DEGENERATE STOCHASTIC DIFFERENTIAL EQUATIONS, FLOWS AND HYPOELLIPTICITY*

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1. Introduction.

In this article we shall study stochastic hereditary systems on $\mathbf{R}^d$, their flows and regularity of their solutions with respect to d-dimensional Lebesgue measure. More specifically we will state and outline the proofs of several results on the following issues:

I)Existence of smooth densities for solutions of stochastic hereditary equations whose covariances degenerate polynomially (anywhere) on hypersurfaces in $\mathbf{R}^d$.

II)Existence of smooth densities for diffusions with degeneracies of infinite order on a collection of hypersurfaces in $\mathbf{R}^d$.

III)Extension and refinement of Hörmander's hypoellipticity theorem for a large class of highly degenerate second order parabolic operators: Hörmander's Lie algebra condition is allowed to fail exponentially fast on the degeneracy hypersurfaces, which are imbedded in submanifolds of dimension less than d. The exponential decay rate near the degeneracy surface is found to be *optimal*.

Our proofs are based on the Malliavin calculus and require new sharp estimates for Itô processes in Euclidean space.

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2. Degenerate SDE's.

We shall consider stochastic differential equations (SDE's) on $\mathbf{R}^d$ driven by n-dimensional Brownian motion $W := (W_1, \cdots, W_n)$, with coefficients that may or may not depend on the history of the solution $x(t) \in \mathbf{R}^d$. More specifically we look at the following two types of SDE's:

Stochastic Hereditary Equations (SHE):

$$\left.\begin{aligned} dx(t) = G_0(x_t)dt + \sum_{i=1}^{n} g_i(x(t-r))\, dW_i(t), \quad 0 \le t \le a \\ x(t) = \eta(t), \qquad -r \le t \le 0. \end{aligned}\right\} \qquad (SHE)$$

where $x_t(s) := x(t+s), \quad -r \le s \le 0, \quad t \ge 0$, the segment of x on $[t-r, t]$.

Stochastic ODE's (SODE's, no memory):

$$\left.\begin{aligned} &dx(t) = g_0(x(t))dt + \sum_{i=1}^{n} g_i(x(t)) \circ dW_i(t), \quad 0 < t < a \\ &x(0) = x_0 \in \mathbf{R}^d, \end{aligned}\right\} \qquad (SODE)$$

Both (SHE) and (SODE) are defined on the canonical complete filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$, where

Ω is the space of all continuous paths $w : \mathbf{R}^+ \to \mathbf{R}^n$, $w(0) = 0$, in Euclidean space $\mathbf{R}^n$, with compact open topology;

$\mathcal{F}$ is the completed Borel σ-field of Ω;

P is Wiener measure on Ω;

$dW_i(t)$ and $\circ dW_i(t)$ denote Itô and Stratononvich stochastic differentials respectively;

r is a positive real;

$C := C([-r, 0], \mathbf{R}^d)$ is the Banach space of all continuous paths $\eta : [-r, 0] \to \mathbf{R}^d$ on d-dimensional Euclidean space $\mathbf{R}^d$, with sup norm

$$\|\eta\|_\infty := \sup_{-r \le s \le 0} |\eta(s)|;$$

and $|\cdot|$ is the Euclidean norm on $\mathbf{R}^d$.

The following conditions will be required:

Smoothness conditions:

(S_1) $G_0 : C \to \mathbf{R}^d$ is a continuous map such that at each $\eta \in C$ it possesses Fréchet derivatives $D^{(k)}G_0(\eta)$ for all $k \geq 1$, which are globally bounded in $\eta \in C$.

(S_2) $g := (g_1, \cdots, g_n) : \mathbf{R}^d \to \mathbf{R}^{d\times n}$ is C^∞ and bounded into the space $\mathbf{R}^{d\times n}$ of $d \times n$ matrices with the Euclidean norm. All derivatives $D^{(k)}g_i(v)$, $k \geq 1, 1 \leq i \leq n$, are globally bounded in $v \in \mathbf{R}^d$.

