SEMILINEAR SPDEs AS DYNAMICAL SYSTEMS

Salah Mohammed

http://sfde.math.siu.edu/

Institut Mittag-Leffler
Royal Swedish Academy of Sciences

Sweden: September 25, 2007

Department of Mathematics, SIU-C, Carbondale, Illinois, USA
Acknowledgment

- Joint work with T.S. Zhang and H. Zhao.
Acknowledgment

- Joint work with T.S. Zhang and H. Zhao.

- Research supported by (US) NSF Grants DMS-9703852, DMS-9975462, DMS-0203368 and DMS-0705970.
Examples of semilinear spdes.
Outline

- Examples of semilinear SPDEs.
- Smooth cocycles in Hilbert space. Stationary trajectories.

Existence of cocycles generated by linear and semilinear stochastic evolution equations (SSEs). (Kolmogorov’s continuity theorem fails).

Linearization of a cocycle along a stationary trajectory.

Ergodic theory of cocycles in Hilbert space.

Hyperbolicity of stationary trajectories (via Lyapunov exponents). (M.Z.Z.)
Examples of semilinear spdes.

Smooth cocycles in Hilbert space. Stationary trajectories.

Existence of cocycles generated by linear and semilinear stochastic evolution equations (sees). (Kolmogorov’s continuity theorem fails). [Mo.2].
Examples of semilinear SPDEs.

Smooth cocycles in Hilbert space. Stationary trajectories.

Existence of cocycles generated by linear and semilinear stochastic evolution equations (sees). (Kolmogorov’s continuity theorem fails). [Mo.2].

Linearization of a cocycle along a stationary trajectory.
Examples of semilinear SPDEs.

Smooth cocycles in Hilbert space. Stationary trajectories.

Existence of cocycles generated by linear and semilinear stochastic evolution equations (sees). (Kolmogorov’s continuity theorem fails). [Mo.2].

Linearization of a cocycle along a stationary trajectory.

Ergodic theory of cocycles in Hilbert space.
Examples of semilinear SPDEs.

Smooth cocycles in Hilbert space. Stationary trajectories.

Existence of cocycles generated by linear and semilinear stochastic evolution equations (sees). (Kolmogorov’s continuity theorem fails). [Mo.2].

Linearization of a cocycle along a stationary trajectory.

Ergodic theory of cocycles in Hilbert space.

Hyperbolicity of stationary trajectories (via Lyapunov exponents).
Examples of semilinear spdes.

Smooth cocycles in Hilbert space. Stationary trajectories.

Existence of cocycles generated by linear and semilinear stochastic evolution equations (sees). (Kolmogorov’s continuity theorem fails). [Mo.2].

Linearization of a cocycle along a stationary trajectory.

Ergodic theory of cocycles in Hilbert space.

Hyperbolicity of stationary trajectories (via Lyapunov exponents).

Stable manifolds. ([M.Z.Z]).
Notation

- \((\Omega, \mathcal{F}, P) := \) probability space; e.g. Wiener space.
**Notation**

- $(\Omega, \mathcal{F}, P) :=$ probability space; e.g. *Wiener space*.
- $\theta : \mathbb{R} \times \Omega \to \Omega$ group of $P$-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$; e.g. *Wiener shift*:
  \[
  \theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \ \omega \in \Omega.
  \]
Notation

- \((\Omega, \mathcal{F}, P) := \text{probability space}; \text{ e.g. Wiener space}.\)
- \(\theta : \mathbb{R} \times \Omega \rightarrow \Omega \ \text{group of } P\text{-preserving ergodic transformations on } (\Omega, \mathcal{F}, P); \text{ e.g. Wiener shift:}\)
  \[\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \ \omega \in \Omega.\]
- \(H := \text{real (separable) Hilbert space, norm } | \cdot |.\)
(\Omega, \mathcal{F}, P) := \text{probability space; e.g. Wiener space.}

\theta : \mathbb{R} \times \Omega \to \Omega \text{ group of } P\text{-preserving ergodic transformations on } (\Omega, \mathcal{F}, P); \text{ e.g. Wiener shift:}

\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega.

H := \text{real (separable) Hilbert space, norm } | \cdot |.

\mathcal{B}(H) := \text{Borel } \sigma\text{-algebra of } H.
Notation

- \((\Omega, \mathcal{F}, P) := \) probability space; e.g. Wiener space.
- \(\theta : \mathbb{R} \times \Omega \to \Omega\) group of \(P\)-preserving ergodic transformations on \((\Omega, \mathcal{F}, P)\); e.g. Wiener shift:
  \[
  \theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \ \omega \in \Omega.
  \]
- \(H := \) real (separable) Hilbert space, norm \(| \cdot |\).
- \(\mathcal{B}(H) := \) Borel \(\sigma\)-algebra of \(H\).
- \(M := \) \(d\)-dimensional smooth (oriented) compact Riemannian manifold with boundary \(\partial M\).
Notation

- \((\Omega, \mathcal{F}, P) :=\) probability space; e.g. Wiener space.
- \(\theta : \mathbb{R} \times \Omega \to \Omega\) group of \(P\)-preserving ergodic transformations on \((\Omega, \mathcal{F}, P)\); e.g. Wiener shift:
  \[
  \theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \ \omega \in \Omega.
  \]
- \(H :=\) real (separable) Hilbert space, norm \(| \cdot |\).
- \(\mathcal{B}(H) :=\) Borel \(\sigma\)-algebra of \(H\).
- \(M :=\) \(d\)-dimensional smooth (oriented) compact Riemannian manifold with boundary \(\partial M\).
- \(d\xi :=\) Riemannian volume on \(M\).
\( \Delta := \text{Laplacian on } M. \)
Notation-Contd

- $\Delta := \text{Laplacian on } M$.

- $H^k_0(M, \mathbb{R}) := \text{Sobolev space of all functions } u : M \to \mathbb{R} \text{ (vanishing on } \partial M) \text{ with all derivatives up to order } k \text{ square-integrable with respect to } d\xi$. 

