SEMILINEAR SPDEs
AS DYNAMICAL SYSTEMS

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Sweden: September 25, 2007

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Acknowledgment

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- Research supported by (US) NSF Grants DMS-9703852, DMS-9975462, DMS-0203368 and DMS-0705970.

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- Hyperbolicity of stationary trajectories (via Lyapunov exponents).
- Stable manifolds. ([M.Z.Z]).

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- $(\Omega, \mathcal{F}, P) :=$ probability space; e.g. Wiener space.

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$$\theta(t, \omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R},\ \omega \in \Omega.$$

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- $d\xi :=$ Riemannian volume on M.

Notation-Contd

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- $H_0^k(M, \mathbf{R}) :=$ Sobolev space of all functions $u : M \to \mathbf{R}$ (vanishing on ∂M) with all derivatives up to order k square-integrable with respect to $d\xi$. $H_0^k(M, \mathbf{R})$ is a Hilbert space under usual Sobolev norm.

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- $L^{(j)}(H) :=$ continuous H-valued j-multilinear maps on H.

Examples: Affine Linear SEEs

Affine Linear SEEs (Additive Noise):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + B_0\,dW(t),\;\; t>0 \\ u(0,x) &= x \in H. \end{aligned}\right\}$$

A hyperbolic: $0 \notin \sigma(A)$–discrete bounded below.

W Brownian motion with covariance Hilbert space K.

$B_0 : K \to H$, Hilbert Schmidt. Mild solutions.

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See has stationary solution, and affine linear semiflow on H.

Reaction-Diffusion Equations

Stochastic Reaction-Diffusion Equation:

$$du = \frac{1}{2}\Delta u\, dt + (1 - |u|^{\alpha})u\, dt + \sum_{i=1}^{\infty} \sigma_i u\, dW_i(t),$$

W_i := independent standard Brownian motions on $\mathbf{R}$.

$\sigma_i \in H_0^s(M, \mathbf{R})$, $s > 2 + d/2$; $\sum_{i=1}^{\infty} \|\sigma_i\|_{H_0^s}^2 < \infty$.

Dirichlet boundary conditions. Weak solutions.

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Dirichlet boundary conditions. Weak solutions.

Has C^1 stochastic semiflow on $H := L^2(M, \mathbf{R})$ for $\alpha < \frac{4}{d}$.

Lipschitz semiflow for α even integer.

Stochastic Heat Equation

Stochastic Heat Equation:

$$du(t) = \frac{1}{2}\Delta u(t)\,dt + \sum_{i=1}^{\infty} \sigma_i u(t)\,dW_i(t) + f(u(t))\,dt$$

$$u(0) = \psi \in H_0^k(M)$$

W_i as above; $\sigma_i \in H_0^s(M, \mathbf{R})$, $\sum_{i=1}^{\infty} \|\sigma_i\|_{H_0^s}^2 < \infty$,
$s > k + d/2$; $d := dim M$; $f : \mathbf{R} \to \mathbf{R}$ is C_b^∞.
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Dirichlet boundary conditions. Weak solutions.

Has C^∞ stochastic semiflow on $H_0^k(M)$ for $k > \dfrac{d}{2}$.

Semilinear Parabolic SPDEs

Semilinear Parabolic SPDEs:

In stochastic heat equation replace Δ by a second order self-adjoint elliptic linear differential operator:

$$L := \sum_{i,j=1}^{d} a_{ij}(\xi) \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{d} b_i(\xi) \frac{\partial}{\partial \xi_i}$$

on M.

Dirichlet boundary condition. Weak solutions.

Smooth coefficients $a_{i,j} : M \to \mathbf{R}$, $b_i : M \to \mathbf{R}$.

Parabolic SPDEs-contd

View parabolic spde as a semilinear stochastic evolution equation (see):

$$du(t) = -Au(t)\,dt + F(u(t))\,dt + \sum_{i=1}^{\infty} B_i u(t)\,dW_i(t)$$

$$u(0) = x \in H := H_0^k(M).$$

$$A := -L,\ B_i(u) := \sigma_i u,\ \ F(u) := f \circ u,\ \ u \in H.$$

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Let $k > \frac{d}{2}$. Then Nemytskii operator $F : H \to H$ is C^∞.

Smooth stochastic semiflow on $H_0^k(M)$.

Burgers Equation

Considered by many authors in recent years. (e.g. [E.K.M.S]).

One-dimensional *stochastic Burgers equation*:

$$du + u\frac{\partial u}{\partial \xi}\,dt = \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2}\,dt + \sum_{i=1}^{\infty}\sigma_i(\xi)\,dW_i(t)$$

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W_i independent one dimensional Brownian motions.

$\sigma_i \in C^2([0,1]);\ \|\sigma_i\|_{C^2} \leq \frac{C}{i^2},\ i \geq 1.$ Mild solutions.

Has C^1 stochastic semiflow on $L^2([0,1], \mathbf{R})$.

The Cocycle

$k =$ non-negative integer, $\epsilon \in (0, 1]$. H Hilbert.

A $C^{k,\epsilon}$ perfect cocycle (U, θ) on H is a measurable random field $U : \mathbf{R}^+ \times H \times \Omega \longrightarrow H$ such that:

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A $C^{k,\epsilon}$ perfect cocycle (U,θ) on H is a measurable random field $U : \mathbf{R}^+ \times H \times \Omega \longrightarrow H$ such that:

- For each $\omega \in \Omega$, the map

$$\mathbf{R}^+ \times H \ni (t,x) \mapsto U(t,x,\omega) \in H$$

is continuous; for fixed $(t,\omega) \in \mathbf{R}^+ \times \Omega$, the map

$$H \ni x \mapsto U(t,x,\omega) \in H$$

is $C^{k,\epsilon}$ ($D^k U(t,x,\omega)$ is C^ϵ in x on bounded sets in H).

