A Delayed Option Pricing Formula

Salah Mohammed
http://sfde.math.siu.edu/

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- Joint work with M. Arriojas, Y. Hu and G. Pap.

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- Hedging strategy.

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Objective: To derive an option pricing formula under stock-dynamics with finite memory. (Theorem 4).

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European call options can only be exercised at the maturity date.

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on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions.

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$C([-L, 0], \mathbf{R}) :=$ Banach space of continuous functions $[-L, 0] \to \mathbf{R}$ given the sup norm.

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Brownian motion: W-one-dimensional standard, adapted to $(\mathcal{F}_t)_{0 \leq t \leq T}$.

Feasibility of Delayed Stock Model

Model is feasible: Admits pathwise unique solution such that $S(t) > 0$ almost surely for all $t \geq 0$ whenever the initial path $\varphi(t) > 0$ for all $t \in [-L, 0]$.

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(i) $h : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ is continuous.

(ii) $g : \mathbf{R} \to \mathbf{R}$ is continuous.

(iii) Delays a and b are positive and fixed.

Theorem 1

Assume Hypotheses (E). Then the delayed stock model

$$\left.\begin{aligned} dS(t) &= h(t, S(t-a))S(t)\,dt + g(S(t-b))S(t)\,dW(t), \\ &\qquad\qquad\qquad\qquad t \in [0,T] \\ S(t) &= \varphi(t), \quad t \in [-L, 0] \end{aligned}\right\} \tag{1}$$

admits a pathwise unique solution S *for a given* $\mathcal{F}_0$*-measurable initial process* $\varphi : \Omega \to C([-L,0], \boldsymbol{R})$*. If* $\varphi(0) > 0$ *a.s., then* $S(t) > 0$ *a.s. for all* $t \geq 0$.

Proof of Theorem 1

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Define minimum delay $l := \min\{a, b\} > 0$.

Let $t \in [0, l]$. The delayed stock model gives

$$\left.\begin{aligned} dS(t) &= S(t)[h(t, \varphi(t-a))\,dt + g(\varphi(t-b))\,dW(t)] \\ &\qquad\qquad\qquad\qquad\qquad t \in [0, l] \\ S(0) &= \varphi(0). \end{aligned}\right\} \tag{1}$$

Proof of Theorem 1– Cont'd

Define the semimartingale

$$N(t) := \int_0^t h(u, \varphi(u-a))\, du + \int_0^t g(\varphi(u-b))\, dW(u),$$

for $t \in [0, l]$.

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Then (1) becomes

$$dS(t) = S(t)\, dN(t), \quad t > 0, \qquad S(0) = \varphi(0),$$

with the unique solution:

Proof of Theorem 1– Cont'd

$$
\begin{aligned}
S(t) &= \varphi(0)\exp\{N(t) - \frac{1}{2}[N,N](t)\}, \\
&= \varphi(0)\exp\Big\{\int_0^t h(u,\varphi(u-a))\,du \\
&\qquad + \int_0^t g(\varphi(u-b))\,dW(u) \\
&\qquad - \frac{1}{2}\int_0^t g(\varphi(u-b))^2\,du\Big\},
\end{aligned}
$$

for $t \in [0, l]$. This implies that $S(t) > 0$ almost surely for all $t \in [0, l]$, when $\varphi(0) > 0$ a.s..

Proof of Theorem 1– Cont'd

Similarly, since $S(l) > 0$, then $S(t) > 0$ for all $t \in [l, 2l]$ a.s.. Therefore $S(t) > 0$ for all $t \geq 0$ a.s., by induction using forward steps of lengths l.

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Above argument also gives existence and pathwise uniqueness of the strong solution to the delayed stock model. $\diamond$

Remark 1

In the delayed stock model, we need only require $\varphi(0) \geq 0$ (or $\varphi(0) > 0$) to conclude that a.s. $S(t) \geq 0$ for all $t \geq 0$ (or $S(t) > 0$ for all $t \geq 0$, resp.).

An Extension of the Model

Another feasible model for the stock price is

$$dS(t) = f(t, S^{t-a})S(t)\,dt + g(S(t-b))S(t)\,dW(t), \qquad t \in [0, T],$$

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where $f : [0,T] \times C([-L,T], \mathbf{R}) \to \mathbf{R}$ is a continuous functional; and $S^t \in C([-L,T], \mathbf{R})$, $t \in [-L,T]$, is defined by

$$S^t(s) := S(t \wedge s),\ \ s \in [-L,T],$$

for $S \in C([-L,T], \mathbf{R})$.

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- Positive delays a, b; h, g continuous.
- $g(v) \neq 0$ whenever $v \neq 0$.

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- Obtain an equivalent local martingale measure (via Girsanov’s theorem).
- Establish completeness and no-arbitrage property of the market.
- Obtain a hedging strategy.