$H^k_0(M, \mathbb{R})$ is a Hilbert space under usual Sobolev norm.
\( \Delta := \) Laplacian on \( M \).

\( H^k_0(M, \mathbb{R}) := \) Sobolev space of all functions \( u : M \rightarrow \mathbb{R} \) (vanishing on \( \partial M \)) with all derivatives up to order \( k \) square-integrable with respect to \( d\xi \). \( H^k_0(M, \mathbb{R}) \) is a Hilbert space under usual Sobolev norm.

\( L^{(j)}(H) := \) continuous \( H \)-valued \( j \)-multilinear maps on \( H \).
**Examples: Affine Linear SEEs**

**Affine Linear SEEs (Additive Noise):**

\[
\begin{align*}
\frac{du(t, x)}{dt} &= -Au(t, x) dt + B_0 dW(t), \quad t > 0 \\
u(0, x) &= x \in H.
\end{align*}
\]

\(A\) hyperbolic: \(0 \notin \sigma(A)\)–discrete bounded below.

\(W\) Brownian motion with covariance Hilbert space \(K\).

\(B_0 : K \to H\), Hilbert Schmidt. **Mild solutions.**
Examples: Affine Linear SEE s

**Affine Linear SEE s (Additive Noise):**

\[ du(t, x) = -Au(t, x) \, dt + B_0 \, dW(t), \quad t > 0 \]
\[ u(0, x) = x \in H. \]

A hyperbolic: \( 0 \notin \sigma(A) \) – discrete bounded below.

\( W \) Brownian motion with covariance Hilbert space \( K \).

\( B_0 : K \rightarrow H \), Hilbert Schmidt. **Mild solutions.**

See has stationary solution, and affine linear semiflow on \( H \).
Stochastic Reaction-Diffusion Equation:

\[ du = \frac{1}{2} \Delta u \, dt + (1 - |u|^\alpha)u \, dt + \sum_{i=1}^{\infty} \sigma_i u \, dW_i(t), \]

\[ W_i := \text{independent standard Brownian motions on } \mathbb{R}. \]

\[ \sigma_i \in H^s_0(M, \mathbb{R}), \ s > 2 + d/2; \ \sum_{i=1}^{\infty} \|\sigma_i\|_{H^s_0}^2 < \infty. \]

Dirichlet boundary conditions. **Weak solutions.**
Stochastic Reaction-Diffusion Equation:

\[ du = \frac{1}{2} \Delta u \, dt + (1 - |u|^\alpha)u \, dt + \sum_{i=1}^{\infty} \sigma_i u \, dW_i(t), \]

\( W_i := \) independent standard Brownian motions on \( \mathbb{R} \).

\( \sigma_i \in H^s_0(M, \mathbb{R}) \), \( s > 2 + d/2 \); \( \sum_{i=1}^{\infty} \| \sigma_i \|_{H^s_0}^2 < \infty \).

Dirichlet boundary conditions. Weak solutions.

Has \( C^1 \) stochastic semiflow on \( H := L^2(M, \mathbb{R}) \) for \( \alpha < \frac{4}{d} \).

Lipschitz semiflow for \( \alpha \) even integer.
Stochastic Heat Equation:

\[ du(t) = \frac{1}{2} \Delta u(t) \, dt + \sum_{i=1}^{\infty} \sigma_i u(t) \, dW_i(t) + f(u(t)) \, dt \]

\[ u(0) = \psi \in H^k_0(M) \]

\( W_i \) as above; \( \sigma_i \in H^s_0(M, \mathbb{R}) \), \( \sum_{i=1}^{\infty} \| \sigma_i \|_{H^s_0}^2 < \infty \), 
\( s > k + d/2; \, d := \text{dim}M; \, f : \mathbb{R} \to \mathbb{R} \) is \( C^\infty_b \).

Dirichlet boundary conditions. Weak solutions.
Stochastic Heat Equation:

\[ du(t) = \frac{1}{2} \Delta u(t) \, dt + \sum_{i=1}^{\infty} \sigma_i u(t) \, dW_i(t) + f(u(t)) \, dt \]

\[ u(0) = \psi \in H_0^k(M) \]

\( W_i \) as above; \( \sigma_i \in H_0^s(M, \mathbb{R}) \), \( \sum_{i=1}^{\infty} \| \sigma_i \|_{H_0^s}^2 < \infty \), \( s > k + d/2 \); \( d := \text{dim}M \); \( f : \mathbb{R} \to \mathbb{R} \) is \( C^\infty_b \).

Dirichlet boundary conditions. **Weak solutions.**

Has \( C^\infty \) stochastic semiflow on \( H_0^k(M) \) for \( k > \frac{d}{2} \).
Semilinear Parabolic SPDEs:

In stochastic heat equation replace $\Delta$ by a second order self-adjoint elliptic linear differential operator:

$$L := \sum_{i,j=1}^{d} a_{ij}(\xi) \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{d} b_i(\xi) \frac{\partial}{\partial \xi_i}$$

on $M$.

Dirichlet boundary condition. Weak solutions.

Smooth coefficients $a_{i,j} : M \to \mathbb{R}$, $b_i : M \to \mathbb{R}$. 
View parabolic spde as a **semilinear stochastic evolution equation** (see):

\[
du(t) = -Au(t) \, dt + F(u(t)) \, dt + \sum_{i=1}^{\infty} B_i u(t) \, dW_i(t)
\]

\[
u(0) = x \in H \equiv H_0^k(M).
\]

\[
A \equiv -L, \quad B_i(u) \equiv \sigma_i u, \quad F(u) \equiv f \circ u, \quad u \in H.
\]
View parabolic spde as a semilinear stochastic evolution equation (see):

\[ du(t) = -Au(t) \, dt + F(u(t)) \, dt + \sum_{i=1}^{\infty} B_i u(t) \, dW_i(t) \]

\[ u(0) = x \in H := H_0^k(M). \]

A := \(-L, B_i(u) := \sigma_i u, F(u) := f \circ u, u \in H. \]

Let \( k > \frac{d}{2}. \) Then Nemytskii operator \( F : H \to H \) is \( C^\infty. \)

Smooth stochastic semiflow on \( H_0^k(M). \)
Burgers Equation

Considered by many authors in recent years. (e.g. [E.K.M.S]).

