) - H(t, \eta_2)| + \|G(t, \eta_1) - G(t, \eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

$\eta_x^{t_1} :=$ solution starting off at $\theta \in L^2(\Omega,C;\mathcal{F}_{t_1})$ at $t = t_1$ for the sfde:

$$\eta_x^{t_1}(t) = \begin{cases} \eta(0) + \int_{t_1}^t H(u,x_u^{t_1}) \, du + \int_{t_1}^t G(u,x_u^{t_1}) \, dW(u), & t > t_1 \\ \eta(t-t_1), & t_1-r \leq t \leq t_1. \end{cases}$$
This gives a two-parameter family of mappings

\[ T_{t_2}^{t_1} : L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}), \quad t_1 \leq t_2, \]

\[ T_{t_2}^{t_1}(\theta) := \theta x_{t_2}^{t_1}, \quad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \quad (1) \]

Uniqueness of solutions gives the \textit{two-parameter} semigroup property:

\[ T_{t_2}^{t_1} \circ T_{t_1}^0 = T_{t_2}^0, \quad t_1 \leq t_2. \quad (2) \]

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

**Theorem II.1** (Markov Property) ([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field \( \{ x_t : t \geq 0, \eta \in C \} \) is a Feller process on \( C \) with transition probabilities

\[ p(t_1, \eta, t_2, B) := P(\eta x_{t_2}^{t_1} \in B) \quad t_1 \leq t_2, \quad B \in \text{Borel } C, \quad \eta \in C. \]

i.e.

\[ P(x_{t_2} \in B | \mathcal{F}_{t_1}) = p(t_1, x_{t_1}(\cdot), t_2, B) = P(x_{t_2} \in B | x_{t_1}) \text{ a.s.} \]

Further, if \( H \) and \( G \) do not depend on \( t \), then the trajectory is time-homogeneous:

\[ p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \leq t_1 \leq t_2, \quad \eta \in C. \]

**Proof.**

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. \( \square \)
3. The Semigroup

In the autonomous sfde
\[
\begin{align*}
&dx(t) = H(x_t) dt + G(x_t) dW(t) \quad t > 0 \\
&x_0 = \eta \in C
\end{align*}
\]
(XIV)
suppose the coefficients \( H : C \to \mathbb{R}^d \), \( G : C \to \mathbb{R}^{d \times m} \) are globally bounded and globally Lipschitz.

\( C_b := \) Banach space of all bounded uniformly continuous functions \( \phi : C \to \mathbb{R} \), with the sup norm
\[
\|\phi\|_{C_b} := \sup_{\eta \in C} |\phi(\eta)|, \quad \phi \in C_b.
\]

Define the operators \( P_t : C_b \to C_b, t \geq 0 \), on \( C_b \) by
\[
P_t(\phi)(\eta) := E\phi^n(x_t) \quad t \geq 0, \eta \in C.
\]

A family \( \phi_t, t > 0 \), converges weakly to \( \phi \in C_b \) as \( t \to 0^+ \) if \( \lim_{t \to 0^+} \phi_t, \mu >= \phi, \mu > \) for all finite regular Borel measures \( \mu \) on \( C \). Write \( \phi := w - \lim_{t \to 0^+} \phi_t \). This is equivalent to
\[
\begin{align*}
\phi_t(\eta) &\to \phi(\eta) \text{ as } t \to 0^+, \text{ for all } \eta \in C \\
\{\|\phi_t\|_{C_b} : t \geq 0\} &\text{ is bounded}.
\end{align*}
\]
(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

**Theorem II.2**([Mo], Pitman Books, 1984)

(i) \( \{P_t\}_{t \geq 0} \) is a one-parameter contraction semigroup on \( C_b \).
(ii) \( \{P_t\}_{t \geq 0} \) is weakly continuous at \( t = 0 \):

\[
\begin{cases}
    P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0^+ \\
    \{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C\} \text{ is bounded by } \|\phi\|_{C_b}.
\end{cases}
\]

(iii) If \( r > 0 \), \( \{P_t\}_{t \geq 0} \) is never strongly continuous on \( C_b \) under the sup norm.