Discounted Stock

Let

$$\widetilde{S}(t) := \frac{S(t)}{B(t)} = e^{-rt}S(t), \qquad t \in [0, T],$$

be the discounted stock price. Then by the product rule:

$$\begin{aligned} d\widetilde{S}(t) &= e^{-rt}dS(t) + S(t)(-re^{-rt})\,dt \\ &= \widetilde{S}(t)\Big[\big\{h(t, S(t-a)) - r\big\}\,dt \\ &\qquad +g\big(S(t-b)\big)\,dW(t)\Big]. \end{aligned}$$

Discounted Stock – Cont'd

Define

$$\widehat{S}(t) := \int_0^t \left\{ h(u, S(u-a)) - r \right\} du + \int_0^t g\big(S(u-b)\big)\, dW(u),$$

for $t \in [0, T]$.

Discounted Stock – Cont'd

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for $t \in [0, T]$.

Then

$$d\widetilde{S}(t) = \widetilde{S}(t)\, d\widehat{S}(t), \quad 0 < t < T. \tag{2}$$

Discounted Stock – Cont'd

Since $\widetilde{S}(0) = \varphi(0)$, then

$$\widetilde{S}(t) = \varphi(0) + \int_0^t \widetilde{S}(u)\, d\widehat{S}(u), \qquad t \in [0, T]. \tag{3}$$

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To establish an equivalent local martingale measure, recall Girsanov's theorem:

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Let $W(t)$, $t \in [0,T]$, *be a standard Wiener process on* $(\Omega, \mathcal{F}, P)$. *Let* Σ *be a predictable process such that* $\int_0^T |\Sigma(u)|^2 du < \infty$ *a.s.. Define*

$$\varrho(t) := \exp\left\{ \int_0^t \Sigma(u)\, dW(u) - \frac{1}{2}\int_0^t |\Sigma(u)|^2\, du \right\},$$

for $t \in [0,T]$. *Suppose that* $E_P(\varrho(T)) = 1$, *where* E_P *denotes expectation with respect to the probability measure* P. *Define the probability measure* Q *on* $(\Omega, \mathcal{F})$ *by* $dQ := \varrho(T)\, dP$.

Theorem 2 – Cont'd

Then the process

$$\widehat{W}(t) := W(t) - \int_0^t \Sigma(u)\,du, \qquad t \in [0,T],$$

is a standard Wiener process under the measure Q.

Backward Conditioning

Apply Girsanov's theorem with the process

$$\Sigma(u) := -\frac{\{h(u, S(u-a)) - r\}}{g\big(S(u-b)\big)}, \qquad u \in [0, T].$$

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The hypothesis on g implies that Σ is well-defined, since by Theorem 1, $S(t) > 0$ for all $t \in [0, T]$ a.s.. Clearly $\Sigma(t)$, $t \in [0, T]$, is a predictable process.

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The process $S(t)$, $t \in [0, T]$, is a.s. bounded because it is sample continuous. The hypothesis on g implies that $1/g(v)$, $v \in (0, \infty)$, is bounded on bounded intervals. Thus $\int_0^T |\Sigma(u)|^2 du < \infty$ a.s..

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Set $\mathcal{F}_t := \mathcal{F}_0$ for $t \leq 0$.

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Let $l := \min\{a, b\}$, minimum delay.
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Then $\Sigma(u)$, $u \in [0, T]$, is measurable with respect to the σ-algebra $\mathcal{F}_{T-l}$.

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Then $\Sigma(u)$, $u \in [0, T]$, is measurable with respect to the σ-algebra $\mathcal{F}_{T-l}$.

Hence, the stochastic integral $\int_{T-l}^{T} \Sigma(u)\, dW(u)$ *conditioned on* $\mathcal{F}_{T-l}$ has a normal distribution with mean zero and variance $\int_{T-l}^{T} \Sigma(u)^2\, du$.

Backward Conditioning– Cont'd

By normality (e.g. moment generating function):

$$E_P\left(\exp\left\{\int_{T-l}^{T}\Sigma(u)\,dW(u)\right\}\Big|\,\mathcal{F}_{T-l}\right) = \exp\left\{\frac{1}{2}\int_{T-l}^{T}|\Sigma(u)|^2\,du\right\}$$

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a.s.. Hence

$$E_P\left(\exp\left\{\int_{T-l}^{T}\Sigma(u)dW(u) - \frac{1}{2}\int_{T-l}^{T}|\Sigma(u)|^2du\right\}\Bigg|\,\mathcal{F}_{T-l}\right) = 1, \quad \text{a.s..}$$

Backward Conditioning– Cont'd

This implies:

$$E_P\left(\exp\left\{\int_0^T \Sigma(u)\,dW(u) - \frac{1}{2}\int_0^T |\Sigma(u)|^2 du\right\}\,\middle|\,\mathcal{F}_{T-l}\right)$$
$$= \exp\left\{\int_0^{T-l} \Sigma(u)\,dW(u) - \frac{1}{2}\int_0^{T-l} |\Sigma(u)|^2\,du\right\}$$

a.s..