One-dimensional stochastic Burgers equation:

\[ du + u \frac{\partial u}{\partial \xi} \, dt = \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} \, dt + \sum_{i=1}^{\infty} \sigma_i(\xi) \, dW_i(t) \]
Considered by many authors in recent years. (e.g. [E.K.M.S]).

One-dimensional stochastic Burgers equation:

\[ du + u \frac{\partial u}{\partial \xi} \, dt = \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} \, dt + \sum_{i=1}^{\infty} \sigma_i(\xi) \, dW_i(t) \]

\( W_i \) independent one dimensional Brownian motions.

\( \sigma_i \in C^2([0, 1]) \); \( \| \sigma_i \|_{C^2} \leq \frac{C}{i^2}, \ i \geq 1 \). Mild solutions.

Has \( C^1 \) stochastic semiflow on \( L^2([0, 1], \mathbb{R}) \).
The Cocycle

$k = \text{non-negative integer, } \epsilon \in (0, 1]. H \text{ Hilbert.}

A $C^{k,\epsilon}$ perfect cocycle $(U, \theta)$ on $H$ is a measurable random field $U : \mathbb{R}^+ \times H \times \Omega \to H$ such that:

For each $\epsilon$, the map $\mathbb{R}^+ \times H \times (t;x) \mapsto U(t;x;\epsilon)$ is continuous; for fixed $(t;\epsilon)$, the map $H \times x \mapsto U(t;x;\epsilon)$ is $C^k$ in $x$ on bounded sets in $H$.

SEMILINEAR SPDEs AS DYNAMICAL SYSTEMS – p.12/71
The Cocycle

$k = \text{non-negative integer, } \varepsilon \in (0, 1]. H \text{ Hilbert.}

A $C^{k,\varepsilon}$ perfect cocycle $(U, \theta)$ on $H$ is a measurable random field $U : \mathbb{R}^+ \times H \times \Omega \to H$ such that:

- For each $\omega \in \Omega$, the map

$$\mathbb{R}^+ \times H \ni (t, x) \mapsto U(t, x, \omega) \in H$$

is continuous; for fixed $(t, \omega) \in \mathbb{R}^+ \times \Omega$, the map

$$H \ni x \mapsto U(t, x, \omega) \in H$$

is $C^{k,\varepsilon}$ ($D^k U(t, x, \omega)$ is $C^\varepsilon$ in $x$ on bounded sets in $H$).
The Cocycle-Contd

\[ U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega) \]

for all \( t_1, t_2 \in \mathbb{R}^+ \), all \( \omega \in \Omega \).
The Cocycle-Contd

- \( U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega) \)
  for all \( t_1, t_2 \in \mathbb{R}^+, \) all \( \omega \in \Omega. \)
- \( U(0, x, \omega) = x \) for all \( x \in H, \omega \in \Omega. \)
The Cocycle Property

$U(t_1, \cdot, \omega)$

$U(t_2, \cdot, \theta(t_1, \omega))$

$U(t_1, x, \omega)$

$U(t_1 + t_2, x, \omega)$

$\theta(t_1, \cdot)$

$\theta(t_2, \cdot)$

$\theta(t_1 + t_2, \omega)$
A random variable $Y : \Omega \to H$ is a \textit{stationary point} for the cocycle $(U, \theta)$ if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $t \in \mathbb{R}^+$ and every $\omega \in \Omega$. 
A random variable $Y : \Omega \to H$ is a stationary point for the cocycle $(U, \theta)$ if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $t \in \mathbb{R}^+$ and every $\omega \in \Omega$.

Denote a stationary trajectory by

$$U(t, Y) = Y(\theta(t)).$$
A random variable \( Y : \Omega \rightarrow H \) is a **stationary point** for the cocycle \((U, \theta)\) if

\[
U(t, Y(\omega), \omega) = Y(\theta(t, \omega))
\]

for all \( t \in \mathbb{R}^+ \) and every \( \omega \in \Omega \).

Denote a stationary trajectory by

\[
U(t, Y) = Y(\theta(t)).
\]

For sde’s: a non-anticipating stationary point corresponds to an invariant measure for the one-point motion.
Linearization

Linearize a $C^{k,\epsilon}$ cocycle $(U, \theta)$ along a stationary random point $Y$:

Get an $L(H)$-valued cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$. Follows from cocycle property of $U$ and chain rule:
Linearization

Linearize a $C^{k,\varepsilon}$ cocycle $(U, \theta)$ along a stationary random point $Y$:

Get an $L(H)$-valued cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$.

Follows from cocycle property of $U$ and chain rule:

$$DU(t_1 + t_2, Y(\omega), \omega)$$

$$= DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \circ DU(t_1, Y(\omega), \omega)$$

for all $\omega \in \Omega, t_1, t_2 \geq 0$. 
Linearization-contd

Assume $U(t, \cdot, \omega)$ locally compact and

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq 1} \| DU(t_2, Y(\theta(t_1)), \theta(t_1)) \|_{L(H)} < \infty.$$ 

Apply **Oseledec-Ruelle Theorem** to linearized cocycle ([Ru.2]):
Linearization-contd

Assume $U(t, \cdot, \omega)$ locally compact and

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq 1} \|DU(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(H)} < \infty.$$ 

Apply **Oseledec-Ruelle Theorem** to linearized cocycle $([\text{Ru}.2])$:

Get a sequence of closed finite-codimensional Oseledec spaces

$$\cdots E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = H,$$

all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$. 