The Cocycle-Contd

- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$
  for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

The Cocycle-Contd

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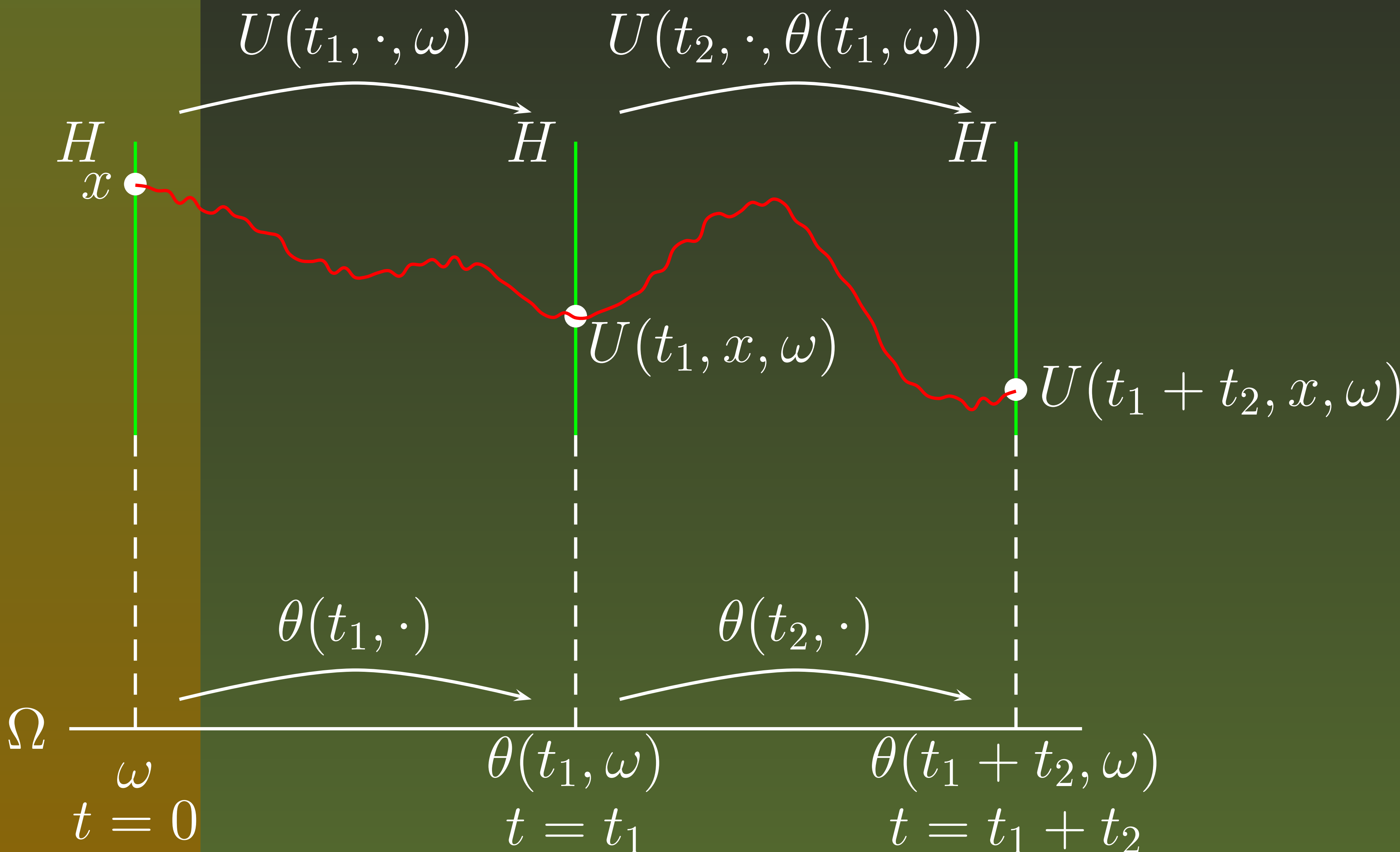
- $U(0, x, \omega) = x$ for all $x \in H, \omega \in \Omega$.

The Cocycle Property



Stationary Point

A random variable $Y : \Omega \to H$ is a *stationary point* for the cocycle (U, θ) if

$$U(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

for all $t \in \mathbf{R}^+$ and every $\omega \in \Omega$.

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Denote a stationary trajectory by

$$U(t, Y) = Y(\theta(t)).$$

For sde's: a non-anticipating stationary point corresponds to an invariant measure for the one-point motion.

Linearization

Linearize a $C^{k,\epsilon}$ cocycle (U, θ) along a stationary random point Y:

Get an $L(H)$-valued cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$.

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Follows from cocycle property of U and chain rule:

$$DU(t_1 + t_2, Y(\omega), \omega)$$
$$= DU(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \circ DU(t_1, Y(\omega), \omega)$$

for all $\omega \in \Omega, t_1, t_2 \geq 0$.

Linearization-contd

Assume $U(t,\cdot,\omega)$ *locally compact* and

$$E \log^+ \sup_{0\le t_1,t_2\le 1} \|DU(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(H)} < \infty.$$

Apply Oseledec-Ruelle Theorem to linearized cocycle ([Ru.2]):

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Apply Oseledec-Ruelle Theorem to linearized cocycle ([Ru.2]):
Get a sequence of closed finite-codimensional Oseledec spaces

$$\cdots E_{i+1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = H,$$

all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$.

Linearization-contd

Obtain Lyapunov spectrum

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\};$$

$$\lim_{t\to\infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(x)|$$

$$= \begin{cases} \lambda_i & \text{if } x \in E_i(\omega) \backslash E_{i+1}(\omega), \\ -\infty & \text{if } x \in E_\infty(\omega). \end{cases}$$