(S_3) $g_0 : \mathbf{R}^d \to \mathbf{R}^d$ is C^∞ and bounded with all derivatives $D^{(k)}g_0(v)$, $k \geq 1$, globally bounded in $v \in \mathbf{R}^d$.

Polynomial Degeneracy Condition:

(PD) Suppose there is a C^2 real-valued function $\phi : \mathbf{R}^d \to \mathbf{R}$, positive reals a, b and a neighborhood U of the surface $\phi^{-1}(0)$ such that

(i) $|\nabla\phi(v)| \geq b$ for all $v \in U$;

(ii)

$$g(v)g(v)^* \geq \begin{cases} \alpha I, & \text{if } v \notin U \\ |\phi(v)|^{2p} I, & \text{if } v \in U. \end{cases}$$

Note that condition $(PD)'$(ii) above implies that

(ii)′ $v \in \phi^{-1}(0)$ whenever $\hat{g}(v) := \inf\{|g(v)^*(e)| : e \in \mathbf{R}^d, \ \ |e| = 1\} = 0$.

Under Conditions (S_1) and (S_2) it is known that the stochastic hereditary equation (SHE) has a unique pathwise solution $x \in \mathcal{L}^2(\Omega, C([-r, a], \mathbf{R}^d)$ with initial path $\eta \in C([-r, 0], \mathbf{R}^d)$, (cf. Mohammed [Mo], 1984, pp. 36-39 and pp. 151-152, Kusuoka and Stroock [K-S]).

Our first result is as follows.

Theorem 1.

Assume (PD) for some $p \geq 1$, and the smoothness conditions (S_1), (S_2). Suppose that the initial path $\eta \in C([-r,0],\mathbf{R}^d)$ satisfies

$$\int_{s_1}^{s_2} [\phi(\eta(s))]^2 \, ds > 0,$$

for every $s_1, s_2 \in [-r,0]$ such that $s_1 < s_2$. Then, for each $0 < t \leq a$, the random variable $x(t)$ has a distribution which is absolutely continuous with respect to d-dimensional Lebesgue measure and has a C^∞ density.

Theorem 1 also holds when the drift G_0 is allowed to be depend on time and on the *whole* history $x|[-r,t]$. Using similar techniques to the ones outlined below one can also treat the case of a finite number of *moving degeneracy points* $v_i(t) \in \mathbf{R}^d$ of polynomial order. ([B-M], 1993).

Although the trajectory $x_t, t \geq 0$, gives a Feller process on C, (SHE) *never admits a stochastic flow on C when r is positive*; e.g. for $H \equiv 0$, $n = d = 1, g =$ identity, the trajectory of (SHE) has no Borel measurable version $X : \mathbf{R}^+ \times \Omega \times C \to C$ with the property that $X(t,\omega,\cdot) : C \to C$ is continuos or even locally bounded or linear ([Mo 1-2], 1984, 1986). This pathology poses difficulties in the computation of the Malliavin covariance matrix for the solution $x(t)$ of (SHE). By contrast, the diffusion equation (SODE) has a smooth flow of diffeomorphisms on $\mathbf{R}^d$. This flow is used to compute the necessary lower bounds on the covariance matrix for (SODE) in highly degenerate cases.

3. Diffusions with Exponential Degeneracies.

In (SODE) we impose the smoothness conditions (S_2), (S_3). In what follows we shall describe the type of degeneracy of infinite order under which the solution $x(t)$ of (SODE) admits a smooth density with respect to Lebesgue measure on $\mathbf{R}^d$.

For any positive integer m, let $G^{(m)}$ be the matrix with columns consisting of

$$g_1; \cdots ; g_n;$$

together with all vector fields of the form

$$[g_{i_1}, g_{i_2}]_{i_1,i_2=0}^n ; \cdots ; [g_{i_1}, [g_{i_2}, [g_{i_3}, \cdots , [g_{i_{m-1}}, g_{i_m}]] \cdot \cdot]]_{i_1,i_2,\cdots,i_m=0}^n ,$$

arranged in any specified order. The symbol $[\cdot, \cdot]$ denotes the Lie bracket operation on smooth vector fields on $\mathbf{R}^d$.

