ABSOLUTE CONTINUITY OF STATIONARY SOLUTIONS OF INTERACTING RANDOM EVOLUTION EQUATIONS *

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Consider the following equation

$$(*) \qquad \frac{\partial x(u,t)}{\partial t} = A(x(u,t)-u) + \int_X a(x(u,t),x(v,t),\xi(t))\nu(dv),\ u \in X,$$

where X is a separable Banach space, ν is a probability measure on X, $\xi(t)$ is a stationary random process with values in some metric space Y, $a: X \times X \times Y \to X$ is a measurable function and A is the generator of a contraction semigroup on X.

The system (*) is considered as stochastic equation with interaction. Here $x(u,t)$ is identified with the position of a particle indexed by $u \in X$ at time t. The evolution of the flow of particles is as follows. The flow is attracted to the identity map on X (see term $A(x(u,t)-u)$); and there is random interaction between the particles described by the function a and the measure ν. The process $\xi(t)$ represents random perturbations.

Let μ be a probability measure on X. Consider a measure-valued process

$$\mu_t(\omega) = \mu \circ (x(\cdot,t,\omega))^{-1},$$

generated by transforming μ under the flow $x(\cdot,t,\omega): X \to X$.

The purpose of this article is to prove the existence of a stationary solution of (*) and to find sufficient conditions to guarantee the absolute continuity of $\mu_t(\omega)$ with respect to the initial measure μ.

Similar problems in the finite-dimensional case were studied in [7, 15, 16], where the initial measure μ is equivalent to Lebesgue measure.

When $x(t)$ is a solution of an infinite-dimensional ODE (without interaction), the study of the absolute continuity of the image-measure $\mu \circ (x(t))^{-1}$ with respect to the initial measure μ has a rich history (cf. [3, 5, 13, 20] and the references therein). An analogue of the infinite-dimensional Liouville theorem is proved in the above-mentioned papers for certain classes of differentiable measures (particularly Gaussian ones).

In §2 we prove a theorem on the equivalence of μ and $\mu \circ (x(\cdot,t))^{-1}$, where x is the solution of the Cauchy problem (*) with initial condition $x(u,0)=u$ and without noise ($\xi(t) \equiv 0$).

Using this result, the general problem for stationary solution is studied in §3.

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§1. Infinite-dimensional Sobolev Spaces

In this section we introduce some assumptions on the space X and the measure μ. We also construct the Sobolev spaces $W_p^1(X,\mu)$.

Let X be a real separable complete LCS, μ be a probability measure on the Borel σ-algebra $\mathcal{B}(X)$, H is a separable Hilbert space densely imbedded into X by a continuous operator $j : H \hookrightarrow X$. Assume that μ is Fomin differentiable [4] along the directions of $j(H)$. Denote by (ρ, h) the logarithmic derivative of measure μ with respect to the direction $j(h)$. Suppose that $\{(\rho,h), h \in H\}$ has a weak order p for any $p < \infty$, i.e.

$$\forall p > 0\ \exists C_p > 0 : \int_X |(\rho,h)|^p d\mu \le C_p \|h\|_H^p,\ h \in H.$$

Definition 1.1. The Sobolev derivative $D_{p,h}$, $p \in (1,+\infty)$, along the direction $h \in H$ is defined as the closure in the space $L_p(X,\mu)$ of the operator

$$\nabla_h :\ f \mapsto (\nabla f, jh)$$

defined on the set of all functions f which are continuous and bounded together with their Fréchet derivative ∇f with respect to the space H.

By virtue of the differentiability of the measure μ, this closure is well defined [2].

Definition 1.2. A function $f \in L_p(X,\mu)$ belongs to the Sobolev space $W_p^1(X,\mu)$ if the following conditions are satisfied:

(i) for any $h \in H$, the derivative $D_{p,h} f$ is defined;
(ii) there exists a random element $g_f \in L_p(X,\mu,H)$ such that

$$\forall h \in H :\ D_{p,h} f = (g_f, h)_H \text{ a.e.}$$

The element g_f is called the Sobolev derivative of the function f and is denoted by $D_p f$. If this does not lead to misunderstanding, we will omit the index p in the expressions $D_{p,h} f$ and $D_p f$.

Let E be a separable Hilbert space. The Sobolev derivatives

$$D_{h,p} :\ L_p(X,\mu,E) \to L_p(X,\mu,E),$$
$$D_p :\ L_p(X,\mu,E) \to L_p(X,\mu,E \otimes H)$$

and the Sobolev spaces $W_p^1(X,\mu,E)$ for E-valued elements are defined by analogy.

Definition 1.3. A function $f \in L_\infty(X,\mu,E)$ belongs to the class $W_\infty^1(X,\mu,E)$, if the following conditions are satisfied:

(i) $f \in \cap_{p>1} W_p^1(X,\mu,E)$
(ii) the functions f and Df are essentially bounded random elements with values in E and $E \otimes H$, respectively.

Note that for fixed x the value $Df(x)$ can be considered as Hilbert-Schmidt operator from E to H.

Definition 1.4. A function $f \in W^1_\infty(X,\mu,E)$ belongs to the class $WL^1_\infty(X,\mu,E)$, if

$$\exists C > 0\, \forall x \in X\, \forall h \in H :\ \|f(x+h) - f(x)\|_E \le C\|h\|_H$$

It will be assumed that a measure μ satisfies the following condition:

B. *For any $h \in H$ the logarithmic derivative (ρ, h) belongs to $\cap_p W^1_{p>1}(X,\mu)$ and there exists a bounded random operator B acting in H [19] such that $Bh = -D(\rho,h)$ and $\beta_\mu = \operatorname{esssup}_x \|B(x)\|_{op} < \infty$, where $\|\cdot\|_{op}$ is operator norm.*

Example 1.1. Let (X,μ,H) be an abstract Wiener space [21]. Then a logarithmic derivative satisfies the condition **B** and $B = id_H$.

Example 1.2. Let $X = H = \mathbb{R}^d, j(x) = x$. Assume that μ is differentiable. Then [4] it is absolutely continuous w.r.t. the Lebesgue measure λ^d. The measure μ satisfies the assumption **B** iff $\mu(dx) = e^{-V(x)}dx$, where $V \in W^2_{p,loc}(\mathbb{R}^d), p > 1$ and $\operatorname{esssup}_x \|V''(x)\| < \infty$. In this case

$$(\rho, h) = -(V'; h),\ \ h \in \mathbb{R}^d$$

and

$$B = V''.$$

We will frequently use a space W^1_∞. So let us describe a space $W^1_\infty(\mathbb{R}^d, \mu, \mathbb{R}^m)$ carefully.