SEMILINEAR SPDEs AS DYNAMICAL SYSTEMS – p.17/71
Linearization-contd

Obtain Lyapunov spectrum

\[ \{ \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1 \}; \]

\[
\lim_{t \to \infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(x)|
\]

\[ = \begin{cases} 
\lambda_i & \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega), \\
-\infty & \text{if } x \in E_\infty(\omega).
\end{cases} \]
Obtain Lyapunov spectrum

\[ \{ \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1 \}; \]

\[ \lim_{t \to \infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(x)| \]

\[ = \begin{cases} 
\lambda_i & \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega), \\
-\infty & \text{if } x \in E_\infty(\omega).
\end{cases} \]

\[ E_i(\omega) = \{ x \in H : \lim_{t \to \infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(x)| \leq \lambda_i \}, \]

\[ i \geq 1. \]
Linearization: Spectral Theorem

\[ DU(t, Y(\omega), \omega) \]

\[ E_1 = H \]
\[ E_2(\omega) \]
\[ E_3(\omega) \]

\[ \theta(t, \cdot) \]

\[ H \]
\[ E_2(\theta(t, \omega)) \]
\[ E_3(\theta(t, \omega)) \]
Hyperbolicity

A stationary point \( Y(\omega) \) of \((U, \theta)\) is \textit{hyperbolic} if the linearized cocycle \( DU(t, Y(\omega), \omega), \theta(t, \omega) \) has a non-zero Lyapunov spectrum

\[
\{ \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1 \}.
\]

That is

\[
\lambda_i \neq 0 \quad \text{for all} \quad i \geq 1.
\]
Hyperbolicity

A stationary point $Y(\omega)$ of $(U, \theta)$ is *hyperbolic* if the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-zero Lyapunov spectrum

$$\{ \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1 \}.$$ 

That is

$$\lambda_i \neq 0 \quad \text{for all} \quad i \geq 1.$$ 

(*Expect hyperbolicity to be a “generic” property.*)
Hyperbolicity

A stationary point $Y(\omega)$ of $(U, \theta)$ is hyperbolic if the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-zero Lyapunov spectrum

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}.$$ 

That is

$$\lambda_i \neq 0 \quad \text{for all} \quad i \geq 1.$$ 

(Expect hyperbolicity to be a “generic” property.)

Ergodicity: $\lambda_1 < 0.$
\{ \mathcal{U}(\omega), \mathcal{S}(\omega) : \omega \in \Omega^* \} := \text{unstable and stable subspaces associated with the linearized cocycle} \ (DU, \theta) \ (\text{[Mo.3], [M.S]}).
\{ \mathcal{U}(\omega), \mathcal{S}(\omega) : \omega \in \Omega^* \} := \text{unstable and stable subspaces associated with the linearized cocycle } (DU, \theta) ([\text{Mo.3}], [\text{M.S}]).

Then get a measurable invariant splitting

\[ H = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \quad \omega \in \Omega^*, \]

\[ DU(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)), \]

\[ DU(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \]

for all \( t \geq 0. \)
Hyperbolicity-Contd

Have **exponential dichotomies:**

\[
|DU(t, Y(\omega), \omega)(x)| \geq |x|e^{\delta_1 t}
\]

for all \( t \geq \tau_1^*, x \in \mathcal{U}(\omega); \)

\[
|DU(t, Y(\omega), \omega)(x)| \leq |x|e^{-\delta_2 t}
\]

for all \( t \geq \tau_2^*, x \in \mathcal{S}(\omega), \) with \( \tau_i^* = \tau_i^*(x, \omega) > 0, \) \text{random times and } \delta_i > 0, \text{ fixed, } i = 1, 2.\]
Hyperbolicity-Contd

\[ DU(t, Y(\omega), \omega) \]

\[ S(\omega) \]

\[ U(\omega) \]

\[ 0 \]

\[ \theta(t, \cdot) \]

\[ \Omega \]

\[ \omega \]

\[ \theta(t, \omega) \]
Linear SEE\text{s}

Existence of semiflows for mild solutions of linear sees:

\[ du(t, x, \cdot) = -Au(t, x, \cdot) \, dt + Bu(t, x, \cdot) \, dW(t), \quad t > 0 \]

\[ u(0, x, \omega) = x \in H. \]
Existence of semiflows for mild solutions of linear sees:

\[ du(t, x, \cdot) = -Au(t, x, \cdot) \, dt + Bu(t, x, \cdot) \, dW(t), \quad t > 0 \]

\[ u(0, x, \omega) = x \in H. \]

\( A : D(A) \subset H \rightarrow H \) closed linear operator on a separable real Hilbert space \( H \).

\( A \) has complete orthonormal system of eigenvectors \( \{e_n : n \geq 1\} \) with corresponding (bounded below) (non-zero) eigenvalues \( \{\mu_n, n \geq 1\} \); i.e.,

\[ Ae_n = \mu_n e_n, \quad n \geq 1; \]

e.g. \( A = -\Delta \) on compact smooth Riemannian manifold.
\(-A\) generates a strongly continuous semigroup of bounded linear operators

\[ T_t : H \to H, \ t \geq 0. \]

\(W(t), \ t \geq 0, \ E\)-valued cylindrical Brownian motion on canonical filtered Wiener space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\).

\(K \subset E\) Hilbert-Schmidt embedding. ([D.Z]).
Linear SEEs-Contd

$(-A)$ generates a strongly continuous semigroup of bounded linear operators

$$T_t : H \to H, \ t \geq 0.$$ 

$W(t), \ t \geq 0$, $E$-valued cylindrical Brownian motion on canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. $K \subset E$ Hilbert-Schmidt embedding. ([D.Z]).

$L_2(K, H) := \text{Hilbert space}$ of all Hilbert-Schmidt operators $S : K \to H$; H-S norm

$$\|S\|_2 := \left[ \sum_{k=1}^{\infty} |S(f_k)|^2 \right]^{1/2}.$$
Linear SEEs-Contd

\( f_k, \ k \geq 1, \) cons in \( K. \)
\[ | \cdot | := \text{norm on } H. \ \mathcal{L}_2(H) := \mathcal{L}_2(H, H). \]

\( B : H \rightarrow \mathcal{L}_2(K, H) \) bounded (affine) linear operator.