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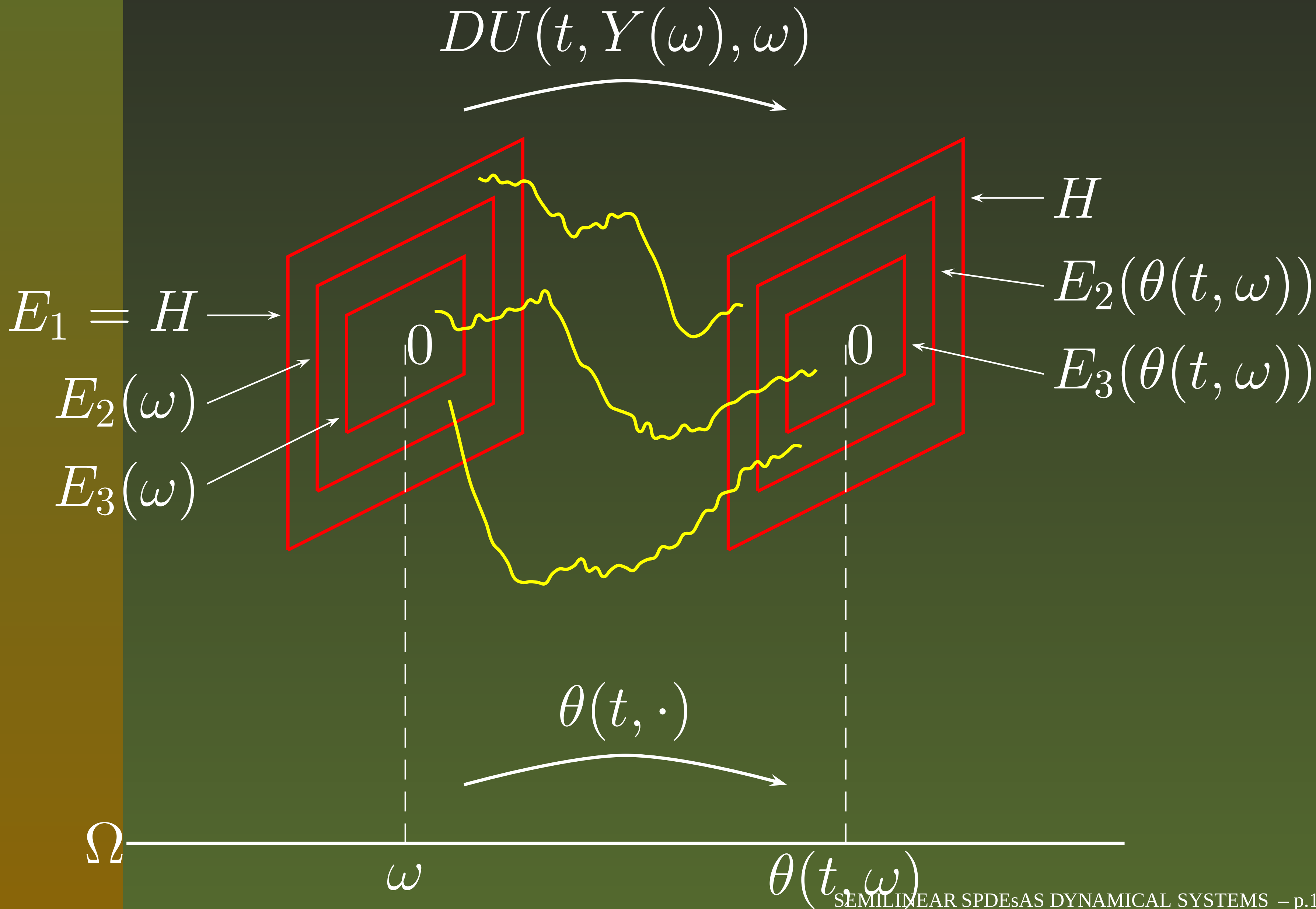
$$E_i(\omega) = \{x \in H : \lim_{t\to\infty} \frac{1}{t} \log |DU(t, Y(\omega), \omega)(x)| \leq \lambda_i\},$$

$i \geq 1.$

Linearization: Spectral Theorem



Hyperbolicity

A stationary point $Y(\omega)$ of (U, θ) is *hyperbolic* if the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-zero Lyapunov spectrum

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}.$$

That is

$$\lambda_i \neq 0 \qquad \text{for all} \quad i \geq 1.$$

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Ergodicity: $\lambda_1 < 0$.

Hyperbolicity-Contd

$\{\mathcal{U}(\omega), \mathcal{S}(\omega) : \omega \in \Omega^*\}$:= unstable and stable subspaces associated with the linearized cocycle (DU, θ) ([Mo.3], [M.S]).

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$\{\mathcal{U}(\omega), \mathcal{S}(\omega) : \omega \in \Omega^*\}$:= unstable and stable subspaces associated with the linearized cocycle (DU, θ) ([Mo.3], [M.S]).

Then get a measurable invariant splitting

$$H = \mathcal{U}(\omega) \oplus \mathcal{S}(\omega), \qquad \omega \in \Omega^*,$$

$$DU(t, Y(\omega), \omega)(\mathcal{U}(\omega)) = \mathcal{U}(\theta(t, \omega)),$$

$$DU(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)),$$

for all $t \geq 0$.

Hyperbolicity-Contd

Have exponential dichotomies:

