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# Dynamics of stochastic $2D$ Navier–Stokes equations

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**Abstract**

In this paper, we study the dynamics of a two-dimensional stochastic Navier–Stokes equation on a smooth domain, driven by linear multiplicative white noise. We show that solutions of the $2D$ Navier–Stokes equation generate a perfect and locally compacting $C^{1,1}$ cocycle. Using multiplicative ergodic theory techniques, we establish the existence of a discrete non-random Lyapunov spectrum for the cocycle. The Lyapunov spectrum characterizes the asymptotics of the cocycle near an equilibrium/stationary solution. We give sufficient conditions on the parameters of the Navier–Stokes equation and the geometry of the planar domain for hyperbolicity of the zero equilibrium, uniqueness of the stationary solution (viz. ergodicity), local almost sure asymptotic stability of the cocycle, and the existence of global invariant foliations of the energy space.


*Keywords:* Stochastic Navier–Stokes equation; Cocycle; Lyapunov exponents; Stable manifolds; Invariant manifolds

## 1. Introduction

Two-dimensional stochastic Navier–Stokes equations (SNSE's) are often used to describe the time evolution of an incompressible fluid in a smooth bounded planar domain.

In this article, we characterize the long-time asymptotics of the following two-dimensional stochastic Navier–Stokes equation with Dirichlet boundary conditions:

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$$\left.\begin{aligned} &du - \nu\Delta u\,dt + (u\cdot\nabla)u\,dt + \nabla p\,dt = \gamma u\,dt + \sum_{k=1}^{\infty}\sigma_k u(t)\,dW_k(t), \quad t>0,\\ &(\operatorname{div} u)(t,x) = 0, \quad x\in D,\ t>0,\\ &u(t,x) = 0, \quad x\in\partial D,\ t>0,\\ &u(0,x) = u_0(x), \quad x\in D, \end{aligned}\right\} \tag{1.1}$$

where $D$ is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial D$, $u(t,x)\in\mathbf{R}^2$ denotes the velocity field at time $t$ and position $x\in D$, $p(t,x)$ denotes the pressure field, and $\nu>0$ the viscosity coefficient. Moreover, the random force field is driven by independent one-dimensional standard Brownian motions $W_k$, $k\geqslant 1$, and a deterministic linear drift term $\gamma u\,dt$ with a fixed parameter $\gamma$. The Brownian motions $W_k$, $k\geqslant 1$, are defined on a complete filtered Wiener space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},P)$. We assume that the noise parameters $\sigma_k$, $k\geqslant 1$, are such that $\sum_{k=1}^{\infty}\sigma_k^2<\infty$.

To formulate the dynamics of the above stochastic Navier–Stokes equation, we introduce the following standard spaces:

Consider the Hilbert space

$$V := \left\{v\in H_0^1\left(D,\mathbf{R}^2\right):\ \nabla\cdot v = 0 \text{ a.e. in } D\right\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v|^2\,dx\right)^{\frac{1}{2}}$$

and inner product $\langle\langle\cdot,\cdot\rangle\rangle$. Denote by $H$ the closure of $V$ in the $L^2$-norm

$$|v|_H := \left(\int_D |v|^2\,dx\right)^{\frac{1}{2}}.$$

The inner product on $H$ will be denoted by $\langle\cdot,\cdot\rangle$.

Denote by $P_H$ the Helmholtz–Hodge projection from the Hilbert space $L^2(D,\mathbf{R}^2)$ onto $H$. Define the (Stokes) operator $A$ in $H$ by the formula

$$Au := -\nu P_H\Delta u, \quad u\in H^2\left(D,\mathbf{R}^2\right)\cap V,$$

and the nonlinear operator $B$ by

$$B(u,v) := P_H\big((u\cdot\nabla)v\big),$$

whenever $u$, $v$ are such that $(u\cdot\nabla v)$ belongs to the space $L^2$. We will often use the short notation $B(u) := B(u,u)$.

By applying the operator $P_H$ to each term of the above stochastic Navier–Stokes equation (SNSE), we can rewrite the equation in the following abstract form:

$$du(t) + Au(t)\,dt + B\big(u(t)\big)\,dt = \gamma u(t)\,dt + \sum_{k=1}^{\infty} \sigma_k u(t)\,dW_k(t) \tag{1.2}$$

in $L^2(0, T; V')$ with the initial condition

$$u(0) = u_0 \in H, \tag{1.3}$$

where $V'$ is the dual of $V$.

Finally, and for the remainder of the article, we will adopt the following convention:

**Definition 1.1** *(Perfection).* A family of propositions $\{P(\omega)\colon\ \omega \in \Omega\}$ is said to **hold perfectly in** $\omega$ if there is a sure event $\Omega^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$ and $P(\omega)$ is true **for every** $\omega \in \Omega^*$.

There is a large amount of literature on the stochastic Navier–Stokes equation. We will only refer to some of it. A good reference for stochastic Navier–Stokes equations driven by additive noise is the book [4] and the references therein; see also [5]. The existence and uniqueness of solutions of stochastic $2D$ Navier–Stokes equations with multiplicative noise were obtained in [9,17]. Ergodic properties, invariant measures, asymptotic compactness and absorbing sets of stochastic $2D$ Navier–Stokes equations are studied in [8,12,11,2]. The results in [12] address important aspects of the ergodic theory and invariant measures for $2D$ stochastic Navier–Stokes equations with (additive) periodic random "kicks". The small noise large deviation of the stochastic $2D$ Navier–Stokes equations was established in [17] and the large deviation of occupation measures was considered in [10]. Related results on the dynamics of semilinear stochastic partial differential equations are given in [6,7,3].

The purpose of this paper is to study the dynamics of the two-dimensional stochastic Navier–Stokes equation (1.1) driven by multiplicative noise. In particular, we will establish the following:

- Existence of a perfect locally compacting $C^{1,1}$ cocycle (semiflow) generated by all solutions of the stochastic Navier–Stokes equation;
- Long-time asymptotics for the stochastic semiflow given by a countable non-random Lyapunov spectrum of the linearized cocycle at the equilibrium (viz. stationary solution);
- Existence of countable families of $C^{1,1}$ local and global flow-invariant submanifolds through the equilibrium (when $\gamma = 0$);
- Sufficient conditions for hyperbolicity of the equilibrium; viz. existence of flow-invariant local stable/unstable manifolds in the neighborhood of the equilibrium;
- Sufficient conditions on the parameters $\nu, \gamma, \sigma_i, i \geqslant 1$, and the geometry of the domain $D$ to guarantee uniqueness of the (zero) equilibrium.

We believe that it is possible to modify the arguments in this article so as to cover the case of additive noise, white in time and sufficiently smooth in space.