Stochastic integral in (see) as in ([D.Z]).
Linear SEEs-Contd

\( f_k, \ k \geq 1, \) cons in \( K. \)

\(| \cdot | := \) norm on \( H. \) \( L_2(H) := L_2(H, H). \)

\( B : H \rightarrow L_2(K, H) \) bounded (affine) linear operator.

Stochastic integral in (see) as in ([D.Z] ).

\( \theta : \mathbb{R} \times \Omega \rightarrow \Omega \) standard \( P \)-preserving ergodic Wiener shift on \( \Omega. \) \( (W, \theta) \) is a helix:

\[ W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega)) \]

for all \( t_1, t_2 \in \mathbb{R}, \ \omega \in \Omega. \)
A mild solution of the linear see is a family of $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H$, $x \in H$, s.t.

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot) \, dW(s), \quad t \geq 0.$$  

Integral equation holds $x$-almost surely, $x \in H$. 

**Mild Solutions**
A *mild solution* of the linear see is a family of 
\((\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(H)))\)-measurable, \((\mathcal{F}_t)_{t \geq 0}\)-adapted processes \(u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H, \ x \in H, \) s.t.

\[
    u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot) \, dW(s), \quad t \geq 0.
\]

Integral equation holds \(x\)-almost surely, \(x \in H\).

Is \(u(t, x, \cdot)\) pathwise continuous linear in \(x\)?
Kolmogorov Fails!

Kolmogorov’s continuity theorem fails for random field $I : L^2([0, 1], \mathbb{R}) \to L^2(\Omega, \mathbb{R})$

$$I(x) := \int_0^1 x(t) \, dW(t), \quad x \in L^2([0, 1], \mathbb{R}).$$
Kolmogorov Fails!

**Kolmogorov’s continuity theorem fails** for random field $I : L^2([0, 1], \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R})$

$$I(x) := \int_0^1 x(t) \, dW(t), \quad x \in L^2([0, 1], \mathbb{R}).$$

No continuous (or even Borel measurable linear!) selection

$$L^2([0, 1], \mathbb{R}) \times \Omega \rightarrow \mathbb{R}$$

$$(x, \omega) \mapsto I(x, \omega)$$

of $I$ ([Mo.1]).
Lift semigroup $T_t$, $t \geq 0$, to a strongly continuous semigroup of bounded linear operators

$\tilde{T}_t : L_2(K, H) \rightarrow L_2(K, H), t \geq 0$, via composition

$\tilde{T}_t(C) := T_t \circ C$, $C \in L_2(K, H), t \geq 0$. 

Lift stochastic integral

$Z_t^0 \sim T_t s (f[B^v(s)]) x g)$ dW(s), $x \in H, t \geq 0$;}

Denote lifting by $Z_t^0 T_t s B^v(s) dW(s)$.
Lifting

- Lift semigroup $T_t, t \geq 0$, to a strongly continuous semigroup of bounded linear operators
  
  $\tilde{T}_t : L^2(K, H) \to L^2(K, H), t \geq 0$, via composition
  
  $\tilde{T}_t(C) := T_t \circ C, \ C \in L^2(K, H), t \geq 0.$

- Lift stochastic integral

  $$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\}) \, dW(s), \ x \in H, t \geq 0,$$

  to $L^2(H)$ for adapted square-integrable $v : \mathbb{R}^+ \times \Omega \to L^2(H)$. Denote lifting by

  $$\int_0^t T_{t-s} Bv(s) \, dW(s).$$
Lifting-contd

That is:

\[
\left[ \int_{0}^{\tau} T_{t-s} Bv(s) \, dW(s) \right](x) = \\
\int_{0}^{\tau} \tilde{T}_{t-s}(\{[B \circ v(s)](x)\}) \, dW(s)
\]

for all \( t \geq 0, \ x \text{-a.s..} \)
Regularity Hypotheses

- **Hypothesis (A1):**

\[
\sum_{n=1}^{\infty} \mu_n^{-1} \| B(e_n) \|_{L_2(K,H)}^2 < \infty.
\]
Regularity Hypotheses

- **Hypothesis (A1):**
  \[ \sum_{n=1}^{\infty} \mu_n^{-1} \| B(e_n) \|_{L_2(K,H)}^2 < \infty. \]

- **Hypothesis (A2):**
  For some \( \alpha \in (0, 1) \), \( A^{-\alpha} \) is trace-class, i.e.,
  \[ \sum_{n=1}^{\infty} \mu_n^{-\alpha} < \infty. \]
Regularity Hypotheses

- **Hypothesis (A1):**
  \[
  \sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|_{L_2(K,H)}^2 < \infty.
  \]

- **Hypothesis (A2):**
  For some \(\alpha \in (0, 1)\), \(A^{-\alpha}\) is trace-class, i.e.,
  \[
  \sum_{n=1}^{\infty} \mu_n^{-\alpha} < \infty.
  \]

- **Hypothesis (A3):**
  \(A^{-1}\) is trace-class and \(T_t \in L(H), \ t \geq 0\), is a strongly continuous contraction semigroup.
**Hypothesis (B):**

\( B : H \rightarrow L_2(K, H) \) extends to a bounded linear operator \( B \in L(H, L(E, H)) \); \( \sum_{k=1}^{\infty} \|B_k\|^2 < \infty \), where \( B_k \in L(H) \) is defined by

\[ B_k(x) := B(x)(f_k), \quad x \in H, \quad k \geq 1. \]

No restriction on \( \dim M \) under (A1) for examples of SPDEs: e.g. \( B \in L_2(H, L_2(K, H)) \).
Assume hypothesis (B) and any one of hypotheses (A1), (A2) or (A3). Then the mild solution of the linear see has a Borel (strongly) measurable $(\mathcal{F}_t)_{t \geq 0}$-adapted version $\phi : \mathbb{R}^+ \times \Omega \rightarrow L(H)$ with the following properties:
Assume hypothesis (B) and any one of hypotheses (A1), (A2) or (A3). Then the mild solution of the linear see has a Borel (strongly) measurable \((\mathcal{F}_t)_{t \geq 0}\)-adapted version \(\phi : \mathbb{R}^+ \times \Omega \rightarrow L(H)\) with the following properties:

- Under (A2),

\[
E \sup_{0 \leq t \leq a} \|\phi(t, \cdot)\|_{L(H)}^{2p} < \infty,
\]

whenever \(p \in (1, \alpha^{-1}], a \in \mathbb{R}^+\).
Theorem 1-Contd: “Chaos''!