Observe that a function $e^{-V(x)}$ is locally separated from zero and infinity by Sobolev embedding theorems (cf. [1]). Therefore $W^1_\infty(\mathbb{R}^d,\mu,\mathbb{R}^m) = W^1_\infty(\mathbb{R}^d,\lambda^d,\mathbb{R}^m)$. So let us characterize only $W^1_\infty(\mathbb{R}^d,\lambda^d,\mathbb{R}^m)$. Recall Rademacher's theorem:

Theorem 1.1 [11]. *If a function $f : \mathbb{R}^d \to \mathbb{R}^m$ satisfies Lipschitz condition then f is differentiable λ^d-almost sure.*

Using this theorem it can be easily shown that f belongs to $W^1_\infty(\mathbb{R}^d,\lambda^d,\mathbb{R}^m)$ if and only if

1) f is essentially bounded;
2) there exists a modification $\tilde{f} = f$ λ^d-a.e. which satisfies the Lipschitz condition.

In addition, Sobolev derivative of f is equal to the usual derivative of $\tilde{f}$ a.s.

Example 1.3. Let $X = \mathbb{R}^\infty, \mu = \bigotimes_{k=1}^\infty \mu_k$, where $\mu_k(dx_k) = e^{-V(x_k)}dx_k$ is a measure on $\mathbb{R}$ which satisfies the conditions of the Ex.1.2. Let $H = l_2$, j is the inclusion map of l_2 into $\mathbb{R}^\infty$. Then μ is differentiable w.r.t. H [4] and μ satisfies hypothesis **B** with a constant $\beta_\mu = \operatorname{esssup}_{x\in\mathbb{R}} |V''(x)|$.

Example 1.4. The measure that satisfies **B** can be constructed as a distribution of random series (compare with Ex.1.3, see also [12]). For example, let $\{\xi_n | n \ge 1\}$ be i.i.d. random variables with a density of the form $e^{-V(x)}$, $\beta = \sup_x |V''(x)| < \infty$. Let $\{e_n | n \ge 1\}$ be an orthonormal basis of $H = L_2([0,1])$. It can be shown that a random process $\xi(t) = \sum_{k=1}^{\infty} \xi_k \int_0^t e_k(s)ds$ has a continuous modification and its distribution μ in $C_0([0,1]) = \{f \in C([0,1]) | f(0) = 0\}$ has the following properties [17]:

1) μ is differentiable along the directions of $j(H)$, where $H = L_2([0,1])$, $j(h) = \int_0^{\cdot} h(s)ds$.
2) μ satisfies hypothesis **B** with a constant β.

In particular, if $\xi_k \sim N(0,1)$ are normally distributed then $\xi(t)$ is a Wiener process and H is its Cameron-Martin space.

§2. Deterministic equation with interaction

In this section we consider transformations of a measure by a flow that is generated by deterministic equations with interaction. The corresponding result will be used in the study of the Eq.(*) in §3, but is of interest in its own right.

Let X, μ, H be as in the previous section. If there is no ambiguity used in we will identify H and $j(H) \subset X$.

Consider the equation

$$\begin{cases} \frac{\partial x(u,t)}{\partial t} & = A(x(u,t) - u) + \int_X a(x(u,t), x(v,t), t)\nu(dv),\ t \in [0,T] \\ x(u,0) & = u,\ u \in X, \end{cases} \tag{2.1}$$

where ν is a probability measure on X, $a : X \times X \times [0,T] \to H$ is a measurable bounded function, A is a generator of a strongly continuous semigroup $S(t), t \geq 0$ in H.

Definition 2.1. A function $x = x(u,t) : X \times [0,T] \to X$ is called a mild solution of (2.1) if

1) $x(u,t) - u \in H,\ u \in X,\ t \in [0,T]$;
2) a function $X \times [0,T] \ni u \to x(u,t) - u \in H$ is measurable;
3) $x(u,t) = u + \int_0^t S(t-z) \int_X a(x(u,z), x(v,z), z)\nu(dv)dz,\ u \in X, t \in [0,T]$.

The transformations of measures in infinite dimensional spaces by flows generated by ordinary differential equations where studied by many authors. See [3, 5, 13, 20] and references therein. There are some differences between (2.1) and cited works. The main (and very essential) difference is that the equation (2.1) is not homogeneous in space (see term $A(x(u,t) - u)$). Another difference, but not so significant, is the presence of interaction and the unbounded operator.

The main result of this section is the following theorem.

Theorem 2.1. *Assume that a measurable function $a : X \times X \times [0,T] \to H$ satisfies the conditions:*

1) $\exists L > 0\ \forall u, v \in X\ \forall g, h \in H\ \forall t \in [0,T]$:
$$\|a(u+g, v+h, t) - a(u,v,t)\|_H \leq L(\|g\|_H + \|h\|_H),$$
2) $\exists C > 0\ \forall u, v \in X\ \forall t \in [0,T] : \|a(u,v,t)\|_H \leq C$,
3) $\exists K > 0\ \forall v \in X\ \forall t \in [0,T] : a(\cdot, v, t) \in W^1_\infty(X, \mu, H)$ *and* $\operatorname{esssup}_u \|Da(u,v,t)\|_{HS} \leq K$.

Suppose that a semigroup $S(t), t \geq 0$ satisfies the condition

$$\exists M, \alpha > 0 \ \forall t > 0 : \quad \|S(t)\|_{op} \leq Me^{-\alpha t} \tag{2.2}$$

and

$$2MK < \alpha. \tag{2.3}$$

Then there exists the unique mild solution of

$$\begin{cases} x(u,0) & = u \\ \frac{\partial x(u,t)}{\partial t} & = A(x(u,t) - u) + \int_X a(x(u,t), x(v,t), t)\nu(dv) \end{cases} \tag{2.4}$$

and for each $t \in [0,T]$:

$$\mu \circ (x(\cdot,t))^{-1} \sim \mu. \tag{2.5}$$

Proof. For simplicity only the homogeneous in time equation will be considered, i.e. a case $a(u,v,t) \equiv a(u,v)$.

Existence and Uniqueness. A solution of

$$x(u,t) = u + \int_0^t S(t-z) \int_X a(x(u,z), x(v,z))\nu(dv)dz \tag{2.6}$$

can be obtained by iterations, similarly to [7]. So let us show only main steps of the proof.

Put

$$\begin{aligned} x_0(u,t) &:= u, \\ x_{n+1}(u,t) &:= u + \int_0^t S(t-z) \int_X a(x_n(u,z), x_n(v,z))\nu(dv)dz. \end{aligned} \tag{2.7}$$

Then

$$\begin{aligned} \|x_{n+1}(u,t) - x_n(u,t)\|_H \leq C_1 \int_0^t \Big(\|x_n(u,z) - x_{n-1}(u,z)\|_H + \\ + \int_X \|x_n(v,z) - x_{n-1}(v,z)\|_H \nu(dv) \Big) dz, \end{aligned} \tag{2.8}$$

where C_1 is some constant, which does not depend on n.