## 2. Preliminaries

Let us identify the Hilbert space $H$ in Section 1 with its dual $H'$. We then consider the stochastic Navier–Stokes equation (1.1) in the framework of the Gelfand triple:

$$V \subset H \cong H' \subset V'$$

where $V'$ is the dual of $V$. Thus, we may consider the Stokes operator $A$ as a bounded linear map from $V$ into $V'$. Moreover, we also denote by $\langle\cdot,\cdot\rangle : V \times V' \to \mathbf{R}$, the canonical bilinear pairing between $V$ and $V'$. Hence, using integration by parts, we have

$$\langle Au, w\rangle = \nu \sum_{i,j=1}^{2} \int_D \partial_i u_j \partial_i w_j \, dx = \nu \langle\langle u, w\rangle\rangle \tag{2.1}$$

for $u = (u_1, u_2) \in V$, $w = (w_1, w_2) \in V$.

Define the real-valued trilinear form $b$ on $H \times H \times H$ by setting

$$b(u, v, w) := \sum_{i,j}^{2} \int_D u_i \partial_i v_j w_j \, dx, \tag{2.2}$$

whenever the integral in (2.2) makes sense. In particular, if $u, v, w \in V$, then

$$b(u, v, w) = \big\langle B(u, v), w\big\rangle = \big\langle (u \cdot \nabla)v, w\big\rangle = \sum_{i,j}^{2} \int_D u_i \partial_i v_j w_j \, dx.$$

Using integration by parts, it is easy to see that

$$b(u, v, w) = -b(u, w, v), \tag{2.3}$$

for all $u, v, w \in V$. Thus,

$$b(u, v, v) = 0 \tag{2.4}$$

for all $u, v \in V$.

Throughout the paper, we will denote various generic positive constants by the same letter $c$, although the constants may differ from line to line. We now list some well-known estimates for $b$ which will be used frequently in the sequel (see [19,15] for example):

$$\big|b(u, v, w)\big| \leqslant c\|u\|_V \cdot \|v\|_V \cdot \|w\|_V, \quad u, v, w \in V, \tag{2.5}$$

$$\big|b(u, v, w)\big| \leqslant c|u|_H \cdot \|v\|_V \cdot |Aw|_H, \quad u \in H,\ v \in V,\ w \in D(A), \tag{2.6}$$

$$\big|b(u, v, w)\big| \leqslant c\|u\|_V \cdot |v|_H \cdot |Aw|_H, \quad u \in V,\ v \in H,\ w \in D(A), \tag{2.7}$$

$$\big|b(u, v, w)\big| \leqslant 2\|u\|_V^{\frac{1}{2}} \cdot |u|_H^{\frac{1}{2}} \cdot \|w\|_V^{\frac{1}{2}} \cdot |w|_H^{\frac{1}{2}} \cdot \|v\|_V, \quad u, v, w \in V. \tag{2.8}$$

Moreover, combining (2.3) and (2.8), we obtain

$$\begin{aligned} \big|B(u, w)\big|_{V'} &= \sup_{\|v\|_V \leqslant 1} \big|b(u, w, v)\big| = \sup_{\|v\|_V \leqslant 1} \big|b(u, v, w)\big| \\ &\leqslant 2\|u\|_V^{\frac{1}{2}} \cdot |u|_H^{\frac{1}{2}} \cdot \|w\|_V^{\frac{1}{2}} \cdot |w|_H^{\frac{1}{2}} \end{aligned} \tag{2.9}$$

for all $u, w \in V$.

## 3. Existence of the cocycle

In this section, we will show that strong solutions of the stochastic NSE generate a Fréchet $C^{1,1}$ locally compacting cocycle (viz. stochastic semiflow) $u : \mathbf{R}^+ \times H \times \Omega \to H$ on the Hilbert space $H$. Our approach is to use a variational technique which transforms the SNSE into a *random* NSE that we then analyze using a combination of Galerkin approximations and a priori estimates (cf. [19,15]).

Consider the SNSE

$$\begin{cases} du(t,f) + Au(t,f)\,dt + B\big(u(t,f)\big)\,dt = \gamma u(t,f)\,dt + \sum_{k=1}^{\infty} \sigma_k u(t,f)\,dW_k(t), & t>0, \\ u(0,f) = f \in H. \end{cases} \tag{3.1}$$

It is known that for each $f \in H$, the SNSE (3.1) admits a unique strong solution $u(\cdot, f) \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega \times [0,T]; V)$ [1]. Writing (3.1) in integral form, we have

$$\begin{aligned} u(t,f) = f - \int_0^t Au(s,f)\,ds - \int_0^t B\big(u(s,f)\big)\,ds + \gamma \int_0^t u(s,f)\,ds \\ + \sum_{k=1}^{\infty} \int_0^t \sigma_k u(s,f)\,dW_k(s), \end{aligned} \tag{3.2}$$

for all $t \in [0,T]$.

Let $Q : [0,\infty) \times \Omega \to \mathbf{R}$ be the solution of the one-dimensional linear stochastic ordinary differential equation

$$\left. \begin{aligned} &dQ(t) = \gamma Q(t)\,dt + \sum_{k=1}^{\infty} \sigma_k Q(t)\,dW_k(t), \quad t \geqslant 0, \\ &Q(0) = 1. \end{aligned} \right\} \tag{3.3}$$

By Itô's formula, we have

$$Q(t) = \exp\left\{ \sum_{k=1}^{\infty} \sigma_k W_k(t) - \frac{t}{2} \sum_{k=1}^{\infty} \sigma_k^2 + \gamma t \right\}, \quad t \geqslant 0. \tag{3.4}$$

This implies that

$$E\|Q\|_\infty < \infty,$$

where

$$\|Q\|_\infty \equiv \big\|Q(\cdot,\omega)\big\|_\infty := \sup_{0 \leqslant t \leqslant T} Q(t,\omega), \quad \omega \in \Omega,$$

for any finite positive $T$. Define

$$v(t,f) := u(t,f)Q^{-1}(t), \quad t \geqslant 0. \tag{3.5}$$

Applying Itô's formula to the relation $u(t,f) = v(t,f)Q(t)$, $t \geqslant 0$, and using (3.3), it is easy to see that $v(t) \equiv v(t,f)$ satisfies the random NSE

$$\left.\begin{aligned} &dv(t) = -Av(t)\,dt - Q(t)B\big(v(t)\big)\,dt, \quad t \geqslant 0,\\ &v(0) = f \in H. \end{aligned}\right\} \tag{3.6}$$

Our next proposition gives a priori bounds on solutions of the random NSE (3.6).