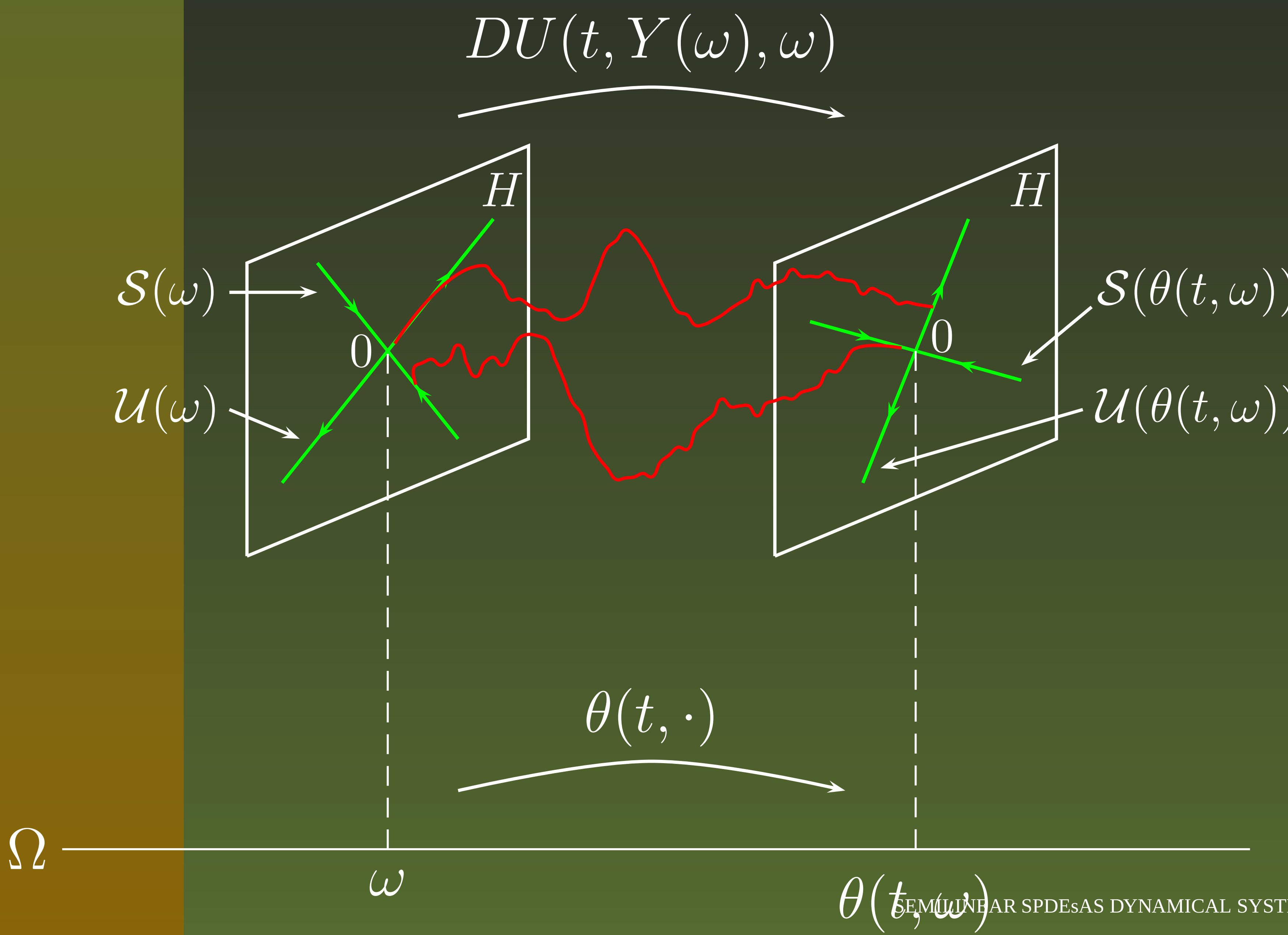
$$|DU(t, Y(\omega), \omega)(x)| \geq |x| e^{\delta_1 t}$$

for all $t \geq \tau_1^*, x \in \mathcal{U}(\omega)$;

$$|DU(t, Y(\omega), \omega)(x)| \leq |x| e^{-\delta_2 t}$$

for all $t \geq \tau_2^*, x \in \mathcal{S}(\omega)$, with $\tau_i^* = \tau_i^*(x, \omega) > 0$, random times and $\delta_i > 0$, fixed, $i = 1, 2$.

Hyperbolicity-Contd



Linear SEEs

Existence of semiflows for mild solutions of linear sees:

$$du(t,x,\cdot) = -Au(t,x,\cdot)\,dt + Bu(t,x,\cdot)\,dW(t),$$
$$t > 0$$

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$A : D(A) \subset H \to H$ closed linear operator on a separable real Hilbert space H.

A has complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding (bounded below) (non-zero) eigenvalues $\{\mu_n, n \geq 1\}$; i.e.,
$Ae_n = \mu_n e_n,\ \ n \geq 1;$
e.g. $A = -\Delta$ on compact smooth Riemannian manifold.

Linear SEEs-Contd

$(-A)$ generates a strongly continuous semigroup of bounded linear operators

$$T_t : H \to H, \; t \geq 0.$$

$W(t), t \geq 0$, E-valued cylindrical Brownian motion on canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
$K \subset E$ Hilbert-Schmidt embedding. ([D.Z]).

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$L_2(K, H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$; H-S norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|^2\right]^{1/2},$$

Linear SEEs-Contd

f_k, $k \geq 1$, cons in K.
$|\cdot| :=$ norm on H. $L_2(H) := L_2(H, H)$.

$B : H \to L_2(K, H)$ bounded (affine) linear operator.

Stochastic integral in (see) as in ([D.Z]).

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$\theta : \mathbf{R} \times \Omega \to \Omega$ standard P-preserving ergodic Wiener shift on Ω. (W, θ) is a *helix*:

$$W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$$

for all $t_1, t_2 \in \mathbf{R}$, $\omega \in \Omega$.

Mild Solutions

A *mild solution* of the linear see is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \longrightarrow H,\ x \in H$, s.t.

$$u(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B u(s, x, \cdot)\, dW(s), \quad t \geq 0.$$

Integral equation holds x-almost surely, $x \in H$.

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Integral equation holds x-almost surely, $x \in H$.

Is $u(t, x, \cdot)$ pathwise continuous linear in x?

Kolmogorov Fails!

Kolmogorov's continuity theorem fails for random field $I : L^2([0,1],\mathbf{R}) \to L^2(\Omega,\mathbf{R})$

$$I(x) := \int_0^1 x(t)\,dW(t), \quad x \in L^2([0,1],\mathbf{R}).$$

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No continuous (or even Borel measurable linear!) selection

$$L^2([0,1],\mathbf{R}) \times \Omega \to \mathbf{R}$$

$$(x,\omega) \mapsto I(x,\omega)$$

of I ([Mo.1]).

Lifting

- Lift semigroup $T_t, t \geq 0$, to a strongly continuous semigroup of bounded linear operators $\tilde{T}_t : L_2(K, H) \to L_2(K, H), t \geq 0$, via composition $\tilde{T}_t(C) := T_t \circ C,\ C \in L_2(K, H), t \geq 0.$

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- Lift stochastic integral

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s),\ x \in H,\, t \geq 0,$$

to $L_2(H)$ for adapted square-integrable $v : \mathbf{R}^+ \times \Omega \to L_2(H)$. Denote lifting by $\int_0^t T_{t-s}Bv(s)\, dW(s)$.

Lifting-contd

That is:

$$\left[\int_0^t T_{t-s} B v(s)\, dW(s)\right](x) = \int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$$

for all $t \geq 0$, x-a.s..