Define $\lambda^{(m)}(x)$ to be the smallest eigenvalue of $G^{(m)}(x) \, G^{(m)*}(x)$ for each $x \in \mathbf{R}^d$. Clearly $\lambda^{(m)}(x)$ is independent of the specific ordering of the columns above. Furthermore $\lambda^{(m)}(x) > 0$ for some $m \geq 1$ if and only if the parabolic operator $\frac{1}{2}\sum_{i=1}^{n} g_i^2 + g_0 + \frac{\partial}{\partial t}$ satisfies Hörmander's general Lie algebra condition at (t, x) for some $t > 0$ (and hence for every $t \in R$).

Definition.

A point $x \in \mathbf{R}^d$ is said to be a *Hörmander point* for the diffusion (SODE) if there is an integer $m \geq 1$ such that $\lambda^{(m)}(x) > 0$. Otherwise x is called a *non-Hörmander point.*

Note that the set H of all Hörmander points is open in $\mathbf{R}^d$. Its compliment H^c is the set of non-Hörmander points and is closed in $\mathbf{R}^d$. It is *not* a smooth submanifold of $\mathbf{R}^d$. It may have corners.

We can now state the following

Exponential Degeneracy Condition (ED)(p):

For a given point $x \in \mathbf{R}^d$ suppose there exists $m \geq 1$, an open neighborhood U of x, a C^2 function $\phi : U \to \mathbf{R}$, and an exponent $p \in (-1, 0)$ such that

(i) $\phi(x) = 0$ and $\nabla\phi(x) \cdot g_i(x) \neq 0$, for at least one $i = 1, \ldots, n$,

(ii) $\lambda^{(m)}(y) \geq \exp(-|\phi(y)|^p)$, for all $y \in U$.

Under the above exponential degeneracy condition we have the following theorem:

Theorem 2.

In (SODE) assume the smoothness conditions (S_2), (S_3) *and suppose that for each non-Hörmander point* $x \in R^d$ *there exists* $p \in (-1, 0)$ *such that the exponential degeneracy condition (ED)(p) holds. Then, for each* $t > 0$, *the diffusion* $x(t)$ *has a distribution which is absolutely continuous with respect to d-dimensional Lebesgue measure and has a* C^∞ *density.*

4. Hörmander's Theorem for Infinitely Degenerate Parabolic PDE's.

Consider the second-order partial differential operator

$$L := \frac{1}{2}\sum_{i=1}^{n} g_i^2 + g_0 + c.$$

where $c : \mathbf{R}^d \to \mathbf{R}$ is a smooth bounded function with all derivatives globally bounded.

Define $G^{(m)}, \lambda^{(m)}$ as in Section 3.

Let $Lie(g_0, g_1, \ldots, g_n)$ be the Lie algebra generated by the vector fields $g_0, g_1, \ldots, g_n$. By Hörmander's theorem ([H], Theorem 1.1), L is hypoelliptic on $\mathbf{R}^d$ if $Lie(g_0, g_1, \ldots, g_n)(x)$ is d-dimensional for every $x \in \mathbf{R}^d$. This condition characterizes hypoellipticity for L when its coefficients are real analytic. Such a characterization is not valid if the coefficients of L are smooth but *not analytic*. In fact we have the following example due to Kusuoka and Stroock:

Example ([K-S]):

Consider the differential operator

$$L_\sigma := \frac{\partial^2}{\partial x_1^2} + \sigma^2(x_1)\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

where σ is a C^∞ real-valued even function, non-decreasing on $[0, \infty)$, and vanishing (only) at zero. Then L_σ is hypoelliptic on $\mathbf{R}^3$ if and only if $\lim_{x \to 0+} x \log \sigma(x) = 0$ ([K-S], Theorem

8.41). E.g. L_σ with $\sigma(x) = \exp(-|x|^p)$ is hypoelliptic if $p \in (-1, 0)$. Hörmander's condition fails for this operator on the hyperplane $x_1 = 0$.