**Proposition 3.1.** *For $f \in H$ and $\omega \in \Omega$, let $v(\cdot, f, \omega) \in C([0,T], H) \cap L^2([0,T], V)$ be a solution of* (3.6) *on* $[0,T]$ *for some $T > 0$. Then for each $\omega \in \Omega$ and any $f \in H$, the following estimates hold*

$$\sup_{0 \leqslant t \leqslant T} \big|v(t,f,\omega)\big|_H \leqslant |f|_H \tag{3.7}$$

*and*

$$\int_0^T \big\|v(t,f,\omega)\big\|_V^2\,dt \leqslant \frac{1}{2\nu}|f|_H^2. \tag{3.8}$$

*Moreover, for each $\omega \in \Omega$, the map $H \ni f \mapsto v(\cdot, f, \omega) \in C([0,T], H) \cap L^2([0,T], V)$ is Lipschitz on bounded sets in $H$.*

**Proof.** Let $f \in H$ and $v(t) \equiv v(t,f,\omega)$, $t \in [0,T]$, be a solution of (3.6). We fix and suppress $\omega \in \Omega$ throughout this proof. Employing the divergence free condition, $\langle B(v), v\rangle = 0$, we obtain

$$\begin{aligned} \big|v(t,f)\big|_H^2 &= |f|_H^2 - 2\int_0^t \big\langle Av(s,f), v(s,f)\big\rangle\,ds - 2\int_0^t Q(s)\big\langle B\big(v(s,f)\big), v(s,f)\big\rangle\,ds \\ &= |f|_H^2 - 2\int_0^t \big\langle Av(s,f), v(s,f)\big\rangle\,ds \\ &= |f|_H^2 - 2\nu\int_0^t \big\|v(s,f)\big\|_V^2\,ds \end{aligned} \tag{3.9}$$

for all $t \in [0,T]$. Hence,

$$\left|v(t,f)\right|_H^2 + 2\nu \int_0^t \left\|v(s,f)\right\|_V^2 \, ds = |f|_H^2, \quad t \in [0,T].$$

This immediately gives (3.7) and (3.8).

It remains to prove the last assertion of the proposition. Let $f, g \in H$, and $t \in [0,T]$ for the rest of the proof. Using the identity

$$b(u,v,v) = 0, \quad u, v \in V,$$

and the chain rule we obtain

$$\begin{aligned}
\left|v(t,f) - v(t,g)\right|_H^2 &= |f-g|_H^2 - 2\int_0^t \left\langle A\big(v(s,f)-v(s,g)\big), v(s,f)-v(s,g)\right\rangle ds \\
&\quad - 2\int_0^t Q(s)\left\langle B\big(v(s,f)\big) - B\big(v(s,g)\big), v(s,f)-v(s,g)\right\rangle ds \\
&= |f-g|_H^2 - 2\nu \int_0^t \left\|v(s,f)-v(s,g)\right\|_V^2 \, ds \\
&\quad - 2\int_0^t Q(s)\Big[b\big(v(s,f), v(s,f), v(s,f)-v(s,g)\big) \\
&\quad - b\big(v(s,g), v(s,g), v(s,f)-v(s,g)\big)\Big] ds \\
&= |f-g|_H^2 - 2\nu \int_0^t \left\|v(s,f)-v(s,g)\right\|_V^2 \, ds \\
&\quad - 2\int_0^t Q(s) b\big(v(s,f)-v(s,g), v(s,g), v(s,f)-v(s,g)\big) \, ds.
\end{aligned} \tag{3.10}$$

Thus, we have

$$\begin{aligned}
\left|v(t,f) - v(t,g)\right|_H^2 &\leqslant |f-g|_H^2 - 2\nu \int_0^t \left\|v(s,f)-v(s,g)\right\|_V^2 \, ds \\
&\quad + 4\|Q\|_\infty \int_0^t \left\|v(s,f)-v(s,g)\right\|_V \left\|v(s,g)\right\|_V \left|v(s,f)-v(s,g)\right|_H ds
\end{aligned}$$

$$\leqslant |f-g|_H^2 - \nu \int_0^t \big\|v(s,f)-v(s,g)\big\|_V^2\,ds$$

$$+ 4\frac{\|Q\|_\infty^2}{\nu} \int_0^t \big\|v(s,g)\big\|_V^2 \big|v(s,f)-v(s,g)\big|_H^2\,ds. \tag{3.11}$$

Applying Gronwall's lemma (Lemma 5.1) to the above inequality and using (3.8), we get

$$\big|v(t,f)-v(t,g)\big|_H^2 + \nu \int_0^t \big\|v(s,f)-v(s,g)\big\|_V^2\,ds$$

$$\leqslant |f-g|_H^2 \exp\left(4\frac{\|Q\|_\infty^2}{\nu}\int_0^T \big\|v(s,g)\big\|_V^2\,ds\right)$$

$$\leqslant |f-g|_H^2 \exp\left(\frac{2}{\nu^2}\|Q\|_\infty^2 |g|_H^2\right). \tag{3.12}$$

This implies that, for each $\omega \in \Omega$, the solution map

$$H \ni f \mapsto v(\cdot, f, \omega) \in C\big([0,T],H\big) \cap L^2\big([0,T],V\big)$$

(when it exists) is Lipschitz on bounded sets in $H$. $\square$

Our next proposition proves the existence of a unique strong global solution to the random NSE (3.6).

**Proposition 3.2.** *Let $f \in H$, $\omega \in \Omega$. Then for each $T > 0$, there exists a unique solution $v(\cdot, f, \omega) \in C([0,T],H) \cap L^2([0,T],V)$ to Eq. (3.6). Furthermore, the solution map $\mathbf{R}^+ \times H \times \Omega \ni (t, f, \omega) \mapsto v(t, f, \omega) \in H$ is jointly measurable, and for each $f$ the process $v(\cdot, f, \cdot) : \mathbf{R}^+ \times \Omega \to H$ is $(\mathcal{F}_t)_{t \geqslant 0}$-adapted.*

**Proof.** The proof is based on Galerkin approximations coupled with a priori estimates (cf. [19,18]).

Let $f \in H$, fix and suppress $\omega \in \Omega$. We use Galerkin approximations to prove existence of a solution to the random NSE

$$\left.\begin{aligned} &dv(t) = -Av(t)\,dt - Q(t)B\big(v(t)\big)\,dt, \quad t > 0,\\ &v(0) = f \in L^2\big(D, \mathbf{R}^2\big) = H,\\ &v(t)|_{\partial D} = 0, \quad t > 0. \end{aligned}\right\} \tag{3.13}$$