Regularity Hypotheses

- *Hypothesis (A1)*:

$$\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|^2_{L_2(K,H)} < \infty.$$

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Regularity Hypotheses

- *Hypothesis (A1)*:

$$\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|_{L_2(K,H)}^2 < \infty.$$

- *Hypothesis (A2)*:

For some $\alpha \in (0,1)$, $A^{-\alpha}$ is trace-class, i.e., $\sum_{n=1}^{\infty} \mu_n^{-\alpha} < \infty$.

- *Hypothesis (A3)*:

A^{-1} is trace-class and $T_t \in L(H)$, $t \geq 0$, is a strongly continuous contraction semigroup.

Regularity Hypotheses-contd

- *Hypothesis (B)*:

  $B : H \to L_2(K, H)$ extends to a bounded linear operator $B \in L(H, L(E, H))$; $\sum_{k=1}^{\infty} \|B_k\|^2 < \infty$, where $B_k \in L(H)$ is defined by

$$B_k(x) := B(x)(f_k), x \in H, k \geq 1.$$

No restriction on $\dim M$ under (A1) for examples of spdes: e.g. $B \in L_2(H, L_2(K, H))$.

Theorem 1: The Linear Flow

Assume hypothesis (B) and any one of hypotheses (A1), (A2) or (A3). Then the mild solution of the linear see has a Borel (strongly) measurable $(\mathcal{F}_t)_{t\geq 0}$-*adapted version* $\phi : \mathbf{R}^+ \times \Omega \to L(H)$ *with the following properties:*

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- Under (A2),

$$E \sup_{0\leq t\leq a} \|\phi(t,\cdot)\|_{L(H)}^{2p} < \infty,$$

whenever $p \in (1, \alpha^{-1}], a \in \mathbf{R}^+$.

Theorem 1-Contd: “Chaos"!

- *For each* $t > 0$ *and almost all* $\omega \in \Omega$, $\phi(t,\omega) - T_t \in L_2(H)$ *has “chaos-type” representation*

$$\phi(t,\cdot) - T_t = \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n} \, dW(s_n) \cdots dW(s_2)\, dW(s_1).$$

Iterated Itô stochastic integrals are lifted integrals in $L_2(H)$, *and series converges absolutely in* $L_2(H)$.

Theorem 1-contd

- *Under (A1) or (A3),*

$$E \sup_{0 \le t \le a} \|\phi(t,\cdot)\|^2_{L(H)} < \infty,$$

Theorem 1-contd

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$$\phi(t+s,\omega) = \phi(t,\theta(s,\omega)) \circ \phi(s,\omega)$$

for all $s, t \geq 0$ *and all* $\omega \in \Omega$*;*

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for all $s, t \geq 0$ *and all* $\omega \in \Omega$*;*

- $\sup_{0\leq s\leq t\leq a} \|\phi(t-s,\theta(s,\omega))\|_{L(H)} < \infty$, *for all* $\omega \in \Omega$ *and all* $a > 0$.

Semilinear SEE

Consider the semilinear stochastic evolution equation:

$$\left.\begin{aligned} du(t) &= -Au(t)dt + F(u(t))dt \\ &\qquad + Bu(t)\,dW(t), \quad t > 0, \\ u(0) &= x \in H \end{aligned}\right\}$$

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Operators A, B satisfy hypothesis (B) and any one of hypotheses (A1), (A2) or (A3) (of Theorem 1).

$F : H \to H$ is (Fréchet) $C^{k,\epsilon}$ ($k \geq 1$), with linear growth:

$$|F(v)| \leq C(1 + |v|), \quad v \in H$$

for some positive constant C.

Mild Solution: Semilinear SEE

Define a *mild solution* of semilinear see as a family of $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{F},\mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot,x,\cdot):\mathbf{R}^+\times\Omega\longrightarrow H,\ \ x\in H$, satisfying:

$$u(t,x,\cdot)=T_t(x)+\int_0^t T_{t-s}(F(u(s,x,\cdot)))\,ds + \int_0^t T_{t-s}Bu(s,x,\cdot)\,dW(s),$$

for all $t\geq 0$, x-a.s. ([D–Z]).

Random Integral Equation

Obtain a C^k perfect cocycle (U, θ) for mild solutions of the semilinear see, via the random integral equation on H:

$$U(t, x, \omega) = \phi(t, \omega)(x) + \int_0^t \phi(t - s, \theta(s, \omega))(F(U(s, x, \omega)))\, ds,$$

each $\omega \in \Omega$, $t \geq 0$, $x \in H$.

Theorem 2

Assume that the operators A, B satisfy hypothesis (B) and (A1) (or (A2) or (A3)). Let $T_t, t > 0$, be compact. Suppose that $F : H \longrightarrow H$ is $C^{k,\epsilon}$ and has linear growth. Then the mild solution of the semilinear see has a Borel measurable version

$$U : \mathbf{R}^+ \times H \times \Omega \longrightarrow H$$

with the following properties:

- *For each $x \in H$, $U(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \longrightarrow H$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted and is a mild solution of the semilinear see.*

Theorem 2-contd

- (U, θ) *is a* $C^{k,\epsilon}$ *perfect cocycle.*

Theorem 2-contd

- (U, θ) *is a* $C^{k,\epsilon}$ *perfect cocycle.*
- *For each* $(t, \omega) \in (0, \infty) \times \Omega$*, the map*

$$H \ni x \mapsto U(t, x, \omega) \in H$$

takes bounded sets into relatively compact sets.

Theorem 2-contd

- *For each* $(t,x,\omega) \in (0,\infty) \times H \times \Omega$, $1 \leq j \leq k$, *the* j*-th Fréchet derivative* $D^{(j)}U(t,x,\omega)$ $\in L^{(j)}(H)$ *is compact, and the map*

$$[0,\infty) \times H \times \Omega \ni$$
$$(t,x,\omega) \mapsto D^{(j)}U(t,x,\omega) \in L^{(j)}(H)$$

is strongly measurable.

$L^{(j)}(H) :=$ *continuous* H*-valued* j*-multilinear maps on* H.