Now integrate l.h.s. and r.h.s. of (2.8) with respect to the parameter u by the measure $\nu(du)$. Thus

$$\int_X \|x_{n+1}(v,t) - x_n(v,t)\|_H \nu(dv) \leq 2C_1 \int_0^t \int_X \|x_n(v,z) - x_{n-1}(v,z)\|_H \nu(dv)dz. \tag{2.9}$$

Iterating (2.9) we get the inequality:

$$\int_X \|x_{n+1}(v,t) - x_n(v,t)\|_H \nu(dv) \leq C_2 \frac{(2C_1 t)^n}{n!} \tag{2.10}$$

Substituting (2.10) into (2.7) and applying Gronwall's lemma, it is not difficult to prove that a sequence $\{x_n | \, n \geq 1\}$ converges uniformly in $(u,t) \in X \times [0,T]$ to some limit $x = x(u,t)$, which satisfies (2.6). The proof of uniqueness is quite standard, to this end the Gronwall lemma and some inequalities similar to (2.8) – (2.10) are used.

To prove (2.5) we need some auxiliary results.

Theorem 2.2. [13] *Assume that* $\Phi(x) = x + F(x) : X \to X$, *where*

(i) *a function* $F : X \to H$ *belongs to* $WL^1_\infty(H)$;
(ii) $\operatorname{esssup}_x \|DF(x)\|_{HS} = \lambda < 1$.

Then

a) $\mu \circ \Phi^{-1} \sim \mu$,
b) *there exists a constant* $C = C(\|F\|_{L_\infty}, \lambda, \beta_\mu)$ *such that*

$$E\left|\frac{d\mu \circ \Phi^{-1}}{d\mu} \ln \frac{d\mu \circ \Phi^{-1}}{d\mu}\right| \leq C,$$

where β_μ *is a constant from* **B**.

Remark 2.1. The constant C does not depend on the dimension of a space X.

Remark 2.2. Theorem 2.2 is well-known in a case when (X, μ, H) is an abstract Wiener space (cf. [14, 20]).

Remark 2.3. For some measures (particularly Gaussian) the condition $F \in W^1_\infty(X, \mu, H)$ implies [10, 18] the existence of a modification $\tilde{F}$ of F such that $\tilde{F} \in WL^1_\infty(X, \mu, H)$. In this cases we can only require $F \in W^1_\infty(X, \mu, H)$ instead of (i) in Theorem 2.2. But the case of the general measure is unknown to the authors.

Let $\{e_n \,|\, n \geq 1\} \subset H$ be an orthonormal basis in H. Put $H_n = \mathcal{L}(e_1, \ldots, e_n)$. Represent X as a direct sum $X = Y_n \oplus j(H_n)$, where $Y_n \subset X$ is some closed subspace. Identify this decomposition with a product $Y_n \times \mathbb{R}^n$ in such a manner that vectors $j(e_1), \ldots, j(e_n)$ become a natural basis of $\mathbb{R}^n$. The point $u \in X$ under the mentioned decomposition we denote by $(u_n, u^n) \in Y_n \times \mathbb{R}^n$.

Let $\mu_n = \mu_n(du_n)$ be a projection of the measure μ to a space Y_n, $\{\mu_{u_n}(du^n), u_n \in Y_n\}$ be a system of conditional measures with respect to the stratification of $X = Y_n \times \mathbb{R}^n$ onto planes of the form $\{u_n\} \times \mathbb{R}^n, u_n \in Y_n$. This conditional measures are differentiable [4] w.r.t. the directions of $\mathbb{R}^n$, it is also not difficult to show that μ_n-almost all measures μ_{u_n} satisfy condition **B** (on $\mathbb{R}^n$).

Consider the family of operators $S_n(t) := P_n S(t) P_n$ on $\mathbb{R}^n$. Obviously the inequality

$$\|S_n(t)\|_{\mathcal{L}(\mathbb{R}^n)} \leq Me^{-\alpha t}, t > 0. \tag{2.11}$$

is held.

Let us approximate (2.6) by solutions of the equations

$$x_n(u,t) = u + \int_0^t S_n(t-z) \int_X a(x_n(u,z), x(v,z))\nu(dv)dz. \tag{2.12}$$

Observe that (2.12) can be considered as an integral equation of the form

$$x_n(u,t) = u + \int_0^t S_n(t-z) a_n(x_n(u,z), z)dz, \tag{2.13}$$

where the function $a_n(\cdot,t)=\int_X P_n a(\cdot,x(v,t))\nu(dv)$ is bounded by the same constant as a and $a_n(\cdot,t)\in WL^1_\infty(H)$, $\sup_t \operatorname{esssup}_n\|Da_n(u,t)\|_{HS}\le K$.

Note, that for each $u=(u_n,u^n)$:

$$x_n(u,t)\in\{u_n\}\times\mathbb{R}^n,$$

Therefore the image-measure $\mu\circ(x_n(\cdot,t))^{-1}$ is a mixture of images of conditional measures:

$$\{\mu\circ(x_n(\cdot,t))^{-1}\}(du_n,du^n)=\mu_n(du_n)\mu_{u_n}(x_n((u_n,\cdot),t))^{-1}(du^n). \tag{2.14}$$

Let u_n be fixed. Identify $\{u_n\}\times\mathbb{R}^n$ and $\mathbb{R}^n$ for simplicity.

The plan of our further proof is as follows.

1) Show that for each u_n,t a mapping $x((u_n,\cdot),t):\mathbb{R}^n\to\mathbb{R}^n$ satisfies conditions of the Theorem 2.2 with a constant λ which is independent of n. This will imply that $\mu\circ(x_n(\cdot,t))^{-1}\sim\mu$ and

$$\sup_n E\left|\frac{d\mu\circ(x_n(\cdot,t))^{-1}}{d\mu}\ln\frac{d\mu\circ(x_n(\cdot,t))^{-1}}{d\mu}\right|<\infty. \tag{2.15}$$

2) Prove that

$$x_n(\cdot,t)\to x(\cdot,t), n\to\infty \text{ in measure } \mu. \tag{2.16}$$

Then the application of the following theorem to (2.15) and (2.16) will complete a proof.