Backward Conditioning– Cont'd

Let k to be a positive integer such that $0 \leq T - kl \leq l$. Successive conditioning using backward steps of length l, and induction give:

$$E_P\left(\exp\left\{\int_0^T \Sigma(u)\,dW(u) - \frac{1}{2}\int_0^T |\Sigma(u)|^2 du\right\}\Bigg|\,\mathcal{F}_{T-kl}\right)$$

$$= \exp\left\{\int_0^{T-kl} \Sigma(u)\,dW(u) - \frac{1}{2}\int_0^{T-kl} |\Sigma(u)|^2\,du\right\}$$

a.s..

Backward Conditioning– Cont'd

Take conditional expectation with respect to $\mathcal{F}_0$ on both sides of above equation:

$$E_P\Bigg(\exp\Bigg\{\int_0^T \Sigma(u)\,dW(u) - \frac{1}{2}\int_0^T |\Sigma(u)|^2 du\Bigg\}\Bigg|\,\mathcal{F}_0\Bigg)$$

$$= E_P\Bigg(\exp\Bigg\{\int_0^{T-kl} \Sigma(u)\,dW(u) - \frac{1}{2}\int_0^{T-kl} |\Sigma(u)|^2 du\Bigg\}\Bigg|\,\mathcal{F}_0\Bigg) = 1$$

a.s..

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Taking the expectation of the above equation, we get

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where

$$\varrho(T) := \exp\left\{ -\int_0^T \frac{\{h(u, S(u-a)) - r\}}{g\big(S(u-b)\big)}\, dW(u) \right. \\ \left. -\frac{1}{2}\int_0^T \left| \frac{h(u, S(u-a)) - r}{g\big(S(u-b)\big)} \right|^2 du \right\}$$

a.s..

Martingale Measure

Therefore, the Girsanov theorem (Theorem 2) applies and the process

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{h(u, S(u-a)) - r\}}{g\big(S(u-b)\big)} du, \quad t \in [0, T],$$

is a standard Wiener process under the measure Q defined by:

$$dQ := \varrho(T)\, dP.$$

Martingale Measure– Cont'd

Since

$$\widehat{S}(t) = \int_0^t g\big(S(u-b)\big)\, d\widehat{W}(u), \qquad t \in [0,T], \qquad (4)$$

then $\widehat{S}(t)$, $t \in [0,T]$, is a continuous Q-local martingale.

Martingale Measure– Cont'd

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then $\widehat{S}(t)$, $t \in [0,T]$, is a continuous Q-local martingale.

By the representation

$$\widetilde{S}(t) = \varphi(0) + \int_0^t \widetilde{S}(u)\, d\widehat{S}(u), \qquad t \in [0,T], \qquad (3)$$

the discounted stock price $\widetilde{S}(t)$, $t \in [0,T]$, is also a continuous Q-local martingale.

No Aribtrage

I.e. Q is an equivalent local martingale measure.

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By well-known results on trading strategies (e.g., Theorem 7.1 in [K.K]), it follows that the market consisting of $\{B(t), S(t) : t \in [0, T]\}$ satisfies the no-arbitrage property:

No Aribtrage

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By well-known results on trading strategies (e.g., Theorem 7.1 in [K.K]), it follows that the market consisting of $\{B(t), S(t) : t \in [0, T]\}$ satisfies the no-arbitrage property: There is no admissible self-financing strategy which gives an arbitrage opportunity.

Completeness

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By proof of Theorem 1, the solution of the delayed stock model (1) satisfies:

$$S(t) = \varphi(0) \exp\left\{ \int_0^t g\big(S(u-b)\big)\, dW(u) + \int_0^t h(u, S(u-a))\, du - \frac{1}{2}\int_0^t g\big(S(u-b)\big)^2 du \right\}$$

a.s. for $t \in [0,T]$.

Completeness– Cont'd

Hence,

$$\begin{aligned}\widetilde{S}(t) =&\varphi(0)\exp\left\{\int_0^t g\big(S(u-b)\big)\,d\widehat{W}(u)\right.\\ &\left.-\frac{1}{2}\int_0^t g\big(S(u-b)\big)^2 du\right\}\end{aligned} \tag{5}$$

for $t \in [0,T]$.

Completeness– Cont'd

By definitions of $\tilde{S}, \widehat{W}, \widehat{S}$ and equation (2), then for $t \geq 0$, $\mathcal{F}_t^S = \mathcal{F}_t^{\widetilde{S}} = \mathcal{F}_t^{\widehat{W}} = \mathcal{F}_t^W$, the σ-algebras generated by $\{S(u) : u \leq t\}$, $\{\widetilde{S}(u) : u \leq t\}$, $\{\widehat{W}(u) : u \leq t\}$, $\{W(u) : u \leq t\}$, respectively. (Clearly, $\mathcal{F}_t^W \subseteq \mathcal{F}_t$.)