For each $t > 0$ and almost all $\omega \in \Omega$, $\phi(t, \omega) - T_t \in L_2(H)$ has “chaos-type” representation

$$\phi(t, \cdot) - T_t = \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots$$

$$\cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n} \, dW(s_n)$$

$$\cdots dW(s_2) \, dW(s_1).$$

Iterated Itô stochastic integrals are lifted integrals in $L_2(H)$, and series converges absolutely in $L_2(H)$. 
Theorem 1-contd

- Under (A1) or (A3),

\[ E \sup_{0 \leq t \leq a} \| \phi(t, \cdot) \|_{L(H)}^2 < \infty, \]
Theorem 1-contd

- Under (A1) or (A3),

\[ E \sup_{0 \leq t \leq a} \| \phi(t, \cdot) \|_{L(H)}^2 < \infty, \]

- \((\phi, \theta)\) is a perfect \(L(H)\)-valued cocycle:

\[ \phi(t + s, \omega) = \phi(t, \theta(s, \omega)) \circ \phi(s, \omega) \]

for all \(s, t \geq 0\) and all \(\omega \in \Omega;\)
Theorem 1-contd

- **Under (A1) or (A3),**
  
  \[
  E \sup_{0 \leq t \leq a} \| \phi(t, \cdot) \|_{L(H)}^2 < \infty,
  \]

- **(\phi, \theta) is a perfect \( L(H) \)-valued cocycle:**
  
  \[
  \phi(t + s, \omega) = \phi(t, \theta(s, \omega)) \circ \phi(s, \omega)
  \]
  
  for all \( s, t \geq 0 \) and all \( \omega \in \Omega \);

- **\( \sup_{0 \leq s \leq t \leq a} \| \phi(t - s, \theta(s, \omega)) \|_{L(H)} < \infty \), for all \( \omega \in \Omega \) and all \( a > 0 \).**
Consider the **semilinear** stochastic evolution equation:

\[
\begin{align*}
    du(t) &= -Au(t)dt + F(u(t))dt \\
    &\quad + Bu(t) dW(t), \quad t > 0, \\
    u(0) &= x \in H
\end{align*}
\]
Consider the **semilinear** stochastic evolution equation:

\[
\begin{align*}
    du(t) &= -Au(t)dt + F(u(t))dt \\
        &\quad + Bu(t)\,dW(t), \quad t > 0, \\
    u(0) &= x \in H
\end{align*}
\]

Operators $A, B$ satisfy hypothesis (B) and any one of hypotheses (A1), (A2) or (A3) (of Theorem 1). $F : H \to H$ is (Fréchet) $C^{k,\epsilon} \ (k \geq 1)$, with linear growth:

\[|F(v)| \leq C(1 + |v|), \quad v \in H\]

for some positive constant $C$. 
Define a *mild solution* of semilinear see as a family of $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H$, $x \in H$, satisfying:

$$
 u(t, x, \cdot) = T_t(x) + \int_0^t T_{t-s}(F(u(s, x, \cdot))) \, ds \\
 + \int_0^t T_{t-s}Bu(s, x, \cdot) \, dW(s),
$$

for all $t \geq 0$, $x$-a.s. ([D–Z]).
Random Integral Equation

Obtain a $C^k$ perfect cocycle $(U, \theta)$ for mild solutions of the semilinear see, via the random integral equation on $H$:

$$U(t, x, \omega) = \phi(t, \omega)(x) + \int_0^t \phi(t - s, \theta(s, \omega))(F(U(s, x, \omega))) \, ds,$$

each $\omega \in \Omega$, $t \geq 0$, $x \in H$. 
Theorem 2

Assume that the operators $A, B$ satisfy hypothesis (B) and (A1) (or (A2) or (A3)). Let $T_t, t > 0$, be compact. Suppose that $F : H \to H$ is $C^{k, \varepsilon}$ and has linear growth. Then the mild solution of the semilinear see has a Borel measurable version

$$U : \mathbb{R}^+ \times H \times \Omega \to H$$

with the following properties:

- For each $x \in H$, $U(\cdot, x, \cdot) : \mathbb{R}^+ \times \Omega \to H$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted and is a mild solution of the semilinear see.
Theorem 2-contd

- \((U, \theta)\) is a \(C^{k,\varepsilon}\) perfect cocycle.
(U, θ) is a $C^{k,\varepsilon}$ perfect cocycle.

For each $(t, \omega) \in (0, \infty) \times \Omega$, the map

$$H \ni x \mapsto U(t, x, \omega) \in H$$

takes bounded sets into relatively compact sets.
Theorem 2-contd

For each \((t, x, \omega) \in (0, \infty) \times H \times \Omega, 1 \leq j \leq k\), the \(j\)-th Fréchet derivative \(D^{(j)}U(t, x, \omega)\) \(\in L^{(j)}(H)\) is compact, and the map

\[
\begin{align*}
[0, \infty) \times H \times \Omega & \ni (t, x, \omega) \\
& \mapsto D^{(j)}U(t, x, \omega) \in L^{(j)}(H)
\end{align*}
\]

is strongly measurable.