Theorem 2.3[9]. *Suppose that X is a separable metric space, μ is a probability measure on X, and $\{f_n:X\to X\mid n\ge 0\}$ is a sequence of measurable mappings. Assume that the following conditions are satisfied:*

1) *for any $n\ge 1$ the measure $\mu\circ f_n^{-1}$ is absolutely continuous w.r.t. the measure μ;*
2) *the sequence of Radon-Nikodym densities $\{d\mu\circ f_n^{-1}/d\mu\}$ is uniformly integrable;*
3) *f_n converges to f_0 in the measure μ as $n\to\infty$.*

Then $\mu\circ f_0^{-1}\ll\mu$.

So, let us consider the equation (2.13). As it was mentioned, it suffices to suppose that $u\in\mathbb{R}^n, S_n(t)\in\mathcal{L}(\mathbb{R}^n), a_n:\mathbb{R}^n\to\mathbb{R}^n$. For simplicity we suppress the index n for a while. Let $\tilde{\mu}$ is a measure in $\mathbb{R}^n$ satisfying condition **B**.

Assume, at first, that a function $a:\mathbb{R}^n\to\mathbb{R}^n$ be continuously differentiable.

Remark 2.3. Generally this is not true and this mapping is differentiable almost sure w.r.t. the measure μ_{u_n} or the Lebesgue measure λ^n (see Ex.1.2).

Then $x(u,t)$ is differentiable in a parameter u and its derivative satisfies the equation

$$\frac{\partial x(u,t)}{\partial u}=\mathbb{I}+\int_0^t S(t-z)\nabla a(x(u,z),z)\frac{\partial x(u,z)}{\partial u}dz,$$

where $1\!\!1$ is identity matrix in $\mathbb{R}^n$. Put $y(t) := \frac{\partial x(u,t)}{\partial u} - 1\!\!1$. Then

$$\|y(t)\| \le \int_0^t Me^{-\alpha(t-z)}K(\|y(z)\|+1)dz.$$

There $\|\cdot\|$ is Hilbert-Schmidt norm.

Multiplying both sides by $e^{\alpha t}$ and applying the Gronwall lemma to a function $\|y(z)\|e^{\alpha z}$ we get the following estimation

$$\|y(t)\|e^{\alpha t} \le \frac{MKe^{MKt}}{\alpha - MK}.$$

Therefore $\|y(t)\| \le \frac{MK}{\alpha - MK} < 1$ due to (2.3). Thus we obtain (2.15) under the supposition that a is continuous differentiable.

Consider the general case. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a smooth function with compact support and $\int_{R^n}\varphi(x)dx = 1$. Set

$$\varphi_m(x) := m^n\varphi(\frac{x}{m}), \quad a_m(u,t) := \int_{\mathbb{R}^n} a(u+v,t)\varphi_m(v)dv.$$

Observe that a_m is continuously differentiable in the first argument and for each $t \in [0,T]$:

$$\sup_u \|a_m(u,t)\| \le \sup_u \|a(u,t)\|;$$

$$\sup_u \|\nabla a_m(u,t)\| \le \text{esssup}_u \|\nabla a(u,t)\|;$$

$$a_m(\cdot,t) \to a(\cdot,t), m \to \infty; \tag{2.17}$$

$$\nabla a_m(\cdot,t) \to \nabla a(\cdot,t), m \to \infty \tag{2.18}$$

in measure $\tilde{\mu}$ (and in each finite absolutely continuous measure, also).

Note that

$$\forall u,t,m : \left\|\frac{\partial x_m(u,t)}{\partial u} - 1\!\!1\right\| \le \frac{MK}{\alpha - MK} < 1. \tag{2.19}$$

Let us prove that

$$x_m(\cdot,t) \to x(\cdot,t), \ m \to \infty \tag{2.20}$$

in measure $\tilde{\mu}$:

$$\begin{aligned}\gamma_m(t) :&= \int_{\mathbb{R}^n} \|x_m(u,t) - x(u,t)\|\tilde{\mu}(du) \le \\ &\le M\int_0^t\int_{\mathbb{R}^n} \|a_m(x_m(u,z),z) - a(x(u,z),z)\|\tilde{\mu}(du)dz \le \\ &\le MK\int_0^t\int_{\mathbb{R}^n} \|x_m(u,z) - x(u,z)\|\tilde{\mu}(du)dz + \\ &+ M\int_0^t\int_{\mathbb{R}^n} \|a_m(x_m(u,z),z) - a(x_m(u,z),z)\|\tilde{\mu}(du)dz = \\ &= MK\int_0^t \gamma_m(z)dz + M\int_0^t\int_{\mathbb{R}^n} \|a_m(u,z) - a(u,z)\|\rho_m(u,t)\tilde{\mu}(du)dz,\end{aligned} \tag{2.21}$$

where $\rho_m(u,t) := \frac{d\tilde{\mu}\circ(x_m(\cdot,t))^{-1}}{d\tilde{\mu}}(u)$.

The inequality (2.19) and Theorem 2.2 implies that a set of functions $\{\rho_m(\cdot,t)|\, m \geq 1, t \in [0,T]\}$ is uniformly integrable w.r.t. the measure $\tilde{\mu}$. Taking into account convergence (2.17) we have a convergence of the second term of r.h.s. of (2.21) to zero.

So

$$\gamma_m(t) = MK \int_0^t \gamma_m(z)dz + o(1),$$

where $o(1) \to 0$ as $m \to \infty$.

By Gronwall's lemma $\gamma_m(t) \to 0, m \to \infty$. Therefore (2.20) holds true and $\tilde{\mu} \circ (x(\cdot,t))^{-1} \ll \tilde{\mu}$ by Theorem 2.3.

It is not difficult to show that
$\forall t \in [0,T]\, \forall u_1, u_2 \in \mathbb{R}^n$:

$$\|x(u_1,t) - x(u_2,t)\| \leq \frac{MK}{MK-\alpha}\|u_1 - u_2\|,$$

i.e. $x(\cdot,t), t \in [0,T]$ is uniformly Lipschitz. So by Rademacher's theorem it is almost sure differentiable in u (see Ex.1.2).

Let us prove that for each $t \in [0,T]$:

$$x_m(u,t) - u \mapsto x(u,t) - u, m \to \infty \tag{2.22}$$

in $W_p^1(\mathbb{R}^m, \tilde{\mu}), p > 1$.

We have the convergence in L_p, already, because of (2.20),

$$\sup_{m,u,t} \|x_m(u,t) - u\| < \infty.$$

and a dominated convergence theorem.

Let $\nabla x(u,t)$ be a solution of the equation:

$$\nabla x(u,t) = \mathbb{1} + \int_0^t S(t-z)\nabla a(x(u,z),z)\nabla x(u,z)dz. \tag{2.23}$$

We will prove that $\nabla x(u,t)$, defined by (2.23), is really a Sobolev gradient of $x(u,t)$.