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Let X be a contingent claim, viz. an integrable non-negative $\mathcal{F}_T^S$-measurable random variable. Consider the Q-martingale

$$M(t) := E_Q(e^{-rT} X \,|\, \mathcal{F}_t^S) = E_Q(e^{-rT} X \,|\, \mathcal{F}_t^{\widehat{W}}),$$

for $t \in [0, T]$.

Completeness– Cont'd

By the martingale representation theorem, there exists an $(\mathcal{F}_t^{\widehat{W}})$-predictable process $h_0(t)$, $t \in [0,T]$, such that

$$\int_0^T h_0(u)^2\,du < \infty \qquad a.s.,$$

and

$$M(t) = E_Q(e^{-rT}X) + \int_0^t h_0(u)\,d\widehat{W}(u), \qquad t \in [0,T].$$

Completeness– Cont'd

Combining the two relations

$$d\widetilde{S}(t) = \widetilde{S}(t)\, d\widehat{S}(t), \; d\widehat{S}(t) = g\big(S(t-b)\big)\, d\widehat{W}(t),$$

gives:

$$d\widetilde{S}(t) = \widetilde{S}(t) g\big(S(t-b)\big)\, d\widehat{W}(u), \; t \in [0, T].$$

Completeness– Cont'd

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gives:

$$d\widetilde{S}(t) = \widetilde{S}(t) g\big(S(t-b)\big)\, d\widehat{W}(u), \; t \in [0, T].$$

Define

$$\pi_S(t) := \frac{h_0(t)}{\widetilde{S}(t) g\big(S(t-b)\big)}, \; \pi_B(t) := M(t) - \pi_S(t)\widetilde{S}(t)$$

for $t \in [0, T]$.

Completeness– Cont'd

Consider the strategy $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$ which consists of holding $\pi_S(t)$ units of the stock and $\pi_B(t)$ units of the bond at time t. The value of the portfolio at any time $t \in [0, T]$ is:

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).$$

Completeness– Cont'd

Consider the strategy $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$ which consists of holding $\pi_S(t)$ units of the stock and $\pi_B(t)$ units of the bond at time t. The value of the portfolio at any time $t \in [0, T]$ is:

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).$$

By the product rule and the definition of the strategy $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$, get

$$\begin{aligned} dV(t) &= e^{rt}dM(t) + M(t)d(e^{rt}) \\ &= \pi_B(t)d(e^{rt}) + \pi_S(t)dS(t), \end{aligned}$$

for $t \in [0, T]$.

Completeness– Cont'd

Hence, $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$ is a self-financing strategy. Moreover, $V(T) = e^{rT}M(T) = X$ a.s.. Therefore, the contingent claim X is attainable; thus the market $\{B(t), S(t) : t \in [0, T]\}$ is complete: (every contingent claim is attainable).

Completeness– Cont'd

Hence, $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$ is a self-financing strategy. Moreover, $V(T) = e^{rT} M(T) = X$ a.s.. Therefore, the contingent claim X is attainable; thus the market $\{B(t), S(t) : t \in [0, T]\}$ is complete: (every contingent claim is attainable).

For the augmented market $\{B(t), S(t), X : t \in [0, T]\}$ to satisfy the no-arbitrage property, the price of the claim X must be

$$V(t) = e^{-r(T-t)} E_Q(X \,|\, \mathcal{F}_t^S)$$

at each $t \in [0, T]$ a.s. See, e.g., [B.R] or Theorem 9.2 in [K.K].

Delayed Option Pricing Formula

Summarize above discussion in the following formula for the fair price $V(t)$ of an option on the delayed stock.

Theorem 3

Suppose that the stock price S is given by the delayed stock model, where $\varphi(0) > 0$ and g satisfies the given hypotheses. Let T be the maturity time of an option (contingent claim) on the stock with payoff function X, i.e., X is an $\mathcal{F}_T^S$-measurable non-negative integrable random variable. Then at any time $t \in [0, T]$, the fair price $V(t)$ of the option is given by the formula

$$V(t) = e^{-r(T-t)} E_Q(X \mid \mathcal{F}_t^S), \qquad (6)$$

Theorem 3 – Cont'd

where Q *denotes the probability measure on* $(\Omega, \mathcal{F})$ *defined by* $dQ := \varrho(T)\, dP$ *with*

$$\varrho(t) := \exp\left\{ -\int_0^t \frac{\{h(u, S(u-a)) - r\}}{g\big(S(u-b)\big)}\, dW(u) - \frac{1}{2}\int_0^t \left| \frac{h(u, S(u-a)) - r}{g\big(S(u-b)\big)} \right|^2 du \right\}$$

for $t \in [0, T]$.