**Proof.**

(i) One parameter semigroup property

\[
P_{t_2} \circ P_{t_1} = P_{t_1 + t_2}, \quad t_1, t_2 \geq 0
\]

follows from the continuation property (2) and time-homogeneity of the Feller process \( x_t \) (Theorem II.1).

(ii) Definition of \( P_t \), continuity and boundedness of \( \phi \) and sample-continuity of trajectory \( \eta x_t \) give weak continuity of \( \{P_t(\phi) : t > 0\} \) at \( t = 0 \) in \( C_b \).

(iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

\[
S_t : C_b \to C_b, \quad t \geq 0,
\]

by

\[
S_t(\phi)(\eta) := \phi(\tilde{\eta}), \quad \phi \in C_b, \quad \eta \in C,
\]

where \( \tilde{\eta} : [-r, \infty) \to \mathbb{R}^d \) is defined by

\[
\tilde{\eta}(t) = \begin{cases}
    \eta(0) & t \geq 0 \\
    \eta(t) & t \in [-r, 0).
\end{cases}
\]

Then \( P_t \) is strongly continuous iff \( S_t \) is strongly continuous. \( P_t \) and \( S_t \) have the same “domain of strong continuity” independently of \( H, G, \) and \( W \). This follows from the global boundedness of \( H \) and \( G \). ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

\[
\lim_{t \to 0^+} E\|\eta x_t - \tilde{\eta}_t\|_{C_b}^2 = 0
\]
uniformly in $\eta \in C$. But $\{S_t\}$ is strongly continuous on $C_b$ iff $C$ is locally compact iff $r = 0$ (no memory)! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any $s_0 \in [-r, 0)$ and consider the function $\phi_0 : C \to \mathbb{R}$ defined by

$$\phi_0(\eta) := \begin{cases} 
\eta(s_0) & \|\eta\|_C \leq 1 \\
\frac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1
\end{cases}$$

Let $C^0_b$ be the domain of strong continuity of $P_t$, viz.

$$C^0_b := \{\phi \in C_b : P_t(\phi) \to \phi \text{ as } t \to 0+ \text{ in } C_b\}.$$ 

Then $\phi_0 \in C_b$, but $\phi_0 \notin C^0_b$ because $r > 0$. \hfill $\Box$

4. The Generator

Define the weak generator $A : D(A) \subset C_b \to C_b$ by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \to 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where $\phi \in D(A)$ iff the above weak limit exists. Hence $D(A) \subset C^0_b$ (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also $D(A)$ is weakly dense in $C_b$ and $A$ is weakly closed. Further

$$\frac{d}{dt}P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all $\phi \in D(A)$ ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator $A$. We need to augment $C$ by adjoining a canonical $d$-dimensional direction. The generator $A$ will be equal to the weak generator of the shift semigroup $\{S_t\}$ plus a second order linear partial differential operator along this new direction. Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbb{R}^d\}$$

$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbb{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

$$7$$
Lemma II.1. ([Mo], Pitman Books, 1984)

Suppose $\phi : C \to \mathbb{R}$ is $C^2$ and $\eta \in C$. Then $D\phi(\eta)$ and $D^2\phi(\eta)$ have unique weakly continuous linear and bilinear extensions

$$D\phi(\eta) : C \oplus F_d \to \mathbb{R}, \quad D^2\phi(\eta) : (C \oplus F_d) \times (C \oplus F_d) \to \mathbb{R}$$

respectively.

Proof.

First reduce to the one-dimensional case $d = 1$ by using coordinates.