Note that a function $\nabla a(\cdot,z)$ is defined up to sets of $\tilde{\mu}$-null measure and $\mu\circ(x(\cdot,z))^{-1} \ll \mu$. So the composition $\nabla a(x(u,z),z)$ is well-defined for a.a. $u \in \mathbb{R}^n$. Therefore for a.a. $u \in \mathbb{R}^n$ a function $S(t-z)\nabla a(x(u,z),z)$ is defined (and uniformly bounded) for almost all z. That's why for a.a. u there exists the unique solution of (2.23), which can be obtained by iterations.

Taking into account (2.20) and closability of a gradient, in order to prove (2.22) it suffices to show that

$$\nabla x_m(\cdot,t) \xrightarrow{L_p} \nabla x(\cdot,t), m \to \infty. \tag{2.24}$$

Moreover, (2.24) will imply that $\nabla x(u,t)$ is a Sobolev gradient. All functions in further estimations are uniformly bounded, so consider only a case $p=1$.

$$\begin{aligned}
&\|\nabla x(\cdot,t)-\nabla x_m(\cdot,t)\|_{L_1}\leq\\
&\leq M\int_0^t\int_{\mathbb{R}^n}(\|\nabla a(x(u,z),z)\|\|\nabla x(u,z)-\nabla x_m(u,z)\|+\\
&+\|\nabla a(x(u,z),z)-\nabla a_m(x_m(u,z),z)\|\|\nabla x_m(u,z)\|)\tilde{\mu}(du)dz\leq\\
&\leq MK\int_0^t\int_{\mathbb{R}^n}\|\nabla x(u,z)-\nabla x_m(u,z)\|\tilde{\mu}(du)dz+\\
&+\frac{M^2K}{MK-\alpha}\int_0^t\int_{\mathbb{R}^n}\|\nabla a(x(u,z),z)-\nabla a_m(x_m(u,z),z)\|\tilde{\mu}(du)dz.
\end{aligned}\tag{2.25}$$

The application of the Gronwall lemma, the dominated Lebesgue theorem and the following proposition proves (2.24) and therefore (2.22).

Proposition 2.1 [13]**.** *Suppose that X and Y are complete separable metric spaces, μ is a probability measure on X, and $\varphi_n : X \to X$ and $f_n : X \to Y$, $n \geq 0$, are measurable mappings. Assume that the following conditions are satisfied:*

(i) $\varphi_n \xrightarrow{\mu} \varphi_0, n\to\infty \quad f_n \xrightarrow{\mu} f_0, n\to\infty$;
(ii) *for any $n\geq 1$ the measure $\mu\circ(\varphi_n)^{-1}$ is absolute continuous with respect to μ;*
(iii) *the sequence of densities $\{d\mu\circ(\varphi_n)^{-1}/d\mu : n\geq 1\}$ is uniformly integrable.*

Then $f_n\circ\varphi_n \xrightarrow{\mu} f_0\circ\varphi_0, n\to\infty$.

So, we have already proved the following statement.

Proposition 2.2. *Assume that*

a) *a function $a:\mathbb{R}^n\times[0,T]\to\mathbb{R}^n$ lies in $W^1_\infty(\mathbb{R}^n,\tilde{\mu})$;*
b) *a measure $\tilde{\mu}$ satisfies the condition **B**;*
c) *a family $S(t)$ satisfies (2.2);*
d) $2M\sup_t \operatorname{esssup}_x\|\nabla a(x,t)\|_{HS}<\alpha$.

Let $x(u,t)$ be a solution of (2.13). Then $\tilde{\mu}\circ(x(\cdot,t))^{-1}\sim\tilde{\mu}$, a function $u\mapsto x(u,t)-u$ lies in W^1_∞, the derivative $\nabla x(u,t)$ satisfies (2.23) and $\sup_t\operatorname{esssup}_u\|\nabla(x(u,t)-u)\|_{HS}\leq \frac{MK}{\alpha-MK}<1$.

Let us return to the proof of the Theorem 2.1.

As a consequence of the Proposition 2.2, Theorem 2.2 and (2.14) we obtain (2.15). Therefore a sequence of the Radon-Nikodym densities $\left\{\frac{d\mu\circ(x_n(\cdot,t))^{-1}}{d\mu}\,|\,n\geq 1\right\}$ is uniformly integrable.

Let $x(u,t)$ be a solution of (2.6), $x_n(u,t)$ be a solution of (2.12). Note that for each $t\in[0,T]$ the convergence of operators $S_n(t)\to S(t), n\to\infty$ is held in a strong sence.

So

$$\|x(u,t) - x_n(u,t)\|_H \leq$$
$$\leq \int_0^t \|(S(t-z) - S_n(t-z)) \int_X a(x(u,z), x(v,z))\nu(dv)\|_H dz +$$
$$+ \int_0^t \|S_n(t-z) \int_X (a(x_n(u,z), x(v,z)) - a(x(u,z), x(v,z)))\nu(dv)\| dz.$$

The first term of the r.h.s. converges to zero as $n \to \infty$ and (2.16) follows from Gronwall's lemma.

Thus, by the Theorem 2.3:

$$\mu \circ (x(\cdot, t))^{-1} \ll \mu.$$

Let $\nabla x(u,t)$ be a solution of

$$\nabla x(u,t) = 1\!\!1_H + \int_0^t S(t-z) \int_X Da(x(u,z), x(v,z)) \nabla x(u,z) \nu(dv) dz. \tag{2.26}$$

Similarly to the proof of the Proposition 2.2 it can be shown that $\nabla x(u,t)$ is well-defined for μ-a.a. $u \in X$ and

$$\|\nabla x(u,t) - 1\!\!1_H\|_{HS} \leq \frac{MK}{\alpha - MK} < 1 \text{ a.s.} \tag{2.27}$$

and for each vector $e \in H$:

$$D_e x_n(u,t) \xrightarrow{\mu} \langle \nabla x(u,t), e \rangle_H. \tag{2.28}$$

So a function $u \mapsto x(u,t) - u$ is Sobolev differentiable w.r.t. the direction e and $D_e(x(u,t) - u) = \langle \nabla x(u,t) - 1\!\!1, e \rangle_H$. This together with (2.27), (2.28) gives us Sobolev differentiability of a function $u \mapsto x(u,t) - u$, besides $D(x(u,t) - u) = \nabla x(u,t) - 1\!\!1_H$. The application of the Theorem 2.2 completes a proof of the Theorem 2.1.

§3. Measure-valued processes generated by stationary flows with interaction

In this section we consider images of measures driven by a flow that generated by stationary solution of stochastic equation with interaction.