Let $\{e_i\}_{i=1}^\infty$ be a complete orthonormal basis of $H$ that consists of eigenvectors of the operator $-A$ under Dirichlet boundary conditions with corresponding eigenvalues $\{\mu_i\}_{i=1}^\infty$; that is $A(e_i) = -\mu_i e_i$, $e_i|_{\partial D} = 0$, $i \geqslant 1$. Let $H_n$ denote the $n$-dimensional subspace of $H$ spanned by $\{e_1, e_2, \ldots, e_n\}$. Define $f_n \in H_n$ by

$$f_n := \sum_{j=1}^{n} \langle f, e_j \rangle e_j.$$

Clearly, the sequence $\{f_n\}_{n=1}^{\infty}$ converges to $f$ in $H$. Now for every integer $n \geqslant 1$, we seek a solution $v_n$ of the random NSE

$$\left.\begin{aligned} &dv_n(t) = -Av_n(t)\,dt - Q(t)B\big(v_n(t)\big)\,dt, \quad t > 0,\\ &v_n(0) = f_n,\\ &v_n(t)|_{\partial D} = 0, \quad t > 0, \end{aligned}\right\} \tag{3.13$^n$}$$

such that

$$v_n(t) := \sum_{j=1}^{n} g_{jn}(t) e_j, \quad t \geqslant 0,$$

for appropriate choice of the real-valued random processes $g_{jn}$. We will show that the Fourier coefficients $g_{jn}(t)$ solve a system of random ordinary differential equations with locally Lipschitz coefficients. To see this, we proceed as follows. Since $v_n$ satisfies the NSE ($3.13^n$), then for each $1 \leqslant j \leqslant n$, we have

$$\begin{aligned} dg_{jn}(t) &= d\big\langle v_n(t), e_j\big\rangle\\ &= -\big\langle Av_n(t), e_j\big\rangle dt - Q(t)\big\langle \big(v_n(t)\cdot\nabla\big)v_n(t), e_j\big\rangle dt\\ &= -\big\langle v_n(t), Ae_j\big\rangle dt - Q(t)\big\langle \big(v_n(t)\cdot\nabla\big)v_n(t), e_j\big\rangle dt\\ &= \mu_j g_{jn}(t)\,dt - Q(t)\big\langle \big(v_n(t)\cdot\nabla\big)v_n(t), e_j\big\rangle dt \end{aligned} \tag{3.14}$$

for all $t > 0$. Consider

$$\begin{aligned} \big(v_n(t)\cdot\nabla\big)v_n(t) &= \left\{\sum_{i=1}^{n} g_{in}(t)(e_i\cdot\nabla)\right\}\left(\sum_{k=1}^{n} g_{kn}(t)e_k\right)\\ &= \sum_{i,k=1}^{n} g_{in}(t)g_{kn}(t)(e_i\cdot\nabla)(e_k). \end{aligned}$$

Hence

$$\begin{aligned} \big\langle\big(v_n(t)\cdot\nabla\big)v_n(t), e_j\big\rangle &= \sum_{i,k=1}^{n} g_{in}(t)g_{kn}(t)\big\langle(e_i\cdot\nabla)(e_k), e_j\big\rangle\\ &= \sum_{i,k=1}^{n} g_{in}(t)g_{kn}(t)b(e_i, e_k, e_j), \quad 1 \leqslant j \leqslant n. \end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.14) gives the following random system of ode's for $g_{jn}(t)$, $1 \leqslant j \leqslant n$,

$$\left.\begin{aligned} &dg_{jn}(t)=\mu_j g_{jn}(t)\,dt-Q(t)\sum_{i,k=1}^{n} g_{in}(t)g_{kn}(t)b(e_i,e_k,e_j)\,dt, \quad t>0,\\ &g_{jn}(0)=\langle f,e_j\rangle. \end{aligned}\right\} \tag{3.16}$$

The vector fields in (3.16) are locally Lipschitz and so the system (3.16) of random ode's admits a unique local solution defined on a local time interval $[0,T_0)$, where $T_0$ is possibly random. Hence the system ($3.13^n$) has a unique local solution defined on $[0,T_0)$. Since $Q$ is jointly measurable and $(\mathcal{F}_t)_{t\geqslant 0}$-adapted, then so are the $g_{jn}$'s.

To show that $T_0=\infty$ a.s., we first derive a priori estimates on $v_n$ (or the $g_{jn}$). Suppose $T_0\equiv T_0(\omega)<\infty$ for some $\omega\in\Omega$. Multiply both sides of ($3.13^n$) by $v_n(t)$, integrate over $D$ and use the relation

$$\big\langle B\big(v_n(t)\big),v_n(t)\big\rangle_H=0,\quad n\geqslant 1,\ t\in[0,T_0),$$

to obtain

$$d\big|v_n(t)\big|_H^2=-2\big\langle Av_n(t),v_n(t)\big\rangle_H\,dt,\quad t\in[0,T_0).$$

Therefore,

$$d\big|v_n(t)\big|^2+2\nu\int_D\big|\nabla v_n(t)\big|^2\,dt=0,\quad 0<t<T_0.$$

Hence,

$$\big|v_n(t)\big|^2-\big|v_n(0)\big|_H^2+2\nu\int_0^t\big\|v_n(s)\big\|_V^2\,ds=0,\quad 0<t<T_0;$$

and so

$$\big|v_n(t)\big|_H^2+2\nu\int_0^t\big\|v_n(s)\big\|_V^2\,ds=|f_n|_H^2\leqslant|f|_H^2,$$

for all $n\geqslant 1$ and all $t\in(0,T_0)$. In particular,

$$\sup_{0\leqslant t<T_0}\big|v_n(t)\big|_H^2\leqslant|f|_H^2 \tag{3.17}$$

and

$$\int_0^t\big\|v_n(s)\big\|_V^2\,ds\leqslant\frac{1}{2\nu}|f|_H^2 \tag{3.18}$$

for all $t\in[0,T_0)$ and all $n\geqslant 1$.