Our objective in this section is to establish a criterion for parabolic hypoellipticity of the operator L sharper than that of Hörmander, in the case where L has smooth (but not analytic) coefficients. We obtain hypoellipticity of the parabolic operator $L + \dfrac{\partial}{\partial t}$ on $\mathbf{R}^{d+1}$ (and hence of L on $\mathbf{R}^d$) under hypotheses that allow Hörmander's general condition for the parabolic operator to *fail at an exponential rate* on a collection of surfaces in $\mathbf{R}^d$.

Theorem 3.

Let D be an open set in $\mathbf{R}^d$. For the operator L assume the smoothness conditions (S_2), (S_3) and suppose that for every non-Hörmander point $x \in D$ there is a $p \in (-1, 0)$ such that the exponential degeneracy condition (ED)(p) holds. Then the differential operator $L + \dfrac{\partial}{\partial t}$ is hypoelliptic on $\mathbf{R} \times D$.

Remarks:

(i) Assume that the non-Hörmander set $H^c \cap D$ is imbedded as a closed subset of a C^2 submanifold in D of dimension less than d. Suppose further that at every point in $H^c \cap D$, at least one of the vector fields $X_1, \cdots, X_n$ is non-tangential to the submanifold. Then the transversality Condition (ED)(p)(i) is satisfied.

(ii) In the Kusuoka-Stroock example

$$L' := \frac{\partial^2}{\partial x_1^2} + \exp\left\{-\frac{1}{|x_1|}\right\} \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \; ,$$

the operator L' is *not hypoelliptic* on $\mathbf{R}^3$. Since hypoellipticity of $L' + \dfrac{\partial}{\partial t}$ on $\mathbf{R}^4$ implies hypoellipticity of L' on $\mathbf{R}^3$, then the lower bound -1 on p in condition ED(p) is *optimal.*

(iii) Oleinik and Radekevich ([O-R], Theorem 2.5.3, cf. [O]) have shown that if the non-Hörmander set H^c of L is compact and L satisfies condition (ED)(p)(i) at all points of H^c, then L is hypoelliptic. The above counterexample shows that if the compactness assumption on H^c is dropped, then a further hypothesis such as (ED)(p)(ii) is required which controls the rate at which the Hörmander condition fails as one approaches the non-Hörmander set.

(iv) The following alternative form of Theorem 3 shows that the non-vanishing condition in (ED)(p) can be weakened, provided that the range of p is restricted appropriately. In the statement of the Theorem 4 below, denote the action of the vector field g_i on a given C^∞ real-valued function ϕ by $g_i\phi$ for $1 \le i \le n$, and the action of the operator $L-c$ by $g_{n+1}\,\phi$. In this case, we have no reason to believe that the required lower bound, $-\frac{2}{(18)^r}$, on p is optimal for any value of $r > 1$!

Theorem 4.

The conclusion of Theorem 3 holds if condition (ED)(p) is replaced by the following:

(ED)´(p) There exists an integer $r \ge 1$, an open neighborhood $U \subseteq D$ of x, a C^∞ function $\phi : U \to \mathbf{R}$, and an exponent $p \in \left(-\frac{2}{(18)^r}, 0\right)$ such that

(i) $\phi(x) = 0$, and there exist $1 \le i_1, i_2, \cdots, i_r \le n+1$ such that

$$g_{i_1} g_{i_2} \cdots g_{i_r} \phi(x) \ne 0.$$

(ii) $\lambda^{(m)}(y) \ge \exp(-|\phi(y)|^p)$, for all $y \in U$.

4. Outlines of Proofs.

Here we will only give broad outlines of the proofs of Theorems 1-3. Complete details can be found in ([B-M 1-2]. The proofs use the Malliavin calculus [Ma] together with some precise probabilistic lower bounds for degenerate Itô processes. See Lemmas 1 and 2 below.

Proof of Theorem 1

In the following steps we outline a proof of Theorem 1. For further details the reader may refer to ([B-M 2]).