Let Y be a Polish space, $\xi(t)$ be a stationary Y-valued continuous random process, $\eta(t)$ be H-valued continuous process with stationary increments. Assume that $\xi(t)$ and $\eta(t)$ are stationary connected, i.e. for any $n \in \mathbb{N}, t_0 \leq t_1 \leq \ldots \leq t_n, \delta \in \mathbb{R}$ the distribution of $\{\xi(t_k), \eta(t_{k+1}) - \eta(t_k), k = 0, \ldots, n-1\}$ and $\{\xi(t_k + \delta), \eta(t_{k+1} + \delta) - \eta(t_k + \delta), k = 0, \ldots, n-1\}$ coincides.

Consider the following stochastic equation

$$dx(u,t) = \left(A(x(u,t) - u) + \int_X a(x(u,t), x(v,t), \xi(t)) \nu(dv) \right) dt + d\eta(t). \tag{3.1}$$

Definition 3.1. A process $x(u,t), u \in X, t \in \mathbb{R}$ is said to be a mild solution of (3.1) if

(i) $\forall u \in X\, \forall t \in \mathbb{R}\, :\; u \to x(u,t) - u \in H$;

(ii) a mapping $X \times \mathbb{R} \times \Omega \ni (u,t,\omega) \to (x(u,t,\omega) - u) \in H$ is measurable function;

(iii) $\forall u \in X\, \forall s,t,\; s < t$

$$x(u,t) = u + S(t-s)(x(u,s) - u) + \int_s^t S(t-z) \int_X a(x(u,z), x(v,z), \xi(z))\nu(dv)dz + \\ + \int_s^t S(t-z)d\eta(z) \text{ a.s.} \tag{3.2}$$

There a stochastic integral $I_{s,t} := \int_s^t S(t-z)d\eta(z)$ is considered only in a case when at least one of the following three conditions is fulfilled:

1) For a.a. ω a process $\eta(t,\omega)$ is continuously differentiable in t. Then set
$$I_{s,t} := \int_s^t S(t-z)\eta'(z)dz.$$
2) $\eta(t)$ is H-valued Wiener process. Then $I_{s,t}$ is well-defined as the integral of an operator-valued function w.r.t. a Wiener process (see[19]).
3) For a.a.ω and all $t \in \mathbb{R}$ an element $\eta(t,\omega)$ belongs to $\mathcal{D}(A)$ and $A\eta(t)$ is continuous process. This is true, for example, if A is bounded. Then define $I_{s,t}$ via integration by parts formula:
$$I_{s,t} := \int_s^t S(t-z)A\eta(z)dz + A\eta(t) - S(t-s)A\eta(s).$$

Remark 3.1. Note that the definitions of $I_{s,t}$ does not contradict to each other.

Definition 3.2. A process $x(u,t)$ is said to be stationary if for each $n \in \mathbb{N}, u_1, \dots, u_n \in X$, $t_1, \dots, t_n \in \mathbb{R}$ and $\delta \in \mathbb{R}$ the distributions of $\{x(u_k,t_k), 1 \le k \le n\}$ and $\{x(u_k, t_k + \delta), 1 \le k \le n\}$ are equal.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that $a : X \times X \times Y \to H$ is measurable function such that*

1) $\exists L > 0\, \forall u,v \in X\, \forall g,h \in H\, \forall y \in Y$
$$\|a(u+g, v+h, y) - a(u,v,y)\|_H \le L(\|g\|_H + \|h\|_H)$$
2) $\exists C > 0\, \forall u,v \in X\, \forall y \in Y :\; \|a(u,v,y)\|_H \le C$
3) $\exists K > 0\, \forall v \in X\, \forall y \in Y :\; a(\cdot, v, y) \in W^1_\infty(X, \mu, H)$ *and*
$$\text{esssup}_u \|Da(u,v,y)\|_{HS} \le K.$$
4) *integral $I_{s,t} = \int_s^t S(t-z)d\eta(z)$ is defined and continuous in $(s,t), s \le t$ and for each $t \in \mathbb{R}$ there exists a limit*
$$\int_{-\infty}^t S(t-z)d\eta(z) := \lim_{s \to -\infty} \int_s^t S(t-z)d\eta(z) \text{ a.s.}$$

Suppose that (2.2) is satisfied and $\alpha > \max\{2ML, 2MK\}$.

Then there exists the unique up to a stochastic equivalence mild solution of (3.1) and for each $t \in \mathbb{R}$:

$$\mu \circ (x(\cdot, t))^{-1} \sim \mu \text{ for a.a. } \omega.$$

Remark 3.2. The sufficient condition for 4) is a condition 2) of the Definition 3.1 or, for example, the following one:

$$\exists \delta > 0 \; E \sup_{t\in[0,\delta]} \|\eta(t)\|_H < \infty.$$

Proof. The proof of the existence is quite standard (cf.[6, 8]), so we give only a sketch.

Let $x_n(u,t)$ be a solution of the equation

$$x_n(u,t) = u + \int_{-n}^{t} S(t-z) \int_X a(x_n(u,z), x_n(v,z), \xi(z))\nu(dv)dz + I_{-n,t}.$$

Modifying a little the proof of the Theorem 2.1 it can be verified that for a.a. ω and all $t \geq -n$ a mapping $u \mapsto x_n(u,t,\omega)$ lies in $W^1_\infty(X,\mu,H)$ and

$$\|D(x_n(u,t) - u)\|_{HS} \leq \frac{MK}{\alpha - MK} < 1 \text{ for } \mu \times P \text{ a.a. } (u,\omega) \tag{3.3}$$

$$\|x_n(u,t) - u\|_H \leq C_0 = C_0(M,K,\alpha,L), \tag{3.4}$$

where C_0 is some constant.

Let us prove that $x_n(u,t)$ tends to some limit as $n \to \infty$.

Set $\gamma_{m,n}(t) := \sup_{u\in X} \|x_n(u,t) - x_m(u,t)\|_H$. Let $m \geq n$, then

$$\begin{aligned}\gamma_{m,n}(t) \leq \int_{-m}^{-n} Me^{-\alpha(t-z)}C_0 dz + 2LM \int_{-n}^{t} e^{-\alpha(t-z)}\gamma_{m,n}(z)dz + \\ + Me^{-\alpha(t+n)}\|I_{-m,-n}\|_H.\end{aligned} \tag{3.5}$$

Multiplying l.h.s. and r.h.s. by $e^{\alpha t}$ and applying the Gronwall lemma to (3.5) it is easy to deduce that for each segment $[a,b] \subset \mathbb{R}$ the sequence $\gamma_{m,n}(t)$ converges to zero as $m,n \to \infty$ uniformly in $t \in [a,b]$ in probability. Denote by $x(u,t)$ the limit of $x_n(u,t)$. The process $x(u,t)$ is a stationary solution of (3.1) by usual arguments. Let us prove the uniqueness.