Use the estimate

$$\left|B(u_1,u_2)\right|_{V'} \leqslant 2\|u_1\|_V^{1/2} \cdot |u_1|_H^{1/2} \cdot \|u_2\|_V^{1/2} \cdot |u_2|_H^{1/2}, \quad u_1,u_2 \in V, \tag{3.19}$$

to get

$$\left|B(u)\right|_{V'} \leqslant 2\|u\|_V \cdot |u|_H, \quad u \in V. \tag{3.20}$$

We now view ($3.13^n$) as an integral equation in $V'$:

$$v_n(t) = f_n - \int_0^t Av_n(s)\,ds - \int_0^t Q(s)B\big(v_n(s)\big)\,ds, \quad 0 \leqslant t < T_0. \tag{3.21}$$

Consider

$$\begin{aligned} \left|Q(s)B\big(v_n(s)\big)\right|_{V'} &\leqslant 2\|Q\|_\infty \cdot \left|v_n(s)\right|_H \cdot \left\|v_n(s)\right\|_V \\ &\leqslant 2\|Q\|_\infty \cdot |f|_H \cdot \left\|v_n(s)\right\|_V, \quad 0 \leqslant s < T_0. \end{aligned} \tag{3.22}$$

In order to show that

$$\lim_{t\to T_0-} v_n(t) = f_n - \int_0^{T_0} Av_n(s)\,ds - \int_0^{T_0} Q(s)B\big(v_n(s)\big)\,ds, \tag{3.23}$$

it is sufficient to prove that the map

$$[0,T_0) \ni t \mapsto \theta(t) := \int_0^t Q(s)B\big(v_n(s)\big)\,ds \in V' \tag{3.24}$$

is uniformly continuous. To do this, let $0 \leqslant t_1 < t_2 < T_0$. Using (3.22), Hölder's inequality and (3.18), we get

$$\begin{aligned} \left|\theta(t_2) - \theta(t_1)\right|_{V'} &= \left|\int_{t_1}^{t_2} Q(s)B\big(v_n(s)\big)\,ds\right|_{V'} \\ &\leqslant \int_{t_1}^{t_2} \left|Q(s)B\big(v_n(s)\big)\right|_{V'}\,ds \\ &\leqslant 2\|Q\|_\infty \cdot |f|_H \cdot \int_{t_1}^{t_2} \left\|v_n(s)\right\|_V\,ds \end{aligned}$$

$$\leqslant 2\|Q\|_\infty \cdot |f|_H \cdot \left\{ \int_{t_1}^{t_2} \|v_n(s)\|_V^2 \, ds \right\}^{1/2} \cdot (t_2 - t_1)^{1/2}$$
$$\leqslant 2\|Q\|_\infty \cdot |f|_H \cdot \frac{1}{\sqrt{2\nu}} \cdot |f|_H \cdot (t_2 - t_1)^{1/2}$$
$$= \sqrt{\frac{2}{\nu}} |f|_H^2 \cdot \|Q\|_\infty (t_2 - t_1)^{1/2}. \tag{3.25}$$

The above inequality implies that the map (3.24) is uniformly $\frac{1}{2}$-Hölder continuous on $[0, T_0)$. Since $v_n(s) \in H_n$, the $n$-dimensional linear span of $\{e_i\}_{i=1}^n$, and $H_n$ is invariant under $A$, then it follows from (3.17) that

$$\begin{aligned} \big|Av_n(s)\big|_{V'} &\leqslant \big|Av_n(s)\big|_H \leqslant \|A|H_n\|_{L(H_n)} \cdot \big|v_n(s)\big|_H \\ &\leqslant \|A|H_n\|_{L(H_n)} \cdot |f|_H \end{aligned} \tag{3.26}$$

for all $s \in [0, T_0)$ and all $n \geqslant 1$. In the above inequality, $A|H_n \in L(H_n)$ is the restriction of $A$ to $H_n$. Hence (3.23) holds for all $n \geqslant 1$.

Define

$$v_n(T_0) := \lim_{t \to T_{0-}} v_n(t) = f_n - \int_0^{T_0} Av_n(s)\, ds - \int_0^{T_0} Q(s)B\big(v_n(s)\big)\, ds, \quad n \geqslant 1.$$

By local existence, we get a local solution $v_n : [T_0, T_0 + \epsilon) \times \Omega \to \mathbf{R}^2$ of the NSE

$$\begin{aligned} dv_n(t) &= -Av_n(t)\, dt - Q(t)B\big(v_n(t)\big)\, dt, \quad T_0 < t < T_0 + \epsilon, \\ v_n(t)|_{t=T_0} &= v_n(T_0) \in H, \end{aligned}$$

for some $\epsilon > 0$. This contradicts the maximality of $T_0$. Hence $T_0 = \infty$ a.s.

We next show that the sequence $\{v_n\}_{n=1}^\infty$ converges to a weak solution $v$ of the random NSE (3.13).

As before, we view Eq. $(3.13^n)$ as an equation in $V'$:

$$\frac{dv^n(t)}{dt} = -Av^n(t) - Q(t)B\big(v^n(t)\big).$$

Therefore, using (2.9), (3.17), (3.18) and the fact that $A|V : V \to V'$ is continuous linear, we have

$$\begin{aligned} \int_0^T \left\| \frac{dv^n(t)}{dt} \right\|_{V'}^2 dt &\leqslant 2\int_0^T \|Av^n(t)\|_{V'}^2\, dt + 2\|Q\|_\infty \int_0^T \big\|B\big(v^n(t)\big)\big\|_{V'}^2\, dt \\ &\leqslant C\int_0^T \|v^n(t)\|_V^2\, dt + C\int_0^T \|v^n(t)\|_V^2 |v^n(t)|_H^2\, dt \end{aligned}$$

$$\leqslant \frac{C}{2\nu}|f|_H^2\big(1+|f|_H^2\big), \tag{3.27}$$

where $C$ is a positive random constant independent of $f$. Since the embedding $V \hookrightarrow H$ is compact, by the proof of Theorem 2.1 in [19, pp. 111–113], it follows from (3.17), (3.18) and (3.27) that there exists a subsequence $v^{n_k}(t)$, $k \geqslant 1$, and $v(t)$ such that $v^{n_k}(\cdot) \to v(\cdot)$ in the weak star topology of $C([0,T],H)$, $v^{n_k}(\cdot) \to v(\cdot)$ weakly in $L^2([0,T],V)$ and moreover $v^{n_k}(\cdot) \to v(\cdot)$ strongly in $L^2([0,T],H)$ as $k \to \infty$. For $w \in H_m$, if $n_k \geqslant m$ we have

$$\begin{aligned}
\big\langle v^{n_k}(t), w\big\rangle &= \big\langle v^{n_k}(0), w\big\rangle - \int_0^t \big\langle A v^{n_k}(s), w\big\rangle\, ds + \int_0^t Q(s) b\big(v^{n_k}(s), w, v^{n_k}(s)\big)\, ds \\
&= \big\langle v^{n_k}(0), w\big\rangle - \int_0^t \big\langle A v^{n_k}(s), w\big\rangle\, ds \\
&\quad + \sum_{i,j=1}^{2} \int_0^t \int_D Q(s) v_i^{n_k}(s)(\partial_i w_j) v_j^{n_k}(s)\, dx\, ds
\end{aligned} \tag{3.28}$$

for all $t \in [0,T]$. Letting $k \to \infty$ in the above relation, we obtain

$$\begin{aligned}
\big\langle v(t), w\big\rangle &= \langle f, w\rangle - \int_0^t \big\langle A v(s), w\big\rangle\, ds + \sum_{i,j=1}^{2} \int_0^t \int_D Q(s) v_i(s)(\partial_i w_j) v_j(s)\, dx\, ds \\
&= (f, w) - \int_0^t \big\langle A v(s), w\big\rangle\, ds - \int_0^t Q(s)\big\langle B\big(v(s)\big), w\big\rangle\, ds
\end{aligned} \tag{3.29}$$

for all $t \in [0,T]$ and all $m \geqslant 1$. Since $m$ is arbitrary and $\bigcup_{m=1}^{\infty} H_m$ is dense in $V$, it follows that $v$ is a solution to Eq. (3.6). Uniqueness follows by setting $f = g$ in (3.12). Since the Galerkin approximations $v_n$ are jointly measurable and $(\mathcal{F}_t)_{t\geqslant 0}$-adapted, it follows that the limiting process $v$ must have the same measurability properties. □