Let $\alpha \in C^* = [C([-r,0],\mathbb{R})]^*$. We will show that there is a weakly continuous linear extension $\overline{\alpha} : C \oplus F_1 \to \mathbb{R}$ of $\alpha$; viz. If $\{\xi^k\}$ is a bounded sequence in $C$ such that $\xi^k(s) \to \xi(s)$ as $k \to \infty$ for all $s \in [-r,0]$, where $\xi \in C \oplus F_1$, then $\alpha(\xi^k) \to \overline{\alpha}(\xi)$ as $k \to \infty$. By the Riesz representation theorem there is a unique finite regular Borel measure $\mu$ on $[-r,0]$ such that

$$\alpha(\eta) = \int_{-r}^{0} \eta(s) \, d\mu(s)$$

for all $\eta \in C$. Define $\overline{\alpha} \in [C \oplus F_1]^*$ by

$$\overline{\alpha}(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbb{R}.$$ 

Easy to check that $\overline{\alpha}$ is weakly continuous. (Exercise: Use Lebesgue dominated convergence theorem.)

Weak extension $\overline{\alpha}$ is unique because each function $v\chi_{\{0\}}$ can be approximated weakly by a sequence of continuous functions $\{\xi^k_0\}$:

$$\xi^k_0(s) := \begin{cases} (ks+1)v, & -\frac{1}{k} \leq s \leq 0 \\ 0 & -r \leq s < -\frac{1}{k}. \end{cases}$$
Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta} : (C \oplus F_1) \times (C \oplus F_1) \to \mathbb{R}$ for any continuous bilinear form $\beta : C \times C \to \mathbb{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of $\beta$ as a continuos linear map $C \to C^*$. Since $C^*$ is weakly complete ([D-S], I.13.22, p. 341), then $\beta$ is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in $C$ into weakly sequentially compact sets in $C^*$. By the Riesz representation theorem (for vector measures), there is a unique $C^*$-valued Borel measure $\lambda$ on $[-r, 0]$ (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s) \, d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in $F_1$ using weakly convergent sequences of type $\{\xi^k\}$. This gives a unique weakly continuous extension $\hat{\beta} : C \oplus F_1 \to C^*$.

Next for each $\eta \in C$, $v \in \mathbb{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \to \mathbb{R}$ to a weakly continuous linear map $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \oplus F_1 \to \mathbb{R}$. Thus $\overline{\beta}$ corresponds to the weakly continuous bilinear extension $\hat{\beta}(\cdot)(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \to \mathbb{R}$ of $\beta$. (Check this as exercise).
Finally use $\beta = D^2 \phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $D^2 \phi(\eta)$.

\[ \Box \]

**Lemma II.2.** ([Mo], Pitman Books, 1984)

For $t > 0$ define $W_t^* \in C$ by

$$W_t^*(s) := \begin{cases} \frac{1}{\sqrt{t}} |W(t + s) - W(0)|, & -t \leq s < 0, \\ 0 & -r \leq s \leq -t. \end{cases}$$

Let $\beta$ be a continuous bilinear form on $C$. Then

$$\lim_{t \to 0^+} \left[ \frac{1}{t} E\beta(\eta x_t - \eta_t, \eta x_t - \eta_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) \right] = 0$$

**Proof.**

Use

$$\lim_{t \to 0^+} E\left\| \frac{1}{\sqrt{t}} (\eta x_t - \eta_t - G(\eta) \circ W_t^*) \right\|^2_C = 0.$$ 

The above limit follows from the Lipschitz continuity of $H$ and $G$ and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of $\beta$, Hölder’s inequality and the above limit. ([Mo], Pitman Books, 1984, pp. 86-87.)

\[ \Box \]

**Lemma II.3.** ([Mo], Pitman Books, 1984)

Let $\beta$ be a continuous bilinear form on $C$ and $\{e_i\}_{i=1}^m$ be any basis for $\mathbb{R}^m$. Then

$$\lim_{t \to 0^+} \frac{1}{t} E\beta(\eta x_t - \eta_t, \eta x_t - \eta_t) = \sum_{i=1}^m \bar{\beta} (G(\eta)(e_i) \chi_{\{0\}}, G(\eta)(e_i) \chi_{\{0\}})$$

for each $\eta \in C$.

**Proof.**

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By taking coordinates reduce to the one-dimensional case $d = m = 1$:

$$\lim_{t \to 0^+} E\beta(W^*_t, W^*_t) = \overline{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with $W$ one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C \otimes \pi C$ in order to view the continuous \textit{bilinear} form $\beta$ as a continuous \textit{linear} functional on $C \otimes \pi C$. At this level $\beta$ commutes with the (Bochner) expectation. Rest of computation is effected using Mercer’s theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94. 