\(L^{(j)}(H) := \) continuous \(H\)-valued \(j\)-multilinear maps on \(H\).
Theorem 2-contd

For any positive $a, \rho$, 

$$E \sup_{0 \leq t \leq a; \frac{|x|}{\rho} \leq 1; 1 \leq j \leq k} \left\{ \|D(j)U(t, x, \cdot)\|_{L(j)(H)} \right\} < \infty,$$

and 

$$E \left\{ \sup_{0 \leq t \leq a; x \in H} \frac{|U(t, x, \cdot)|^{2p}}{(1 + |x|^{2p})} \right\} < \infty$$

for all positive integers $p$. 
The Stable Manifold Theorem

\[ \tilde{\mathcal{F}} := P{-}\text{completion of } \mathcal{F}. \]
The Stable Manifold Theorem

- \( \bar{\mathcal{F}} := P\text{-completion of } \mathcal{F}. \)
- \( B(x, \rho) \) open ball, radius \( \rho \), center \( x \in H \);
The Stable Manifold Theorem

- $\bar{F} := P$–completion of $F$.
- $B(x, \rho)$ open ball, radius $\rho$, center $x \in H$;
- $\bar{B}(x, \rho)$ closed ball.
The Stable Manifold Theorem

- $\bar{F} := P -$completion of $F$.
- $B(x, \rho)$ open ball, radius $\rho$, center $x \in H$;
- $\bar{B}(x, \rho)$ closed ball.
- Semilinear see:

$$
\begin{aligned}
du(t) &= -Au(t) \, dt + F(u(t)) \, dt \\
&\quad + Bu(t) \, dW(t), \quad t > 0,
\end{aligned}
$$

$$
u(0) = x \in H.
$$
Theorem 3: Stable Manifolds

Assume that the operators $A, B$ satisfy hypothesis (B) and (A1) (or (A2) or (A3)). Let $T_t, t > 0$, be compact. Suppose that $F : H \to H$ is $C^{k, \epsilon}$ and has linear growth. Let $Y : \Omega \to H$ be a hyperbolic stationary point of the semilinear see such that $E(|Y(\cdot)|_{\epsilon_0}^H) < \infty$ for some $\epsilon_0 > 0$. 
Theorem 3: Stable Manifolds

Assume that the operators $A, B$ satisfy hypothesis (B) and (A1) (or (A2) or (A3)). Let $T_t, t > 0$, be compact. Suppose that $F : H \to H$ is $C^{k,\epsilon}$ and has linear growth. Let $Y : \Omega \to H$ be a hyperbolic stationary point of the semilinear see such that $E(\|Y(\cdot)\|_{H}^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$.

Denote by

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$$

the Lyapunov spectrum of the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of the semilinear see.
Theorem 3-contd

Let $\lambda_{i_0} :=$ the largest negative Lyapunov exponent of the linearized cocycle, and $\lambda_{i_0-1}$ its smallest positive Lyapunov exponent:

$$\{ \cdots < \lambda_{i+1} < \lambda_i < \cdots \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_1 \}.$$  

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0-1})$:

$$\{ \cdots \lambda_i < \cdots \lambda_{i_0} < -\epsilon_1 < 0 < \epsilon_2 < \lambda_{i_0-1} < \cdots < \lambda_1 \}.$$
Theorem 3-contd

Then the following exist:

- a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$,
- $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:
Theorem 3-contd

Then the following exist:

- a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$,

- $\bar{\mathcal{F}}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k, \varepsilon}$ ($\varepsilon \in (0, \delta)$) submanifolds $\tilde{S}(\omega)$, $\tilde{U}(\omega)$ of $\tilde{B}(Y(\omega), \rho_1(\omega))$ and $\tilde{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:
(a) $\tilde{S}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|U(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_i + \varepsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log |U(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_i$$

for all $x \in \tilde{S}(\omega)$. 

(a) $\tilde{S}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|U(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log |U(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0}$$

for all $x \in \tilde{S}(\omega)$. Each stable subspace $S(\omega)$ of the linearized semiflow $DU$ is tangent at $Y(\omega)$ to the submanifold $\tilde{S}(\omega)$, viz. $T_{Y(\omega)}\tilde{S}(\omega) = S(\omega)$. 
Theorem 3-contd

In particular, codim $\tilde{S}(\omega) = \text{codim } S(\omega)$, is fixed and finite.

(b) \[ \limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{|U(t, x_1, \omega) - U(t, x_2, \omega)|}{|x_1 - x_2|} : \right. \right. \\
\left. \left. x_1 \neq x_2, x_1, x_2 \in \tilde{S}(\omega) \right\} \right] \leq \lambda_{i_0}. \]
(c) (Cocycle-invariance of the stable manifolds):
There exists \( \tau_1(\omega) \geq 0 \) such that

\[
U(t, \cdot, \omega)(\tilde{S}(\omega)) \subseteq \tilde{S}(\theta(t, \omega))
\]

for all \( t \geq \tau_1(\omega) \). Also

\[
DU(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega)), \quad t \geq 0.
\]
(d) $\tilde{U}(\omega)$ is the set of all $x \in \overline{B}(Y(\omega), \rho_2(\omega))$ with the property that there is a unique discrete-time history process $y(\cdot, \omega) : \{-n : n \geq 0\} \to H$ such that $y(0, \omega) = x$ and for each integer $n \geq 1$, one has

$$U(1, y(-n, \omega), \theta(-n, \omega)) = y(-(n - 1), \omega)$$

and

$$|y(-n, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{-(\lambda i_0 - 1 - \epsilon_2)n}.$$
Furthermore, for each $x \in \tilde{U}(\omega)$, there is a unique continuous-time history process also denoted by $y(\cdot, \omega): (-\infty, 0] \to H$ such that $y(0, \omega) = x$, $U(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ for all $s \leq 0$, $0 \leq t \leq -s$, and

$$
\limsup_{t \to \infty} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}.
$$
Furthermore, for each \( x \in \tilde{U}(\omega) \), there is a unique continuous-time history process also denoted by \( y(\cdot, \omega) : (-\infty, 0] \rightarrow H \) such that \( y(0, \omega) = x \), \( U(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega) \) for all \( s \leq 0 \), \( 0 \leq t \leq -s \), and

\[
\limsup_{t \to \infty} \frac{1}{t} \log |y(-t, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0 - 1}.
\]