Assume that $x(u,t), \tilde{x}(u,t)$ are two stationary solutions of (3.1). Then for each $s \leq t$:

$$\|x(u,t) - u\| \leq Me^{-\alpha(t-s)}\|x(u,s) - u\| + \int_s^t Me^{-\alpha(t-z)}Cdz + \|I_{s,t}\|.$$

Making $s \to -\infty$, the first term tends to zero in probability. Thus for each $t \in \mathbb{R}$, $u \in X$ and a.a. ω

$$\|x(u,t) - u\| \leq MC/\alpha + \|I_{-\infty,t}\|. \tag{3.6}$$

Therefore this inequality is satisfied for a.a. ω and $\nu-$a.a. $u \in X$.

Consider now the difference

$$\|x(u,t)-\tilde{x}(u,t)\| \leq \|S(t-s)(x(u,s)-\tilde{x}(u,s))\| + \int_s^t LMe^{-\alpha(t-z)}\Big(\|x(u,z)-\tilde{x}(u,z)\| + \\ + \int_X \|x(v,z)-\tilde{x}(v,z)\|\nu(dv)\Big) \tag{3.7}$$

Put $\gamma(t) := \int_X \|x(v,z)-\tilde{x}(v,z)\|_H \nu(dv)$. This integral is finite because of (3.6). Let us integrate l.h.s. and r.h.s. of (3.7) with respect to $\nu(du)$ and apply Gronwall's lemma. Analogously to the estimation of (3.5) we get:

$$\gamma(t) \leq M\gamma(s)e^{-(\alpha-2LM)(t-s)}.$$

The right hand side of the inequality converges to zero in probability as $s \to -\infty$. So

$$\int_X \|x_1(v,t)-x_2(v,t)\|_H \nu(dv) = 0 \text{ a.s.} \tag{3.8}$$

Substituting (3.8) to (3.7) we obtain equality

$$\forall u \in X \forall t \in \mathbb{R}: \quad x_1(u,t) = x_2(u,t) \text{ a.s.}$$

in a way analogous to that we deduce (3.8) from (3.7).

So the uniqueness is proved.

By (3.3) and Theorems 2.2, 2.3 we obtain the absolute continuity

$$\mu \circ (x(\cdot,t))^{-1} \ll \mu$$

for a.a.ω.

To prove the equivalence

$$\mu \circ (x(\cdot,t))^{-1} \sim \mu$$

it suffices (Theorem 2.2) to verify that for a.a.ω a

$$x_n(u,t,\omega) - u \to x(u,t,\omega) - u,\ n \to \infty \text{ in } W^1_\infty(X,\mu,H). \tag{3.9}$$

The estimations needed in the Theorem 2.2 will follow from (3.3), (3.4).

Convergence (3.9) can be obtained analogously to the proof in the Theorem 2.1. Really, let $\bar{x}_n(u,t)$ be a solution of

$$\bar{x}_n(u,t) = u + \int_{-n}^t S(t-z)\int_X a(\bar{x}_n(u,z), x(v,z), \xi(z))\nu(dv)dz + \int_{-n}^t S(t-z)d\eta(z).$$

Then for a.a.ω a mapping $u \mapsto \bar{x}_n(u,t,\omega)-u$ is Sobolev differentiable $\sup_t \operatorname{esssup}_u \|D(\bar{x}(u,t)-u)\|_{HS} \le \frac{MK}{\alpha - MK} < 1$ and

$$D\bar{x}_n(u,t) = \mathbb{I} + \int_{-n}^{t} S(t-z) \int_X Da(\bar{x}_n(u,z), x(v,z), \xi(z))\nu(dv) D\bar{x}_n(u,z)dz$$

for μ-a.a. u

Remark 3.3. There we understand $D\bar{x}_n(u,t)$ as $D(\bar{x}_n(u,t)-u) + \mathbb{I}_H$, although it is not a Hilbert-Schmidt operator.

It is not difficult to verify that for μ-a.a. $u \in X$ there exists the unique mild stationary solution of the equation

$$dy(u,t) = A(y(u,t) - \mathbb{I}_H)dt + \int_X Da(x(u,t), x(v,t), \xi(t))\nu(dv) y(u,t)dt$$

and

$$\|D\bar{x}_n(u,t) - y(u,t)\|_{HS} \to 0, n \to \infty$$

in measure $\mu \times P$(for details see the proof of the Theorem 2.1).

This convergence implies that for a.a.ω a mapping $u \mapsto x(u,t,\omega) - u$ is stochastic differentiable, $y(u,t) - \mathbb{I}_H = D(x(u,t)-u)$ and

$$\|D(x(u,t)-u)\|_{HS} \le \frac{MK}{\alpha - MK} < 1.$$

Applying Theorem 2.2 completes the proof of the Theorem 3.1.

Remark 3.4. The proof of the theorem is much easier if a function $a(\cdot, v, y) : X \to H$ belongs to a class $\mathcal{H}C^1(H)$. I.e. if for any $x, v \in X, y \in Y$ the function

$$H \ni h \mapsto a(x+h, v, y) \in H$$

is continuously Fréchet differentiable. In this case the Fréchet (and hence a Sobolev) differentiability of $x(u,t)$ can be obtained by standard arguments on differentiability of a stationary solution with respect to a parameter.

Remark 3.5. Sometimes the integral $\int_s^t S(t-z)dw(z)$ can be defined as an H-valued random element in case $w(t)$ is a Wiener process in a wider space than H. Really, let $w(t)$ has a covariance operator Qt, where $Q \in \mathcal{L}(H)$, but Q is not a Hilbert-Schmidt operator. Assume that $\int_0^\infty \|S(t)Q\|_{HS}dt < \infty$. Then $w(t)$ is not H-valued process, but integrals $\int_s^t S(t-z)dw(z)$ and $\int_{-\infty}^t S(t-z)dw(z)$ are defined and are H-valued processes (more details see in [6]). The statement of the Theorem 3.1 is true in this case, also.

Let us consider some examples of functions, semigroups and spaces that illustrate Theorems 2.1, 3.1.

Example 3.1. Here we show an example of a function a which satisfies assumptions of the Theorem 2.1, but is not $\mathcal{H}C^1(H)$ (cf. Remark 3.4).

Let $\{m_k\}$ be a sequence of natural numbers, $\{h_k\} \subset H$, $\|h_k\|_H = 1$, $\{e^*_{k,i}, \ell^*_{k,i}, i = 1, \ldots, m_k\}$ be a sequence of series of elements of X^* such that for each $k \in \mathbb{N}$ vectors $\{e_{k,i} := j(e^*_{k,i}), \ell_{k,i} := j(\ell^*_{k,i}) : i = 1, \ldots, m_k\}$ are orthonormal in H.