Theorem 2-contd

- *For any positive* a, ρ,

$$E \sup_{\substack{0 \le t \le a \\ |x| \le \rho \\ 1 \le j \le k}} \left\{ \| D^{(j)} U(t, x, \cdot) \|_{L^{(j)}(H)} \right\} < \infty,$$

and

$$E \left\{ \sup_{\substack{0 \le t \le a \\ x \in H}} \frac{|U(t, x, \cdot)|^{2p}}{(1 + |x|^{2p})} \right\} < \infty$$

for all positive integers p.

The Stable Manifold Theorem

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Theorem 3: Stable Manifolds

Assume that the operators A, B *satisfy hypothesis (B) and (A1) (or (A2) or (A3)). Let* $T_t, t > 0$, *be compact. Suppose that* $F : H \longrightarrow H$ *is* $C^{k,\epsilon}$ *and has linear growth. Let* $Y : \Omega \longrightarrow H$ *be a hyperbolic stationary point of the semilinear see such that* $E(|Y(\cdot)|_H^{\epsilon_0}) < \infty$ *for some* $\epsilon_0 > 0$.

Theorem 3: Stable Manifolds

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Denote by

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$$

the Lyapunov spectrum of the linearized cocycle $(DU(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ *of the semilinear see.*

Theorem 3-contd

Let $\lambda_{i_0} :=$ *the largest negative Lyapunov exponent of the linearized cocycle, and* λ_{i_0-1} *its smallest positive Lyapunov exponent:*

$$\{\cdots < \lambda_{i+1} < \lambda_i < \cdots \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_1\}.$$

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ *and* $\epsilon_2 \in (0, \lambda_{i_0-1})$*:*

$$\{\cdots \lambda_i < \cdots \lambda_{i_0} < -\epsilon_1 < 0 < \epsilon_2 < \lambda_{i_0-1} < \cdots < \lambda_1\}.$$

Theorem 3-contd

Then the following exist:

- *a sure event* $\Omega^* \in \mathcal{F}$ *with* $\theta(t,\cdot)(\Omega^*) = \Omega^*$ *for all* $t \in \mathbf{R}$,
- $\bar{\mathcal{F}}$*-measurable random variables* $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, *such that for each* $\omega \in \Omega^*$, *the following is true:*

Theorem 3-contd

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- $\bar{\mathcal{F}}$*-measurable random variables* $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1,2$, *such that for each* $\omega \in \Omega^*$, *the following is true:*

There are $C^{k,\epsilon}$ *(*$\epsilon \in (0,\delta)$*) submanifolds* $\tilde{\mathcal{S}}(\omega)$, $\tilde{\mathcal{U}}(\omega)$ *of* $\bar{B}(Y(\omega), \rho_1(\omega))$ *and* $\bar{B}(Y(\omega), \rho_2(\omega))$ *(resp.) with the following properties:*

Theorem 3-contd

(a) $\tilde{S}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ *such that*

$$|U(n, x, \omega) - Y(\theta(n, \omega))| \le \beta_1(\omega)\, e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \ge 0$. *Furthermore,*

$$\limsup_{t \to \infty} \frac{1}{t} \log |U(t, x, \omega) - Y(\theta(t, \omega))| \le \lambda_{i_0}$$

for all $x \in \tilde{S}(\omega)$.

Theorem 3-contd

(a) $\tilde{\mathcal{S}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ *such that*

$$|U(n,x,\omega) - Y(\theta(n,\omega))| \leq \beta_1(\omega)\, e^{(\lambda_{i_0}+\epsilon_1)n}$$

for all integers $n \geq 0$. *Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log |U(t,x,\omega) - Y(\theta(t,\omega))| \leq \lambda_{i_0}$$

for all $x \in \tilde{\mathcal{S}}(\omega)$. *Each stable subspace* $\mathcal{S}(\omega)$ *of the linearized semiflow* DU *is tangent at* $Y(\omega)$ *to the submanifold* $\tilde{\mathcal{S}}(\omega)$, *viz.* $T_{Y(\omega)}\tilde{\mathcal{S}}(\omega) = \mathcal{S}(\omega)$.

Theorem 3-contd

In particular, codim $\tilde{\mathcal{S}}(\omega) =$ *codim* $\mathcal{S}(\omega)$, *is fixed and finite.*

(b) $$\limsup_{t\to\infty} \frac{1}{t} \log \left[\sup \left\{ \frac{|U(t,x_1,\omega) - U(t,x_2,\omega)|}{|x_1 - x_2|} : x_1 \neq x_2,\ x_1, x_2 \in \tilde{\mathcal{S}}(\omega) \right\} \right] \leq \lambda_{i_0}.$$

Theorem 3-contd

(c) (Cocycle-invariance of the stable manifolds):
There exists $\tau_1(\omega) \geq 0$ *such that*

$$U(t, \cdot, \omega)(\tilde{\mathcal{S}}(\omega)) \subseteq \tilde{\mathcal{S}}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$. *Also*

$$DU(t, Y(\omega), \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega)), \quad t \geq 0.$$

Theorem 3-contd

(d) $\tilde{\mathcal{U}}(\omega)$ *is the set of all* $x \in \bar{B}(Y(\omega), \rho_2(\omega))$ *with the property that there is a unique* *discrete-time history process* $y(\cdot, \omega) : \{-n : n \geq 0\} \to H$ *such that* $y(0, \omega) = x$ *and for each integer* $n \geq 1$, *one has*

$$U(1, y(-n, \omega), \theta(-n, \omega)) = y(-(n-1), \omega)$$

and

$$|y(-n, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{-(\lambda_{i_0 - 1} - \epsilon_2)n}.$$

Theorem 3-contd

Furthermore, for each $x \in \tilde{\mathcal{U}}(\omega)$*, there is a unique* *continuous-time history process* *also denoted by* $y(\cdot,\omega) : (-\infty, 0] \to H$ *such that* $y(0,\omega) = x$, $U(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \leq 0$, $0 \leq t \leq -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log |y(-t,\omega) - Y(\theta(-t,\omega))| \leq -\lambda_{i_0-1}.$$