Theorem 3 – Cont'd

The measure Q is a local martingale measure and the market is complete.

Moreover, there is an adapted and square integrable process $h_0(u),\ u \in [0,T]$ such that

$$E_Q(e^{-rT}X \,|\, \mathcal{F}_t^S) = E_Q(e^{-rT}X) + \int_0^t h_0(u)\, d\widehat{W}(u),$$

for $t \in [0,T]$,where $\widehat{W}$ is a standard Q-Wiener process given by

Theorem 3 – Cont’d

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{h(u, S(u-a)) - r\}}{g\big(S(u-b)\big)} du, \quad t \in [0, T],$$

The hedging strategy is given by

$$\pi_S(t) := \frac{h_0(t)}{\widetilde{S}(t) g\big(S(t-b)\big)}, \qquad (7)$$

$$\pi_B(t) := M(t) - \pi_S(t)\widetilde{S}(t),$$

for $t \in [0, T]$.

Delayed B-S Formula

The following result is a consequence of Theorem 3. It gives a Black-Scholes-type formula for the value of a European option on the stock at times prior to maturity.

Delayed B-S Formula

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Formula is explicit during last delay period before maturity, or when delay is larger than maturity interval.

Theorem 4

Assume the conditions of Theorem 3. Let $V(t)$ *be the fair price of a European call option written on the stock* S *with exercise price* K *and maturity time* T. *Let* φ *denote the standard normal distribution function:*

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \qquad x \in \boldsymbol{R}.$$

Then for all $t \in [T-l, T]$ *(where* $l := \min\{a, b\}$*),* $V(t)$ *is given by*

$$V(t) = S(t)\varphi(\beta_+(t)) - Ke^{-r(T-t)}\varphi(\beta_-(t)), \tag{8}$$

Theorem 4 – Cont'd

where

$$\beta_{\pm}(t) := \frac{\log \frac{S(t)}{K} + \int_t^T \left(r \pm \frac{1}{2} g(S(u-b))^2\right) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

If $T > l$ *and* $t < T - l$, *then*

$$V(t) = e^{rt} E_Q \left(H \left(\widetilde{S}(T-l), -\frac{1}{2} \int_{T-l}^{T} g\big(S(u-b)\big)^2 du, \int_{T-l}^{T} g\big(S(u-b)\big)^2 du \right) \Bigg| \mathcal{F}_t \right) \tag{9}$$

Theorem 4 – Cont'd

where H is given by

$$H(x,m,\sigma^2) := xe^{m+\sigma^2/2}\varphi(\alpha_1(x,m,\sigma)) - Ke^{-rT}\varphi(\alpha_2(x,m,\sigma)),$$

and

$$\alpha_1(x,m,\sigma) := \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m + \sigma^2\right],$$

$$\alpha_2(x,m,\sigma) := \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m\right],$$

for $\sigma, x \in \boldsymbol{R}^+$, $m \in \boldsymbol{R}$.

Theorem 4 – Cont'd

The hedging strategy is given by

$$\begin{aligned} \pi_S(t) &= \varphi(\beta_+(t)), \\ \pi_B(t) &= -Ke^{-rT}\varphi(\beta_-(t)), \end{aligned} \tag{10}$$

for $t \in [T - \ell, T]$.

Remarks 2

If $g(x) = 1$ for all $x \in \mathbf{R}^+$ then equation (8) reduces to the classical Black and Scholes formula.

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In contrast with the classical (non-delayed) Black and Scholes formula, the fair price $V(t)$ in the delayed model in Theorem 4 depends not only on the stock price $S(t)$ at the present time t, but also on the whole segment $\{S(v) : v \in [t-b, T-b]\}$. ($[t-b, T-b] \subset [0, t]$ since $t \geq T - l$ and $l \leq b$.)

Proof of Theorem 4

Consider a European call option in the above market with exercise price K and maturity time T. Taking $X = (S(T) - K)^+$ in Theorem 3, the fair price $V(t)$ of the option is given by

$$\begin{aligned} V(t) &= e^{-r(T-t)} E_Q\big(\big(S(T) - K\big)^+ \,|\, \mathcal{F}_t\big) \\ &= e^{rt} E_Q\big(\big(\widetilde{S}(T) - Ke^{-rT}\big)^+ \,|\, \mathcal{F}_t\big), \end{aligned} \tag{11}$$

at any time $t \in [0, T]$.

Proof of Theorem 4 – Cont'd

We now derive an explicit formula for the option price $V(t)$ at any time $t \in [T-l, T]$. The representation (5) of $\widetilde{S}(t)$ implies:

$$\widetilde{S}(T) = \widetilde{S}(t) \exp\left\{ \int_t^T g\big(S(u-b)\big)\, d\widehat{W}(u) - \frac{1}{2}\int_t^T g\big(S(u-b)\big)^2 du \right\}$$

for all $t \in [0, T]$. Clearly $\widetilde{S}(t)$ is $\mathcal{F}_t$-measurable. If $t \in [T-l, T]$, then $-\frac{1}{2}\int_t^T g\big(S(u-b)\big)^2 du$ is also $\mathcal{F}_t$-measurable.