Step 1:

We use piecewise linear approximations of W in (SHE) to compute the Malliavin covariance matrix $C(T)$ of $x(T)$ as

$$C(T) = \int_0^T Z(u) g\big(u, x(u-r)\big) g\big(u, x(u-r)\big)^* Z(u)^* \, du,$$

where the $d \times d$ matrix-valued process $Z : [0,T] \times \Omega \to \mathbf{R}^{d \times d}$ satisfies the advanced *anticipating* Stratonovich integral equation

$$\begin{aligned} Z(t) = I &+ \int_{T \wedge (t+r)}^T Z(u) Dg\big(x(u-r)\big)(\cdot) \circ dW(u) \\ &+ \int_t^T Z(u) \big[\{D_{(2)} G_0(u,x)^*(\cdot)\}'(t)\big]^* \, du, \qquad 0 \le t \le T. \end{aligned}$$

In the above integral equation, $D_{(2)}G_0(u,x)$ is the Fréchet partial derivative of the map

$$(u,x) \to G_0(x_u)$$

with respect to $x \in C\big([-r,u], \mathbf{R}^d\big)$; and $D_{(2)}G_0(u,x)^*$ denotes the adjoint of the map $D_{(2)}G_0(u,x)$ considered as a linear operator from the Cameron Martin subspace of $C\big([-r,u], \mathbf{R}^d\big)$, into $\mathbf{R}^d$.

We solve the above integral equation as follows.

Start with the terminal condition $Z(T) = I$. On the last delay period $[(T-r) \vee 0, T]$ define Z to be the unique solution of the integral equation

$$Z(t) = I + \int_t^T Z(u) \big[\{D_{(2)} H(u,x)^*(\cdot)\}'(t)\big]^* \, du$$

for a.e. $t \in ((T-r) \vee 0, T)$. When $T > r$, use successive approximations to solve the anticipating integral equation, treating the stochastic integral as a predefined random *forcing term.* This gives a unique solution of the integral equation by successive backward steps of length r. Observe that the matrix $Z(t)$ need not be invertible for small t. It is interesting to compare $Z(t)$ with the analogous process for the diffusion case (SODE), (cf.

Step 1 in the proof of Theorems 2 and 3 below). The latter process is invertible *for all times* and its definition *does not* require anticipating stochastic integrals.

Step 2:

Since $D_{(2)}G_0(u,x)$ is globally bounded, then so is $[D_{(2)}G_0(u,x)^*(\cdot)]'(t)$ in (u,x,t). Hence we can choose a *deterministic* time $t_0 < T$ sufficiently close to T such that almost surely $Z(t)$ is invertible and $\|Z(t)^{-1}\| \leq 2$ for a.e. $t \in (t_0, T]$.

Step 3:

The above lower bound on $\|Z(t)\|$ and the representation of $C(T)$ imply that

$$\det C(T) \geq \frac{1}{4}\int_{t_0}^{T} \hat{g}(x(u-r))^2\, du \qquad \text{a.s.}$$

Recall that

$$\hat{g}(v) := \inf\{|g(v)^*(e)| : e \in \mathbf{R}^d, \quad |e| = 1\},$$

for all $v \in \mathbf{R}^d$.

Step 4:

In view of the polynomial degeneracy condition (PD), we prove the *Propagation Lemma*:

Let $-r < a < b < a + r$. Then the statement

$$P\left(\int_a^b |\phi(x(u))|^2\, du < \epsilon\right) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$,

implies that

$$P\left(\int_{a+r}^{b+r} |\phi(x(u))|^2\, du < \epsilon\right) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$.

Step 5:

By successively applying Step 4, we propagate the hypothesis on the initial path η in order to get the estimate:

$$P\Big(\int_{t_0}^{T} |\phi(x(u-r))|^2 du < \epsilon\Big) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$.

Step 6:

Using hypothesis (PD), Step 5, Jensen's inequality, and Lemma 3 of [B-M 3], we obtain

$$P\Big(\int_{t_0}^{T} \hat{g}\big(x(u-r)\big)^2 du < \epsilon\Big) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$.