Theorem 3-contd

Furthermore, for each $x \in \tilde{\mathcal{U}}(\omega)$*, there is a unique* *continuous-time history process* *also denoted by* $y(\cdot,\omega) : (-\infty, 0] \to H$ *such that* $y(0,\omega) = x$, $U(t, y(s,\omega), \theta(s,\omega)) = y(t+s,\omega)$ *for all* $s \le 0$, $0 \le t \le -s$, *and*

$$\limsup_{t\to\infty} \frac{1}{t} \log |y(-t,\omega) - Y(\theta(-t,\omega))| \le -\lambda_{i_0-1}.$$

Each unstable subspace $\mathcal{U}(\omega)$ *of the linearized semiflow* DU *is tangent at* $Y(\omega)$ *to* $\tilde{\mathcal{U}}(\omega)$*, viz.* $T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) = \mathcal{U}(\omega)$*. In particular,* $\dim \tilde{\mathcal{U}}(\omega)$ *is finite and non-random.*

Theorem 3-contd

(e) Let $y(\cdot, x_i, \omega)$ *be the history processes associated with* $x_i = y(0, x_i, \omega) \in \tilde{\mathcal{U}}(\omega)$, $i = 1, 2$. *Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \left[\sup \left\{ \frac{|y(-t, x_1, \omega) - y(-t, x_2, \omega)|}{|x_1 - x_2|} : x_1 \neq x_2,\ x_i \in \tilde{\mathcal{U}}(\omega), i = 1, 2 \right\} \right] \leq -\lambda_{i_0 - 1}.$$

Theorem 3-contd

(f) (Cocycle-invariance of the unstable manifolds): *There exists* $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq U(t,\cdot,\theta(-t,\omega))(\tilde{\mathcal{U}}(\theta(-t,\omega)))$$

for all $t \geq \tau_2(\omega)$.

Theorem 3-contd

(f) (Cocycle-invariance of the unstable manifolds):
There exists $\tau_2(\omega) \geq 0$ *such that*

$$\tilde{\mathcal{U}}(\omega) \subseteq U(t, \cdot, \theta(-t, \omega))(\tilde{\mathcal{U}}(\theta(-t, \omega)))$$

for all $t \geq \tau_2(\omega)$. *Also*

$$DU(t, \cdot, \theta(-t, \omega))(\mathcal{U}(\theta(-t, \omega))) = \mathcal{U}(\omega), \quad t \geq 0;$$

and the restriction $DU(t, \cdot, \theta(-t, \omega))|\mathcal{U}(\theta(-t, \omega))$, $t \geq 0$, *is a linear homeomorphism from* $\mathcal{U}(\theta(-t, \omega))$ *onto* $\mathcal{U}(\omega)$.

Theorem 3-contd

(g) The submanifolds $\tilde{\mathcal{U}}(\omega)$ *and* $\tilde{\mathcal{S}}(\omega)$ *are transversal, viz.*

$$H = T_{Y(\omega)}\tilde{\mathcal{U}}(\omega) \oplus T_{Y(\omega)}\tilde{\mathcal{S}}(\omega).$$

If F *is* C_b^∞*, then the local stable and unstable manifolds* $\tilde{\mathcal{S}}(\omega), \tilde{\mathcal{U}}(\omega)$ *are* C^∞.