Proof of Theorem 4 – Cont’d

Furthermore, when conditioned on $\mathcal{F}_t$, the distribution of $\int_t^T g\big(S(u-b)\big)\, d\widehat{W}(u)$ under Q is the same as that of $\sigma\xi$, where ξ is a Gaussian $N(0,1)$-distributed random variable, and $\sigma^2 = \int_t^T g\big(S(u-b)\big)^2 du$. Consequently, the fair price at time t is given by

$$V(t) = e^{rt} H \left(\widetilde{S}(t), -\frac{1}{2} \int_t^T g\left(S(u-b)\right)^2 du, \int_t^T g\big(S(u-b)\big)^2 du \right),$$

Proof of Theorem 4 – Cont'd

where

$$H(x, m, \sigma^2) := E_Q(xe^{m+\sigma\xi} - Ke^{-rT})^+,$$

for $\sigma, x \in \mathbf{R}^+$, $m \in \mathbf{R}$. Now, an elementary computation yields the following:

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2}\varphi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\varphi(\alpha_2(x, m, \sigma)).$$

Proof of Theorem 4 – Cont'd

Therefore, $V(t)$ takes the form:

$$V(t) = S(t)\varphi(\beta_+) - Ke^{-r(T-t)}\varphi(\beta_-), \qquad (12)$$

where

$$\beta_\pm = \frac{\log\frac{S(t)}{K} + \int_t^T \left(r \pm \frac{1}{2}g(S(u-b))^2\right) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

Proof of Theorem 4 – Cont'd

For $T > l$ and $t < T - l$, from the representation (5) of $\widetilde{S}(t)$, we have

$$\widetilde{S}(T) = \widetilde{S}(T-l)\exp\left\{\int_{T-l}^{T} g\big(S(u-b)\big)\, d\widehat{W}(u) - \frac{1}{2}\int_{T-l}^{T} g\big(S(u-b)\big)^2 du\right\}.$$

Proof of Theorem 4 – Cont'd

Consequently, the option price at time t with $t < T - l$ is given by

$$V(t) = e^{rt} E_Q \left(H \left(\widetilde{S}(T - l), -\frac{1}{2} \int_{T-l}^{T} g\big(S(u-b)\big)^2 du, \right.\right.$$
$$\left.\left. \int_{T-l}^{T} g\big(S(u-b)\big)^2 du \right) \middle| \mathcal{F}_t \right).$$

Proof of Theorem 4 – Cont'd

Consequently, the option price at time t with $t < T - l$ is given by

$$V(t) = e^{rt} E_Q \left(H \left(\widetilde{S}(T - l), -\frac{1}{2} \int_{T-l}^{T} g\big(S(u-b)\big)^2 du, \int_{T-l}^{T} g\big(S(u-b)\big)^2 du \right) \Big| \mathcal{F}_t \right).$$

To calculate the hedging strategy for $t \in [T - \ell, T]$, it suffices to use an idea from [B.R], pages 95–96. This completes the proof of the theorem. $\diamond$

Remark 3

During last delay period $[T-l,T]$, it is possible to rewrite the option price $V(t), t \in [T-l,T]$ in terms of the solution of a random Black-Scholes pde of the form

$$\left.\begin{aligned} \frac{\partial F(t,x)}{\partial t} &= -\frac{1}{2} g(S(t-b))^2 x^2 \frac{\partial^2 F(t,x)}{\partial x^2} - rx \frac{\partial F(t,x)}{\partial x} \\ &\qquad + rF(t,x), \quad 0 < t < T \\ F(T,x) &= (x-K)^+, \quad x > 0. \end{aligned}\right\} \tag{13}$$

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Above time-dependent random final-value problem admits a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted random field $F(t,x)$.

Remark 3 – Cont'd

Using the classical Itô-Ventzell formula ([Kun]) and (6) of Theorem 3, it can be shown that

$$V(t) = e^{-r(T-t)} F(t, S(t)), \quad t \in [T-b, T].$$

Remark 3 – Cont'd

Using the classical Itô-Ventzell formula ([Kun]) and (6) of Theorem 3, it can be shown that

$$V(t) = e^{-r(T-t)} F(t, S(t)), \quad t \in [T-b, T].$$

Note that the above representation is no longer valid if $t \leq T - b$, because in this range, the solution F of the final-value problem (9) is *anticipating* with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

A Stock Model with Variable Delay

Consider an alternative model for the stock price dynamics with variable delay.