Our next result addresses the issue of local compactness of the solution map

$$H \ni f \mapsto v(t, f, \omega) \in H$$

for $t > 0$, $\omega \in \Omega$.

**Proposition 3.3.** *For $t > 0$ and $\omega \in \Omega$, the solution map $H \ni f \mapsto v(t, f, \omega) \in H$ of* (3.6) *sends bounded sets into relatively compact sets in $H$.*

**Proof.** Fix $\omega \in \Omega$ throughout this proof.

First, we show that the map $[0,T] \ni t \mapsto \sqrt{t}\, v(t, f, \omega) \in V$ is $L^\infty$, and provide a bound for $v$ in $L^\infty([0,T],V)$. Let $f \in V$. Then by an argument similar to the proof of Theorem 3.10

in [18, p. 314], one can show that $v(\cdot, f, \omega) \in L^2([0,T], H^2(D)) \cap L^\infty([0,T], V)$ and the following energy equation holds:

$$\frac{d}{dt}\big\|v(t,f,\omega)\big\|_V^2 + 2\big|Av(t,f,\omega)\big|_H^2 = -Q(t)\big\langle B\big(v(t,f,\omega)\big), Av(t,f,\omega)\big\rangle \tag{3.30}$$

for all $t \in (0,T)$. Consequently,

$$\begin{aligned}
&\frac{d}{dt}\big(t\big\|v(t,f,\omega)\big\|_V^2\big) \\
&\quad = -2t\big|Av(t,f,\omega)\big|_H^2 - tQ(t)\big\langle B\big(v(t,f,\omega)\big), Av(t,f,\omega)\big\rangle + \big\|v(t,f,\omega)\big\|_V^2 \\
&\quad \leqslant -2t\big|Av(t,f,\omega)\big|_H^2 + ctQ(t)\big|v(t,f,\omega)\big|_H^{\frac{1}{2}}\big\|v(t,f,\omega)\big\|_V\big|Av(t,f,\omega)\big|_H^{\frac{3}{2}} + \big\|v(t,f,\omega)\big\|_V^2 \\
&\quad \leqslant ctQ(t)^4\big|v(t,f,\omega)\big|_H^2\big\|v(t,f,\omega)\big\|_V^4 + \big\|v(t,f,\omega)\big\|_V^2, \quad t \in (0,T],
\end{aligned} \tag{3.31}$$

where the following Young's inequality "with $\epsilon$":

$$ab \leqslant \frac{a^4}{4\epsilon^4} + \frac{3\epsilon^{4/3}}{4}b^{4/3}, \quad a,b \geqslant 0,$$

and the fact that

$$\big|B(u)\big|_H \leqslant c|u|_H^{\frac{1}{2}}\|u\||Au|_H^{\frac{1}{2}}, \quad u \in V,$$

have been used. By Gronwall's inequality it follows from (3.31) that

$$\begin{aligned}
&\big\|\sqrt{\cdot}\,v(\cdot,f,\omega)\big\|_{L^\infty([0,T],V)}^2 \\
&\quad \leqslant \left(\int_0^T \big\|v(t,f,\omega)\big\|_V^2\,dt\right)\exp\left(c\|Q\|_\infty^4\int_0^T \big|v(t,f,\omega)\big|_H^2\big\|v(t,f,\omega)\big\|_V^2\,dt\right) \\
&\quad \leqslant \frac{1}{2\nu}|f|_H^2\exp\left(c\|Q\|_\infty^4\frac{1}{2\nu}|f|_H^4\right).
\end{aligned} \tag{3.32}$$

Since the right side of (3.32) depends only on the $H$-norm of $f$, by a limiting procedure it is easy to see that (3.32) also holds for all $f \in H$.

Fix $t > 0$, $f \in H$. Then, for $0 < \delta < t$,

$$v(t,f,\omega) = T_\delta\big[v(t-\delta,f,\omega)\big] + \int_{t-\delta}^{t} T_{t-s}B\big(v(s,f,\omega)\big)\,ds. \tag{3.33}$$

Suppose $\delta_k \searrow 0$, $0 < \delta_k < t$, and let $\{f_n\}_{n=1}^\infty \subset H$ be a bounded sequence; i.e., there exists $M > 0$ such that $|f_n| \leqslant M$ for all $n \geqslant 1$.

**Claim.** *There exists a subsequence* $\{\tilde{f}_n\}_{n=1}^{\infty}$ *of* $\{f_n\}_{n=1}^{\infty}$ *such that for each* $k \geqslant 1$, *the sequence* $\{T_{\delta_k}[v(t-\delta_k, \tilde{f}_n, \omega)]\}_{n=1}^{\infty}$ *converges in* $H$.

**Proof.** We use a diagonalization argument. The set $\{v(t-\delta_1, f_n, \omega)\colon n \geqslant 1\}$ is bounded in $H$ because $|v(t-\delta_1, f_n, \omega)|_H \leqslant |f_n|_H \leqslant M$ for all $n \geqslant 1$. So by compactness of $T_{\delta_1} : H \to H$, the sequence $\{T_{\delta_1}[v(t-\delta_1, f_n, \omega)]\}_{n=1}^{\infty}$ has a convergent subsequence. Therefore, there is a subsequence $\{f_n^1\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that the sequence $\{T_{\delta_1}[v(t-\delta_1, f_n^1, \omega)]\}_{n=1}^{\infty}$ converges. Similarly, by compactness of the map

$$H \ni f \mapsto T_{\delta_2}\big[v(t-\delta_2, f, \omega)\big] \in H$$

there is a subsequence $\{f_n^2\}_{n=1}^{\infty} \subset \{f_n^1\}_{n=1}^{\infty}$ such that $\{T_{\delta_2}[v(t-\delta_2, f_n^2, \omega)]\}_{n=1}^{\infty}$ converges in $H$. By induction, there are subsequences $\{f_n^k\}_{n=1}^{\infty}$, $k \geqslant 1$, such that $\{T_{\delta_k}[v(t-\delta_k, f_n^k, \omega)]\}_{n=1}^{\infty}$ converges and $\{f_n^{k+1}\}_{n=1}^{\infty} \subset \{f_n^k\}_{n=1}^{\infty}$ for each $k \geqslant 1$. Let $\tilde{f}_n := f_n^n$, $n \geqslant 1$, be the diagonal subsequence of $\{f_n\}_{n=1}^{\infty}$. Then the sequence $\{T_{\delta_k}[v(t-\delta_k, \tilde{f}_n, \omega)]\}_{n=1}^{\infty}$ converges in $H$ for each $k \geqslant 1$. This proves the claim. □