Step 7:

Combining steps 3 and 6 gives

$$P\big(\det C(T) < \epsilon\big) = o(\epsilon^k)$$

as $\epsilon \to 0+$ for every $k \geq 1$. This implies that $C(T)^{-1}$ exists a.s. and $\det C(T)^{-1} \in \bigcap_{q=1}^{\infty} \mathcal{L}^q(\Omega, \mathbf{R})$. The conclusion of Theorem 1 now follows from Malliavin's theorem ([S]). $\square$

Proof of Theorem 3

Let x^x denote the solution of (SODE) starting at $x \in \mathbf{R}^d$, $C(t,x)$ the Malliavin covariance matrix of $x^x(t)$, and $\|\cdot\|_q$ the $\mathcal{L}^q$-norm on $\mathcal{L}^q(\Omega, \mathbf{R})$, $q \geq 1$. Set $\Delta(t,x) := \det C(t,x)$. In view of ([K-S], Theorem (8.13)), it is sufficient to verify the following :

For every $q \geq 1$ and every x in D, there exists a neighborhood $V \subseteq D$ of x such that

$$\lim_{t\to 0+} t \log\Big\{\sup_{y\in V} \|\Delta(t,y)^{-1}\|_q\Big\} = 0. \tag{$\star$}$$

In order to verify the above statement we introduce the

Definition.

A non-negative random variable X is said to be *exponentially positive* if there exist positive constants c_1 and c_2 such that

$$P(X < \epsilon) < \exp(-c_1\epsilon^{-1})$$

for all $\epsilon \in (0, c_2)$. We call c_1 and c_2 *characteristics* of X.

Observe that for an Itô process with bounded coefficients, the exit time from a ball is an exponentially positive random variable with characteristics depending only on the bound of the coefficients and the radius of the ball ([I-W], Lemma 10.5, p. 398).

We prove the following two key lemmas:

Lemma 1.

Let $y : [0,T] \times \Omega \to \mathbf{R}^d$ be the Itô process

$$dy(t) = \sum_{i=1}^{n} a_i(t)\, dW_i(t) + b(t)\, dt, \quad 0 \le t \le T,$$

where $a_1, \ldots, a_n, b : [0,T] \times \Omega \to \mathbf{R}^d$ are measurable $(\mathcal{F}_t)_{0\le t\le T}$-adapted processes, all bounded a.s. by a deterministic constant c_3. Suppose that $\tau \le T$ is an exponentially positive $(\mathcal{F}_t)_{0\le t\le T}$-stopping time such that at least one diffusion coefficient a_i satisfies the condition: a.s., $|a_i(s)| \ge \delta$, for all $0 \le s \le \tau$, for some deterministic $\delta > 0$. Then for every $m \ge 2$, there exist positive constants c_4, c_5 and T_0 such that for all $t \in (0, T_0)$ and $\epsilon \in (0, c_4\, t^{m+1})$, the following holds

$$P\left(\int_0^{t\wedge\tau} |y(u)|^m\, du < \epsilon\right) < \exp\left\{-c_5\epsilon^{-\frac{1}{m+1}}\right\}.$$

The constants c_4 and c_5 can be chosen to depend only on m, c_3, δ, and the characteristics of τ. The constant T_0 depends only on the characteristics of τ.

The second key lemma is a version of the composition lemma under the exponential degeneracy hypothesis (ED)(p).

Lemma 2.

Let τ be an exponentially positive $(\mathcal{F}_t)_{0\le t\le T}$-stopping time and let $p \in (-1,0)$. Suppose y is an Itô process with a.s bounded coefficients, and suppose further that y and τ satisfy the last estimate in Lemma 1 for some $m > -\dfrac{p}{p+1}$. Then there exist positive constants T_1, c_6, c_7 and $q > 1$ such that for all $t \in (0, T_1)$ and all $\epsilon < \exp\{-c_6 t^{-\frac{1}{q}}\}$, the following holds

$$P\left(\int_0^{t\wedge\tau} \exp(-|y(u)|^p)\, du < \epsilon\right) < \exp\{-c_7 |\log\epsilon|^q\}.$$

Furthermore, the constants T_1, c_6, c_7 and q are completely determined by c_3, c_4, c_5 in Lemma 1, p, m and the characteristics of τ.