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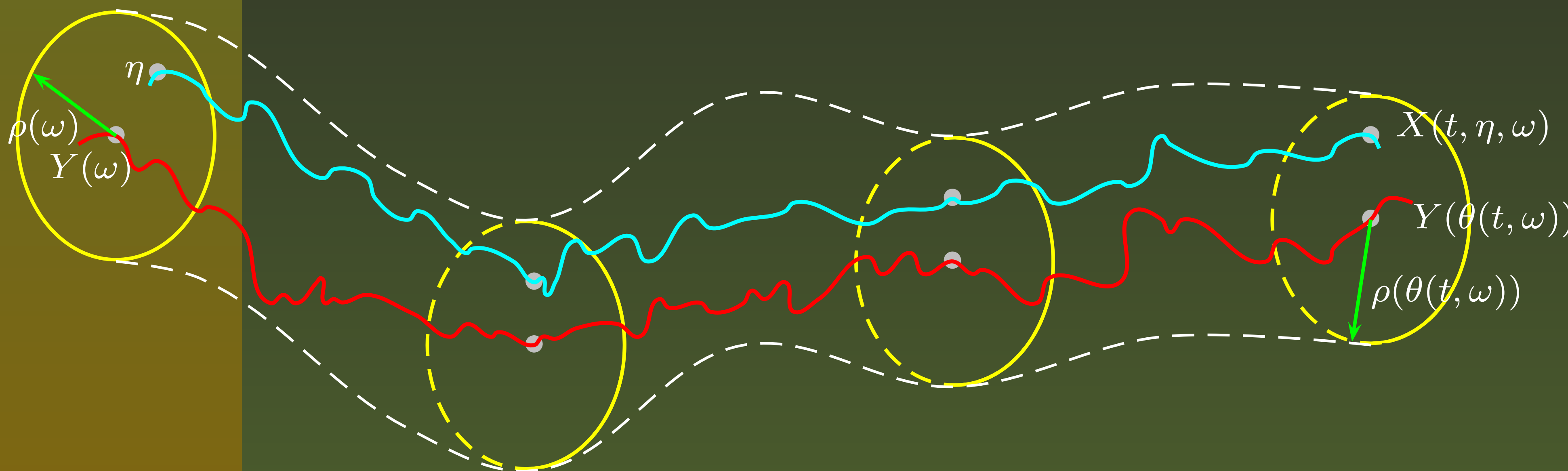
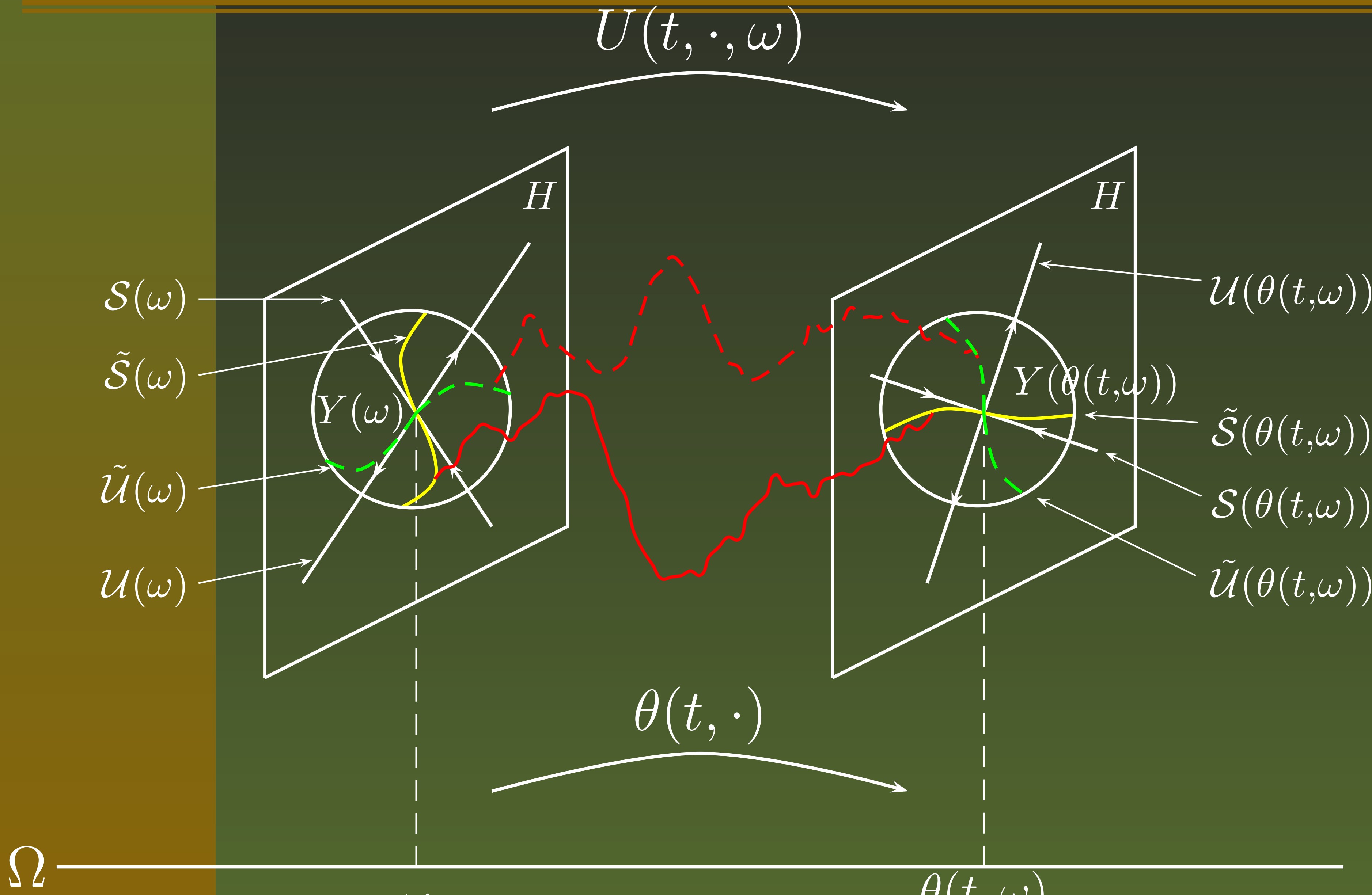
Ergodicity of Y : $\tilde{\mathcal{U}}(\omega) = \{Y(\omega)\}$

A Stationary Tube



Stable/Unstable Manifolds



$t > \tau_1(\omega)$

Examples Revisited

Local stable manifold theorem applies to all examples:

- *Stochastic semilinear heat equation*

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- *Stochastic reaction diffusion equations*
- *Stochastic Burgers equation*

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SKETCH OF PROOF

Proof of Theorem 3: Strategy

- By definition, a *stationary* random point $Y(\omega) \in H$ is invariant under the semiflow U; viz $U(t, Y) = Y(\theta(t, \cdot))$ for all times t.

Proof of Theorem 3: Strategy

- By definition, a *stationary* random point $Y(\omega) \in H$ is invariant under the semiflow U; viz $U(t, Y) = Y(\theta(t, \cdot))$ for all times t.
- Linearize the semiflow U along the stationary point $Y(\omega)$ in H. By stationarity of Y and the cocycle property of U, this gives a linear perfect cocycle $(DU(t, Y), \theta(t, \cdot))$ in $L(H)$.

Strategy-contd

- Ergodicity of θ allows for the notion of hyperbolicity of a stationary point of U via Oseledec-Ruelle theorem:

Strategy-contd

- Ergodicity of θ allows for the notion of hyperbolicity of a stationary point of U via Oseledec-Ruelle theorem:

  Use local compactness of the semiflow for positive t, and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum $\{\lambda_i : i \geq 1\}$ for the linearized cocycle. Y is $hyperbolic$ if $\lambda_i \neq 0$ for every i.

Strategy-contd

- Assume that $\|Y\|^{\epsilon_0}$ is integrable (for small ϵ_0). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of perfect versions of ergodic theorem and Kingman's subadditive ergodic theorem. These refined versions give invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow U.

Strategy-contd

- Establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle U in a neighborhood of the stationary point Y. Estimates follow from the variational construction of the stochastic semiflow.

Strategy-contd

- Introduce the auxiliary perfect cocycle

$$Z(t,\cdot,\omega) := U(t,(\cdot)+Y(\omega),\omega) - Y(\theta(t,\omega)),$$
$$t \in \mathbf{R}^+, \omega \in \Omega.$$

Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/ unstable manifolds for the discrete cocycle $(Z(n,\cdot,\omega),\theta(n,\omega))$ near 0 and hence (by translation) for $U(n,\cdot,\omega)$ near $Y(\omega)$ for all ω sampled from a $\theta(t,\cdot)$-invariant sure event in Ω.

Strategy-contd

- This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between discrete times and further refining the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the *continuous-time* semiflow U near Y.

Strategy-contd

- Final key step:

  Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow U. Use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a *stochastic history process* for U coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow. □

THANK YOU!

THE END!