We will now show that the map $H \ni f \mapsto v(t, f, \omega) \in H$ is compact i.e., takes bounded sets in $H$ into relatively compact sets. It is sufficient to show that for the bounded sequence $\{f_n\}_{n=1}^{\infty} \subset H$ there exists a subsequence $\{\tilde{f}_n\}_{n=1}^{\infty}$ such that $\{v(t, \tilde{f}_n, \omega)\}_{n=1}^{\infty}$ converges. Pick the subsequence $\{\tilde{f}_n\} \subset \{f_n\}$ as in the claim with each sequence $\{T_{\delta_k}[v(t-\delta_k, \tilde{f}_n, \omega)]\}_{n=1}^{\infty}$ convergent. Consider

$$\begin{aligned}
\big|v(t, \tilde{f}_n, \omega) - v(t, \tilde{f}_m, \omega)\big|_H &\leqslant \big|T_{\delta_k}\big[v(t-\delta_k, \tilde{f}_n, \omega)\big] - T_{\delta_k}\big[v(t-\delta_k, \tilde{f}_m, \omega)\big]\big|_H \\
&\quad + \int_{t-\delta_k}^{t} \big|T_{t-s}B\big(v(s, \tilde{f}_n, \omega)\big) - T_{t-s}B\big(v(s, \tilde{f}_m, \omega)\big)\big|_H\, ds \\
&\leqslant \big|T_{\delta_k}\big[v(t-\delta_k, \tilde{f}_n, \omega)\big] - T_{\delta_k}\big[v(t-\delta_k, \tilde{f}_m, \omega)\big]\big|_H \\
&\quad + 2C_M \frac{1}{\sqrt{t-\delta_k}}\sqrt{\delta_k},
\end{aligned} \tag{3.34}$$

for all $k \geqslant 1$ and all $m, n \geqslant 1$, where the following estimate has been used

$$\begin{aligned}
&\big|T_{t-s}B\big(v(s, \tilde{f}_n, \omega)\big) - T_{t-s}B\big(v(s, \tilde{f}_m, \omega)\big)\big|_H \\
&\quad \leqslant C\frac{1}{\sqrt{t-s}}\big[\big\|B\big(v(s, \tilde{f}_n, \omega)\big)\big\|_{V'} + \big\|B\big(v(s, \tilde{f}_m, \omega)\big)\big\|_{V'}\big] \\
&\quad \leqslant C\frac{1}{\sqrt{t-s}}\big[\big|v(s, \tilde{f}_n, \omega)\big|_H \big\|v(s, \tilde{f}_n, \omega)\big\|_{V'} + \big|v(s, \tilde{f}_m, \omega)\big|_H \big\|v(s, \tilde{f}_m, \omega)\big\|_{V'}\big] \\
&\quad \leqslant C\frac{1}{\sqrt{t-s}\sqrt{s}}\Big[\sup_{0\leqslant s\leqslant T}\big|v(s, \tilde{f}_n, \omega)\big|_H \sup_{0\leqslant s\leqslant T}\big\|\sqrt{s}v(s, \tilde{f}_n, \omega)\big\|_{V'} \\
&\qquad + \sup_{0\leqslant s\leqslant T}\big|v(s, \tilde{f}_m, \omega)\big|_H \sup_{0\leqslant s\leqslant T}\big\|\sqrt{s}v(s, \tilde{f}_m, \omega)\big\|_{V'}\Big]
\end{aligned}$$

$$\leqslant C_M(\omega)\frac{1}{\sqrt{t-s}\sqrt{s}},$$

because of (3.32).

For fixed $k \geqslant 1$, the claim implies

$$\limsup_{m,n\to\infty}\left|T_{\delta_k}\left[v(t-\delta_k,\tilde{f}_n,\omega)\right]-T_{\delta_k}\left[v(t-\delta_k,\tilde{f}_m,\omega)\right]\right|_H=0.$$

Take $\limsup_{m,n\to\infty}$ in (3.34) (with $k \geqslant 1$ fixed) and use the above relation to get

$$\limsup_{m,n\to\infty}\left|v(t,\tilde{f}_n,\omega)-v(t,\tilde{f}_m,\omega)\right|_H \leqslant 2C_M\frac{\sqrt{\delta_k}}{\sqrt{t-\delta_k}}, \tag{3.35}$$

for all $k \geqslant 1$.

Now let $k \to \infty$ in (3.35) to obtain

$$\limsup_{m,n\to\infty}\left|v(t,\tilde{f}_n,\omega)-v(t,\tilde{f}_m,\omega)\right|_H \leqslant 2C_M\lim_{k\to\infty}\frac{\sqrt{\delta_k}}{\sqrt{t-\delta_k}}=0.$$

Therefore, $\{v(t,\tilde{f}_n,\omega)\}_{n=1}^{\infty}$ is a Cauchy sequence and hence converges in $H$. This proves compactness of the map $H \ni f \mapsto v(t,f,\omega) \in H$. $\square$

**Theorem 3.1.** *The solution map*

$$H \ni f \to v(t,f,\omega) \in H$$

*is $C^{1,1}$ for $\omega \in \Omega$ and all $t \geqslant 0$, and has bounded Fréchet derivatives on bounded sets in $H$. Furthermore, the Fréchet derivative $Dv(t,f,\omega)(\cdot): H \to H$ is compact for any $t > 0$, $f \in H$ and $\omega \in \Omega$.*

**Proof.** Let $f, g \in H$. Fix and suppress $\omega \in \Omega$ in this proof. Consider the following random integral equation

$$\begin{aligned} z(t,f)(g) &= g-\int_0^t Az(s,f)(g)\,ds-\int_0^t Q(s)\big(z(s,f)(g)\cdot\nabla\big)v(s,f)\,ds \\ &\quad -\int_0^t Q(s)\big(v(s,f)\cdot\nabla\big)z(s,f)(g)\,ds, \quad t\in[0,T]. \end{aligned} \tag{3.36}$$

The existence and uniqueness of the solution $z(t,f)(g)$ of (3.36) can be proved similarly as for Eq. (3.6), using Galerkin approximations (cf. proof of Proposition 3.2). Furthermore, uniqueness of the solution to (3.36) implies that the solution $z(t,f)(g)$ is linear in $g$.