\textbf{Theorem II.3.} ([Mo], Pitman Books, 1984)

In (XIV) suppose $H$ and $G$ are globally bounded and Lipschitz. Let $S : D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. $\phi$ is $C^2$, $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$A(\phi)(\eta) = S(\phi)(\eta) + \overline{D\phi(\eta)} (H(\eta)\chi_{\{0\}})$$

$$+ \frac{1}{2} \sum_{i=1}^m \overline{D^2\phi(\eta)} (G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}).$$

where $\{e_i\}_{i=1}^m$ is any basis for $\mathbb{R}^m$.

\textbf{Proof.}

\textbf{Step 1.}

For fixed $\eta \in C$, use Taylor’s theorem:

$$\phi(\eta x_t) - \phi(\eta) = \phi(\eta_t) - \phi(\eta) + D\phi(\eta_t)(\eta x_t - \eta_t) + R(t)$$

a.s. for $t > 0$; where

$$R(t) := \int_0^1 (1 - u) D^2\phi(\eta_t + u(\eta x_t - \eta_t)) (\eta x_t - \eta_t, \eta x_t - \eta_t) du.$$
Take expectations and divide by \( t > 0 \):

\[
\frac{1}{t} E[\phi(nx_t) - \phi(\eta)] = \frac{1}{t} \left[ S_t(\phi(\eta) - \phi(\eta)) + D\phi(\bar{\eta}_t) \left\{ E\left[ \frac{1}{t} (nx_t - \bar{\eta}_t) \right] \right\} \right] + \frac{1}{t} E(R(t)) \tag{3}
\]

for \( t > 0 \).

As \( t \to 0^+ \), the first term on the RHS converges to \( S(\phi)(\eta) \), because \( \phi \in D(S) \).

**Step 2.**

Consider second term on the RHS of (3). Then

\[
\lim_{t \to 0^+} \left[ E\left\{ \frac{1}{t} (nx_t - \bar{\eta}_t) \right\} \right] (s) = \begin{cases} 
\lim_{t \to 0^+} \frac{1}{t} \int_0^t E[H(nx_u)] \, du, & s = 0 \\
0 & -r \leq s < 0.
\end{cases}
\]

\[= \left[ H(\eta)\chi_{\{0\}} \right](s), \quad -r \leq s \leq 0. \]

Since \( H \) is bounded, then \( \|E\left\{ \frac{1}{t} (nx_t - \bar{\eta}_t) \right\}\|_C \) is bounded in \( t > 0 \) and \( \eta \in C \) (**Exercise**). Hence

\[
w - \lim_{t \to 0^+} \left[ E\left\{ \frac{1}{t} (nx_t - \bar{\eta}_t) \right\} \right] = H(\eta)\chi_{\{0\}} \quad (\notin C).
\]

Therefore by Lemma II.1 and the continuity of \( D\phi \) at \( \eta \):

\[
\lim_{t \to 0^+} D\phi(\bar{\eta}_t) \left\{ E\left[ \frac{1}{t} (nx_t - \bar{\eta}_t) \right] \right\} = \lim_{t \to 0^+} D\phi(\eta) \left\{ E\left[ \frac{1}{t} (nx_t - \bar{\eta}_t) \right] \right\} = \frac{D\phi(\eta)\left( H(\eta)\chi_{\{0\}} \right)}{D\phi(\eta)}
\]