Each unstable subspace \( U(\omega) \) of the linearized semiflow \( DU \) is tangent at \( Y(\omega) \) to \( \tilde{U}(\omega) \), viz. \( T_{Y(\omega)}\tilde{U}(\omega) = U(\omega) \). In particular, \( \dim \tilde{U}(\omega) \) is finite and non-random.
(e) Let \( y(\cdot, x_i, \omega) \) be the history processes associated with \( x_i = y(0, x_i, \omega) \in \tilde{U}(\omega), \ i = 1, 2 \). Then

\[
\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{|y(-t, x_1, \omega) - y(-t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2, \ x_i \in \tilde{U}(\omega), \ i = 1, 2 \right\} \right] \leq -\lambda_{i_0-1}.
\]
(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\tilde{U}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\tilde{U}(\theta(-t, \omega)))$$

for all $t \geq \tau_2(\omega)$. 
(f) **(Cocycle-invariance of the unstable manifolds):**

There exists \( \tau_2(\omega) \geq 0 \) such that

\[
\tilde{U}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\tilde{U}(\theta(-t, \omega)))
\]

for all \( t \geq \tau_2(\omega) \). Also

\[
DU(t, \cdot, \theta(-t, \omega))(U(\theta(-t, \omega))) = U(\omega), \quad t \geq 0;
\]

and the restriction \( DU(t, \cdot, \theta(-t, \omega))|_{U(\theta(-t, \omega))}, \ t \geq 0, \) is a linear homeomorphism from \( U(\theta(-t, \omega)) \)

onto \( U(\omega) \).
(g) The submanifolds $\tilde{U}(\omega)$ and $\tilde{S}(\omega)$ are transversal, viz.

$$H = T_{Y(\omega)}\tilde{U}(\omega) \oplus T_{Y(\omega)}\tilde{S}(\omega).$$

If $F$ is $C^\infty_b$, then the local stable and unstable manifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ are $C^\infty$. 

SEMILINEAR SPDEs AS DYNAMICAL SYSTEMS – p.54/71
(g) The submanifolds $\tilde{U}(\omega)$ and $\tilde{S}(\omega)$ are transversal, viz.

$$H = T_{Y(\omega)}\tilde{U}(\omega) \oplus T_{Y(\omega)}\tilde{S}(\omega).$$

If $F$ is $C_b^\infty$, then the local stable and unstable manifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ are $C^\infty$.

**Ergodicity of** $Y$ : $\tilde{U}(\omega) = \{Y(\omega)\}$
A Stationary Tube
Stable/Unstable Manifolds

$U(t, \cdot, \omega)$

$\theta(t, \cdot)$

$t > \tau_1(\omega)$

$\omega$
Examples Revisited

Local stable manifold theorem applies to all examples:

- *Stochastic semilinear heat equation*
Examples Revisited

Local stable manifold theorem applies to all examples:

- *Stochastic semilinear heat equation*
- *Stochastic semilinear parabolic pdes*
Examples Revisited

Local stable manifold theorem applies to all examples:

- *Stochastic semilinear heat equation*
- *Stochastic semilinear parabolic pdes*
- *Stochastic reaction diffusion equations*
Examples Revisited

Local stable manifold theorem applies to all examples:

- *Stochastic semilinear heat equation*
- *Stochastic semilinear parabolic pdes*
- *Stochastic reaction diffusion equations*
- *Stochastic Burgers equation*
<table>
<thead>
<tr>
<th>Reference</th>
<th>Author(s)</th>
<th>Title and Details</th>
</tr>
</thead>
</table>


<table>
<thead>
<tr>
<th>Authors</th>
<th>Title</th>
<th>Journal</th>
<th>Volume</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mo.3 Mohammed, S.-E. A.</td>
<td>The Lyapunov spectrum and stable manifolds for stochastic linear delay equations</td>
<td><em>Stochastics and Stochastic Reports</em></td>
<td>Vol. 29</td>
<td>89-131</td>
</tr>
<tr>
<td>Reference</td>
<td>Author(s)</td>
<td>Title</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----------</td>
<td>----------</td>
<td>-------</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
SKETCH OF PROOF
Proof of Theorem 3: Strategy

- By definition, a *stationary* random point \( Y(\omega) \in H \) is invariant under the semiflow \( U \); viz
  \[
  U(t, Y) = Y(\theta(t, \cdot))
  \]
  for all times \( t \).
Proof of Theorem 3: Strategy

- By definition, a stationary random point $Y(\omega) \in H$ is invariant under the semiflow $U$; viz $U(t, Y) = Y(\theta(t, \cdot))$ for all times $t$.

- Linearize the semiflow $U$ along the stationary point $Y(\omega)$ in $H$. By stationarity of $Y$ and the cocycle property of $U$, this gives a linear perfect cocycle $(DU(t, Y), \theta(t, \cdot))$ in $L(H)$. 
Ergodicity of $\theta$ allows for the notion of hyperbolicity of a stationary point of $U$ via Oseledec-Ruelle theorem:
Ergodicity of $\theta$ allows for the notion of hyperbolicity of a stationary point of $U$ via Oseledec-Ruelle theorem:

Use local compactness of the semiflow for positive $t$, and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. $Y$ is hyperbolic if $\lambda_i \neq 0$ for every $i$. 
Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small $\epsilon_0$). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of perfect versions of ergodic theorem and Kingman’s subadditive ergodic theorem. These refined versions give invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow $U$. 
Establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle $U$ in a neighborhood of the stationary point $Y$. Estimates follow from the variational construction of the stochastic semiflow.
Strategy-contd

- Introduce the auxiliary perfect cocycle

\[ Z(t, \cdot, \omega) := U(t, (\cdot) + Y(\omega), \omega) - Y(\theta(t, \omega)), \]
\[ t \in \mathbb{R}^+, \omega \in \Omega. \]

Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle \((Z(n, \cdot, \omega), \theta(n, \omega))\) near 0 and hence (by translation) for \(U(n, \cdot, \omega)\) near \(Y(\omega)\) for all \(\omega\) sampled from a \(\theta(t, \cdot)\)-invariant sure event in \(\Omega\).
This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between discrete times and further refining the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the continuous-time semiflow $U$ near $Y$. 
Final key step:

Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow $U$. Use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a stochastic history process for $U$ coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.
THANK YOU!
THE END!