Proofs of Lemmas 1 and 2 are given in [B-M 1].

In the following steps we verify the Kusuoka-Stroock condition ($\star$).

Step 1

The Malliavin covariance is given by

$$C(t,x) = Y^x(t) \int_0^t Z^x(s) g(x^x(s)) g(x^x(s))^* Z^x(s)^*\, ds\, [Y^x(t)]^*\,.$$

where $Y^x(t)$ is the derivative of the stochastic flow $x \to x^x(t,\omega)$ on $\mathbf{R}^d$ with respect to x for a.a. $\omega \in \Omega$, and $Z^x(t) := Y^x(t)^{-1}$.

These matrix-valued processes satisfy the integral equations

$$Y^x(t) = I + \sum_{i=1}^n \int_0^t Dg_i(x^x(s)) Y^x(s) \circ dW_i(s) + \int_0^t Dg_0(x^x(s)) Y^x(s)\, ds,$$

and

$$Z^x(t) = I - \sum_{i=1}^n \int_0^t Z^x(s) Dg_i(x^x(s)) \circ dW_i(s) - \int_0^t Z^x(s) Dg_0(x^x(s))\, ds.$$

See ([K-S], p.3-4), ([B], p.75).

Step 2

We obtain the following estimate by an elementary argument:

For every $q \geq 1$ and every bounded set $V \subset \mathbf{R}^d$ there exists a positive constant c_8 such that for all $t \in (0,T)$ and $x \in V$, we have

$$\|\Delta(t,x)^{-1}\|_{2q}^{2q} \leq c_8 \Big\{ 1 + \sum_{j=1}^{\infty} P\Big(Q(t,x) < j^{-\frac{1}{2dq}} \Big) \Big\},$$

where

$$Q(t,x) := \inf \Big\{ \sum_{i=1}^{n} \int_0^t < Z^x(u) g_i(x^x(u)), h >^2 \ du : h \in \mathbf{R}^d, |h| = 1 \Big\}.$$

Let $m \geq 1$, $x_0 \in \mathbf{R}^d$, $t \in (0,T)$ and let x belong to a fixed bounded neighborhood W of x_0. Define

$$\tau_1 := \inf \Big\{ s > 0 : |x^x(s) - x| \vee \|Z^x(s) - I\| = \frac{1}{2} \Big\} \wedge T \, .$$

Then (cf. [K-S], 1985) there exist positive constants c_9, c_{10} and exponents r_1, $r_2 \in (0,1)$ such that for all $t \in (0,T)$, $x \in W$, and $\epsilon \in (0, c_9)$, the following inequality holds :

$$\begin{aligned} &P(Q(t,x) < \epsilon) \leq \exp(-c_{10} \epsilon^{-r_1}) + \\ &+ \epsilon^{-d} \sup \Big\{ P\Big(\sum_{j=1}^{N} \int_0^{t \wedge \tau_1} < Z^x(u) K_j(x^x(u)), h >^2 \ du < \epsilon^{r_2} \Big) : |h| = 1 \Big\} \end{aligned}$$

where the vector fields $K_1, \ldots, K_N$ are the columns of the matrix function $G^{(m)}$.

Step 3

From the definition of τ_1 and Step 2 we get

$$P(Q(t,x) < \epsilon) \leq \exp(-c_{10} \epsilon^{-r_1}) + \epsilon^{-d} P\Big(\int_0^{t \wedge \tau_1} \lambda^{(m)}(x^x(u)) \, du < \epsilon^{r_2} \Big).$$

Step 4

Suppose x_0 is a Hörmander point. Then for some $m \geq 1$, $\lambda^{(m)}$ is bounded away from zero by some $\delta > 0$ in a neighborhood V of x_0. This fact and Step 3 imply that

$$P(Q(t,x) < \epsilon) \ \leq c_{11} \ \exp(-c_{12}\epsilon^{-c_{13}r_3})$$

provided $t > \dfrac{\epsilon^{r2}}{\delta}$, where $r_3 := r_1 \wedge r_2$; c_{11}, c_{12} and c_{13} are positive constants, independent of $(t,x) \in (0,T) \times V$.