**Step 3.**

\[\text{12}\]
To compute limit of third term in RHS of (3), consider
\[
\left| \frac{1}{t} E D^2 \phi[\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t)](\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) \right|
\]
\[
- \frac{1}{t} E D^2 \phi(\eta)(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t)
\]
\[
\leq (E\|D^2 \phi[\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t)] - D^2 \phi(\eta)\|^2)^{1/2} \left[ \frac{1}{t^2} E\|\eta x_t - \tilde{\eta}_t\|^4 \right]^{1/2}
\]
\[
\leq K(t^2 + 1)^{1/2}[E\|D^2 \phi[\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t)] - D^2 \phi(\eta)\|^2]^{1/2}
\]
\[
\to 0
\]
as \( t \to 0^+ \), uniformly for \( u \in [0,1] \), by martingale properties of the Itô integral and the Lipschitz continuity of \( D^2 \phi \). Therefore by Lemma II.3

\[
\lim_{t \to 0^+} \frac{1}{t} E R(t) = \int_0^1 (1 - u) \lim_{t \to 0^+} \frac{1}{t} E D^2 \phi(\eta)(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) \ du
\]
\[
= \frac{1}{2} \sum_{i=1}^m D^2 \phi(\eta)(G(\eta)(e_i) \chi_{\{i\}}, G(\eta)(e_i) \chi_{\{i\}}).
\]
The above is a weak limit since \( \phi \in D(S) \) and has first and second derivatives globally bounded on \( C \). \( \square \)

5. Quasitame Functions

Recall that a function \( \phi : C \to \mathbb{R} \) is tame (or a cylinder function) if there is a finite set \( \{s_1 < s_2 < \cdots < s_k\} \) in \( [-r,0] \) and a \( C^\infty \)-bounded function \( f : (\mathbb{R}^d)^k \to \mathbb{R} \) such that

\[
\phi(\eta) = f(\eta(s_1), \cdots, \eta(s_k)), \quad \eta \in C.
\]

The set of all tame functions is a weakly dense subalgebra of \( C_b \), invariant under the static shift \( S_t \) and generates Borel \( C \). For \( k \geq 2 \) the tame function \( \phi \) lies outside the domain of strong continuity \( C^0_b \) of \( F_t \) and hence outside \( D(A) \) ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV.2.2, pp. 73-76). To overcome this difficulty we introduce
Definition.

Say $\phi : C \to \mathbb{R}$ is quasitame if there are $C^\infty$-bounded maps $h : (\mathbb{R}^d)^k \to \mathbb{R}$, $f_j : \mathbb{R}^d \to \mathbb{R}^d$, and piecewise $C^1$ functions $g_j : [-r, 0] \to \mathbb{R}, 1 \leq j \leq k-1$, such that

$$\phi(\eta) = h\left( \int_{-r}^{0} f_1(\eta(s))g_1(s)ds, \cdots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s)ds, \eta(0) \right)$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of $C^0_b$, invariant under $S_t$, generates $\mathcal{B}or\, C$ and belongs to $D(A)$. In particular, if $\phi$ is the quasitame function given by (4), then

$$A(\phi)(\eta) = \sum_{j=1}^{k-1} D_j h(m(\eta))\{f_j(\eta(0))g_j(0)-f_j(\eta(-r))g_j(-r)$$

$$-\int_{-r}^{0} f_j(\eta(s))g'_j(s)ds\}$$

$$+ D_k h(m(\eta))(H(\eta)) + \frac{1}{2} \text{trace}[D^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))]$$

for all $\eta \in C$, where

$$m(\eta) := \left( \int_{-r}^{0} f_1(\eta(s))g_1(s)ds, \cdots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s)ds, \eta(0) \right).$$

Remarks.

(i) Replace $C$ by the Hilbert space $M_2$. No need for the weak extensions because $M_2$ is weakly complete. Extensions of $D\phi(v, \eta)$ and $D^2\phi(v, \eta)$ correspond to partial derivatives in the $\mathbb{R}^d$-variable. Tame functions do not exist on $M_2$ but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbb{R}^d$).
Analysis of supermartingale behavior and stability of $\phi(x_t)$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbb{R}^d \times L^2((-\infty, 0], \mathbb{R}^d; \rho)$.

(ii) For each quasitame $\phi$ on $C$, $\phi(x_t)$ is a semimartingale, and the Itô formula holds:

$$d[\phi(x_t)] = A(\phi)(x_t) dt + \overline{D\phi(\eta)}(H(\eta)\chi_{(\eta)}) dW(t).$$