Consider a sequence of bounded functions $\{f_k : \mathbb{R}^{2m_k} \to \mathbb{R}\}$ which satisfy Lipschitz condition:

$$\exists L_k > 0 \, \forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^{2m_k} : \ |f_k(\vec{x}_1) - f_k(\vec{x}_2)| \le L_k \|\vec{x}_1 - \vec{x}_2\|_{\mathbb{R}^{2m_k}}.$$

Set $c_k := \sup_{\vec{x}} |f_k(\vec{x})|$.

As it was mentioned in §1 (Rademacher's theorem), each Lipschitz function is differentiable a.s.

Put

$$K_k := \sup_{\vec{x}'' \in \mathbb{R}^{m_k}} \operatorname{esssup}_{\vec{x}' \in \mathbb{R}^{m_k}} \|\nabla_1 f(\vec{x}', \vec{x}'')\|_{\mathbb{R}^{m_k}},$$

where ∇_1 is a derivative along first m_k coordinates.

Let us introduce notations:

$$\langle \vec{e}^{\,*}_k, u \rangle := (\langle e^*_{k,1}, u \rangle, \ldots, \langle e^*_{k,m_k}, u \rangle),$$
$$\langle \vec{\ell}^{\,*}_k, u \rangle := (\langle \ell^*_{k,1}, u \rangle, \ldots, \langle \ell^*_{k,m_k,u} \rangle),$$

Assume that

$$C := \sum_k C_k < \infty, \ L := \sum_k L_k < \infty, \ K := \sum_k K_k < \infty$$

Then a function

$$a(u,v) = \sum_k f_k(\langle \vec{e}^{\,*}_k, u \rangle, \langle \vec{\ell}^{\,*}_k, v \rangle) h_k \tag{3.10}$$

satisfies assumptions of a Theorem 2.1 with constants C, L and K.

Remark 3.6. Note that $K_k \le L_k m_k$. And we have an upper estimation for K :

$$K \le \sum_k L_k m_k$$

using only Lipschitz constants.

Example 3.2. Let $X = \mathcal{D}'([0,\pi]) = (C^\infty([0,\pi]))'$ be a set of generalized functions on $[0,\pi]$, μ be a mean-zero Gaussian measure on X such that

$$\operatorname{cov}\left({}_X\langle f, \cdot \rangle_{X'}, {}_X\langle g, \cdot \rangle_{X'}\right) = \int_0^\pi f(x) g(x) dx, \ f, g \in C^\infty([0,\pi]).$$

Then Cameron-Martin space H is $L_2([0,\pi])$ and $j : H \to X$ is a canonical inclusion map (cf. Ex.1.1).

Assume that $A = \frac{d^2}{dx^2}, \mathcal{D}(A) = \{f \in W_2^2([0,\pi])/f(0) = f(\pi) = 0\}$.

Then A is a generator of contraction semigroup $S(t)$, which satisfies (2.2) with $M = 1$ and $\alpha = 1$.

Put $\eta(t) = \sum_{k=1}^{\infty} \varphi_k(x) w_k(t)$, where $w_k(t)$ are independent one-dimensional Wiener processes, $\sum_k \|\varphi_k\|_{L_2}^2 < \infty$.

Let $\xi(t) = 0$ and a function $a : X^2 \to H$ is of the form (3.10). Then conditions of a theorem 3.1 are satisfied if $K < \frac{1}{2}$. Denote $x(u,t) - u$ by $y(u,t)$. Observe that $y(u,t)$ is an element of $L_2([0,\pi])$. The equation for $y(u,t)$ is a SPDE of the form

$$dy(u,t,x) = \frac{\partial y(u,t,x)}{\partial x^2} dt +$$
$$+ \sum_k \int_X f_k \left(\langle u, \vec{e}_k^* \rangle + \int_0^\pi y(u,t,r) \vec{e}_k^*(r) dr, \langle v, \vec{\ell}_k^* \rangle + \int_0^\pi y(v,t,r) \vec{\ell}_k^*(r) dr \right) \nu(dv) h_k(x) +$$
$$+ \sum_k \varphi_k(x) dw_k(t),\ x \in [0,\pi], t \in \mathbb{R}.$$

Example 3.3. Let X, μ, H be as in Ex.1.4 and $u(t), t \in [0,1]$ be a canonical process corresponding to a distribution of μ. As in a case of a Wiener process it can be shown that

$$Du(t) = 1\!\!1_{[0,t]}.$$

So

$$Df(u(t)) = f'(u(t)) 1\!\!1_{[0,t]} \tag{3.11}$$

for each $f \in C_b^1(\mathbb{R})$.

The distribution of $u(t)$ is absolute continuous for each $t \in (0,1]$ (for ex. as the sum of independent random variables with absolutely continuous distribution). Thus the formula (3.11) is valid for each bounded Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and $f(u(t)) \in W_\infty^1$. Really this statement can be proved by using Theorem 1.1 and an approximation of $f(u(t))$ by a sequence $f_n(u(t)), n \geq 1$, where $f_n = f * \varphi_n,\ \varphi_n(x) = n\varphi\left(\frac{x}{n}\right),\ \varphi \in C_0^\infty(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(x) dx = 1$.

Let $\xi(t)$ be a stationary process with values in a metric space $Y, K : \mathbb{R}^2 \times Y \to \mathbb{R},\ \nu$ be a probability on X. Consider the equation

$$\frac{\partial x(u,t)(s)}{\partial t} = -\alpha(x(u,t)(s) - u(s)) +$$
$$+ \int_0^s \int_X K(x(u,t)(z), x(v,t)(z), \xi(t)) \nu(dv) dz, s \in [0,1], t \in \mathbb{R}. \tag{3.12}$$

Recall that now inclusion map $j : H \to X$ is of the form $j(h)(\cdot) = \int_0^{\cdot} h(z)dz$. Assume that K is bounded function and it satisfies a Lipschitz condition in first two arguments: $\exists L > 0 \forall y \in Y \forall r_1, r_2 \in \mathbb{R}^2$:

$$|K(r_1, y) - K(r_2, y)| \leq L\|r_1 - r_2\|.$$

Then for any $v \in X, y \in X$ a mapping

$$X \ni u \mapsto K(u(\cdot), v(\cdot), y) \in H$$

belongs to $W^1_\infty(X, \mu, H)$ and

$$D_z(K(u(s), v(s), y)) = K_1'(u(s), v(s), y)\mathbb{1}_{[0,s]}(z).$$

If $\sup_{r_1, r_2, y} \|K_1'(r_1, r_2, y)\|_{HS} < \alpha/2$ then conditions of the Theorem 3.1 are fulfilled and stationary solution of (3.12) transports measure μ to equivalent.

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