Step 5

By Steps 2 and 4,

$$\|\Delta(t,x)^{-1}\|_{2q}^{2q} \leq c_8 \left\{ (\delta t)^{-\frac{2dq}{r_2}} + A(t) \right\},$$

where

$$\begin{aligned} A(t) &:= 1 + \sum_{j=k}^{\infty} c_{11} \exp(-c_{12} j^{r_4}), \\ &\leq 1 + \sum_{j=1}^{\infty} c_{11} \exp(-c_{12} j^{r_4}) < \infty, \end{aligned}$$

$r_4 := \dfrac{c_{13}r_3}{2dq} > 0$, and $k := [(\delta t)^{-\frac{2dq}{r_2}}]$ is the integer part of $(\delta t)^{-\frac{2dq}{r_2}}$. Thus $\|\Delta(t,x)^{-1}\|_q$ may not explode faster than *algebraically* as $t \downarrow 0$, *locally uniformly* with respect to x near x_0. Hence the Kusuoka-Stroock condition $(\star)$ holds.

Step 6

Let x_0 be a non-Hörmander point. By Itô's formula

$$d\phi(x^x(t)) = \sum_{i=1}^{n} \nabla\phi(x^x(t)).g_i(x^x(t))dW_i(t) + (L-c)\phi(x^x(t))\,dt.$$

for x near x_0. The transversality condition in (ED)(p) implies that the process $\phi(x^x(t))$ satisfies the hypotheses of Lemma 1. Applying Lemma 1 and 2, we deduce the existence of an exponentially positive stopping time τ_3 and positive constants c_6, c_7, T_1 and $q' > 1$, all independent of x near x_0 such that for all $t \in (0, T_1)$ and $\epsilon < \exp(-c_6 t^{-\frac{1}{q'}})$

$$P\left(\int_0^{t\wedge\tau_1\wedge\tau_3} \exp(-|\phi(x^x(u))|^p)\,du < \epsilon\right) < \exp\{-c_7|\log\epsilon|^{q'}\}.$$

Step 7

The estimates in Steps 3 and 6, and the condition (ED)(p) yield

$$P(Q(t,x) < \epsilon) \leq \exp(-c_{10}\epsilon^{-r_1}) + \epsilon^{-d}\exp(-c_7|\log\epsilon^{r_2}|^{q'})$$

for $t \in (0, T_1)$ and $\epsilon < \exp(-c_6 t^{-\frac{1}{q'}})$.

Step 8

Combining Step 7 with the first estimate in Step 2 gives

$$\|\Delta(t,x)^{-1}\|_{2q}^{2q} \leq c_8\{\exp(2dqc_6 t^{-\frac{1}{q'}}) + c_{14}\}, \quad 0 < t < T_1$$

where

$$c_{14} := 1 + \sum_{j=1}^{\infty}\left\{\exp\left(-c_{10} j^{\frac{r_1}{2dq}}\right) + j^{1/2q}\exp(-c_7|\log j^{-\frac{r_2}{2dq}}|^{q'})\right\} < \infty.$$

Note that the constants c_6, c_8 and c_{14} can all be chosen to be independent of x in a neighborhood of x_0. Hence $\|\Delta(t,x)^{-1}\|_q$ may not explode faster than *exponentially* as $t \downarrow 0$. Because $q' > 1$, the Kusuoka-Stroock condition $(\star)$ is also satisfied in this case. Thus the operator L is parabolic hypoelliptic. $\square$

Theorem 2 follows immediately from $(\star)$ and Theorem (3.17) in ([K-S]).

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