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Throughout this section, suppose h is a given fixed positive number. Denote $\lfloor t \rfloor := kh$ if $kh \leq t < (k+1)h$. Suppose market consist of a riskless asset ξ with a variable (deterministic) continuous rate of return λ, and a stock S satisfying sdde

$$\left.\begin{array}{rcl} d\xi(t) & = & \lambda(t)\xi(t)\,dt \\ dS(t) & = & f(t, S(\lfloor t \rfloor))S(t)dt + g(t, S(\lfloor t \rfloor))S(t)dW(t) \end{array}\right\} \tag{14}$$

for $t \in (0, T]$.

A Model with Variable Delay – Cont'd

Initial conditions $\xi(0) = 1$ and $S(0) > 0$.

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$f : [0, T] \times \mathbf{R} \to \mathbf{R}$ is continuous.

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$g(t, v) \neq 0$ for all $(t, v) \in [0, T] \times \mathbf{R}$.

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The model is feasible: That is $S(t) > 0$ a.s. for all $t > 0$.

A Model with Variable Delay – Cont'd

Initial conditions $\xi(0) = 1$ and $S(0) > 0$.

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The model is feasible: That is $S(t) > 0$ a.s. for all $t > 0$.

Follows by an argument similar to the proof of Theorem 1.

Theorem 5

Suppose that the stock price S is given by the sdde (14), where $S(0) > 0$ and f, g satisfy given hypotheses. Let T be the maturity time of an option (contingent claim) on the stock with payoff function X, i.e., X is an $\mathcal{F}_T^S$-measurable non-negative integrable random variable. Then at any time $t \in [0, T]$, the fair price $V(t)$ of the option is given by the formula

$$V(t) = E_Q(X \mid \mathcal{F}_t^S) e^{-\int_t^T \lambda(s)\, ds}, \tag{15}$$

where Q denotes the probability measure on $(\Omega, \mathcal{F})$ defined by $dQ := \varrho(T)\, dP$ with

Theorem 5 – Cont'd

$$\varrho(t) := \exp\left\{ -\int_0^t \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))} \, dW(u) - \frac{1}{2}\int_0^t \left| \frac{f(u, S(\lfloor u \rfloor)) - \lambda(u)}{g(u, S(\lfloor u \rfloor))} \right|^2 du \right\}$$

for $t \in [0, T]$*. The measure* Q *is a local martingale measure and the market is complete.*

Theorem 5 – Cont'd

Moreover, there is an adapted and square integrable process $h_1(t),\ t \in [0,T]$, *such that*

$$E_Q\left(\frac{X}{\xi(T)}\,\middle|\,\mathcal{F}_t^S\right) = E_Q\left(\frac{X}{\xi(T)}\right) + \int_0^t h_1(u)\,d\widehat{W}(u), \quad t \in [0,T],$$

where

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))}\,du,\ \ t \in [0,T].$$

Theorem 5 – Cont'd

The hedging strategy is given by

$$\pi_S(t) := \frac{h_1(t)}{\widetilde{S}(t)g(t, S(\lfloor t \rfloor))}, \qquad (16)$$

$$\pi_\xi(t) := M(t) - \pi_S(t)\widetilde{S}(t),$$

for $t \in [0, T]$.

Theorem 5 – Cont'd

The hedging strategy is given by

$$\pi_S(t) := \frac{h_1(t)}{\widetilde{S}(t)g(t, S(\lfloor t \rfloor))}, \tag{16}$$

$$\pi_\xi(t) := M(t) - \pi_S(t)\widetilde{S}(t),$$

for $t \in [0, T]$.

The following result gives a Black-Scholes-type formula for the value of a European option on the stock at any time prior to maturity.

Theorem 6

Assume the conditions of Theorem 5. Let $V(t)$ be the fair price of a European call option written on the stock S with exercise price K and maturity time T. Then for all $t \in \big[T - \lfloor T \rfloor, T\big], V(t)$ is given by

$$V(t) = S(t)\varphi(\beta_+(t)) - K\varphi(\beta_-(t))e^{-\int_t^T \lambda(s)ds}, \quad (17)$$

where

$$\beta_\pm(t) := \frac{\log \frac{S(t)}{K} + \int_t^T \left(\lambda(u) \pm \frac{1}{2} g(u, S(\lfloor u \rfloor))^2\right) du}{\sqrt{\int_t^T g(u, S(\lfloor u \rfloor))^2 du}}.$$

Theorem 6 – Cont'd

If $T > h$ *and* $t < T - \lfloor T \rfloor$*, then*

$$V(t) = e^{\int_0^t \lambda(s)ds} E_Q \Bigg(H \Bigg(\widetilde{S}(T - \lfloor T \rfloor), - \frac{1}{2} \int_{T-\lfloor T \rfloor}^{T} g(u, S(\lfloor u \rfloor))^2 du, \int_{T-\lfloor T \rfloor}^{T} g(u, S(\lfloor u \rfloor))^2 du \Bigg) \Bigg| \mathcal{F}_t \Bigg) \tag{18}$$

where H *is given by*

Theorem 6 – Cont'd

$$H(x,m,\sigma^2) := xe^{m+\sigma^2/2}\varphi(\alpha_1(x,m,\sigma)) - K\varphi(\alpha_2(x,m,\sigma))e^{-\int_0^T \lambda(s)ds},$$

and

$$\alpha_1(x,m,\sigma) := \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + \int_0^T \lambda(s)ds + m + \sigma^2\right],$$

$$\alpha_2(x,m,\sigma) := \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + \int_0^T \lambda(s)ds + m\right],$$

for $\sigma, x \in \boldsymbol{R}^+$, $m \in \boldsymbol{R}$.

Theorem 6 – Cont'd

The hedging strategy is given by

$$\pi_S(t) = \varphi(\beta_+(t)),$$

$$\pi_\xi(t) = -K\varphi(\beta_-(t))e^{-\int_0^T \lambda(s)ds},$$

for $t \in \left[T - \lfloor T \rfloor, T\right]$.

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Stock Dynamics-Simulation

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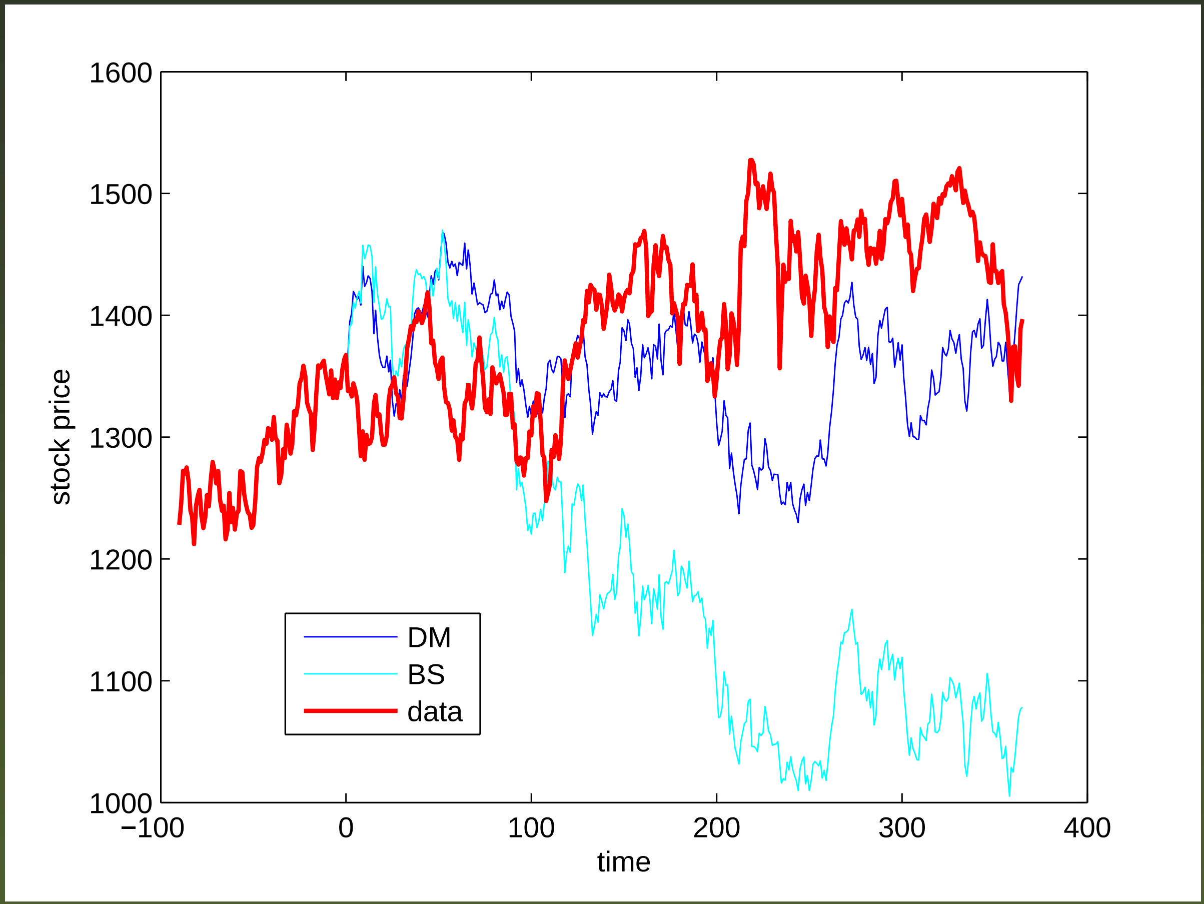

Stock prices when $h =$ constant, $b = 2$, $T = 365$

Stock data: DJX Index at CBOE.

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THANK YOU!