We will now derive some useful estimates for the solution $z(t,f)(g)$ of (3.36). Using the chain rule in (3.36), we obtain

$$
\begin{aligned}
\big|z(t,f)(g)\big|^2 &= |g|_H^2 - 2\nu\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds \\
&\quad - 2\int_0^t Q(s)b\big(z(s,f)(g), v(s,f), z(s,f)(g)\big)\,ds \\
&\quad - 2\int_0^t Q(s)b\big(v(s,f), z(s,f)(g), z(s,f)(g)\big)\,ds \\
&\leqslant |g|_H^2 - 2\nu\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds \\
&\quad + c\|Q\|_\infty \int_0^t \big|z(s,f)(g)\big|_H \big\|v(s,f)\big\|_V \big\|z(s,f)(g)\big\|_V\,ds \\
&\leqslant |g|_H^2 - 2\nu\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds + c\|Q\|_\infty\epsilon\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds \\
&\quad + c\|Q\|_\infty\epsilon^{-1}\int_0^t \big|z(s,f)(g)\big|_H^2 \big\|v(s,f)\big\|_V^2\,ds \\
&= |g|_H^2 - \nu\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds + c\|Q\|_\infty\epsilon^{-1}\int_0^t \big\|v(s,f)\big\|_V^2 \big|z(s,f)(g)\big|_H^2\,ds \\
&\quad + \big(c\|Q\|_\infty\epsilon - \nu\big)\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds, \quad t\in[0,T], \qquad (3.37)
\end{aligned}
$$

where we have used the following "Young inequality with $\epsilon$":

$$ab \leqslant \epsilon a^2 + \epsilon^{-1}b^2, \quad a,b>0,$$

for any $\epsilon > 0$.

Now in (3.37), choose $\epsilon$ sufficiently small (and random) such that

$$c\|Q\|_\infty\epsilon < \nu.$$

So (3.37) implies

$$\big\|z(t,f)(g)\big\|_H^2 \leqslant |g|_H^2 - \nu\int_0^t \big\|z(s,f)(g)\big\|_V^2\,ds + \tilde{c}\|Q\|_\infty \int_0^t \big\|v(s,f)\big\|_V^2 \big|z(s,f)(g)\big|_H^2\,ds$$

for all $t \in [0,T]$.

By Gronwall's lemma (Lemma 5.1), the above inequality gives

$$\sup_{0\leqslant t\leqslant T}\big|z(t,f)(g)\big|_H^2+\nu\int_0^T\big\|z(s,f)(g)\big\|_V^2\,ds\leqslant|g|_H^2\exp\left(c\|Q\|_\infty^2\int_0^T\big\|v(s,f)\big\|_V^2\,ds\right)$$
$$\leqslant|g|_H^2\exp\left(c\|Q\|_\infty^2\frac{1}{2\nu}|f|_H^2\right),\tag{3.38}$$

where (3.8) has been used for the last inequality. Since $z(t,f)(g)$ is linear in $g$, (3.38) implies that $z(t,f)(\cdot)\in L(H)$ for each $t\in[0,T]$, and $z(\cdot,f)(\cdot)\in L(H,L^2([0,T],V))$. Furthermore,

$$\sup_{0\leqslant t\leqslant T}\big\|z(t,f)\big\|_{L(H)}\leqslant\exp\left(\frac{1}{2}\tilde{c}\|Q\|_\infty^2\frac{1}{2\nu}|f|_H^2\right).\tag{3.39}$$

Next we will show that the map

$$H\ni f\to v(t,f,\omega)\in H$$

has a continuous Fréchet derivative given by $Dv(t,f,\omega)=z(t,f,\omega)(\cdot)$. To this end, it suffices to prove that

$$\lim_{h\to0}\sup_{|g|_H\leqslant1}\left|\frac{v(t,f+hg,\omega)-v(t,f,\omega)}{h}-z(t,f)(g)\right|_H=0\tag{3.40}$$

and the map

$$H\ni f\to z(t,f,\omega)\in L(H)$$

is continuous. First, we prove

$$\lim_{h\to0}\sup_{|g|_H\leqslant1}\left\{\sup_{0\leqslant t\leqslant T}\big|v(t,f+hg)-v(t,f)\big|_H^2+\nu\int_0^T\big\|v(s,f+hg)-v(s,f)\big\|_V^2\,ds\right\}=0.\tag{3.41}$$

Using the equations satisfied by $v(t,f)$ and $v(t,f+hg)$, the chain rule and "Young's inequality with $\epsilon$", it follows that

$$\big|v(t,f+hg)-v(t,f)\big|_H^2$$
$$=h^2|g|_H^2-2\nu\int_0^t\big\|v(t,f+hg)-v(t,f)\big\|_V^2\,ds$$

$$
\begin{aligned}
&-2\int_0^t Q(s)b\big(v(s,f+hg)-v(s,f),v(s,f),v(s,f+hg)-v(s,f)\big)\,ds\\
&\leqslant h^2|g|_H^2-2\nu\int_0^t \big\|v(t,f+hg)-v(t,f)\big\|_V^2\,ds\\
&\quad+c\|Q\|_\infty\int_0^t \big|v(s,f+hg)-v(s,f)\big|_H\big\|v(s,f)\big\|_V\big\|v(s,f+hg)-v(s,f)\big\|_V\,ds\\
&\leqslant h^2|g|_H^2-\nu\int_0^t \big\|v(t,f+hg)-v(t,f)\big\|_V^2\,ds\\
&\quad+c\|Q\|_\infty^2\int_0^t \big|v(s,f+hg)-v(s,f)\big|_H^2\big\|v(s,f)\big\|_V^2\,ds
\end{aligned}
\tag{3.42}
$$

for all $t\in[0,T]$. By Gronwall's inequality, we deduce that

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\big|v(t,f+hg)-v(t,f)\big|_H^2+\nu\int_0^T \big\|v(s,f+hg)-v(s,f)\big\|_V^2\,ds\\
&\quad\leqslant h^2|g|_H^2\exp\left(c\|Q\|_\infty^2\int_0^t \big\|v(s,f)\big\|_V^2\,ds\right)
\end{aligned}
\tag{3.43}
$$

for all $f,g\in H$ and $h\in\mathbf{R}$. This immediately implies (3.41). Set

$$
\begin{aligned}
U(t,f,g,h)&=\frac{v(t,f+hg,\omega)-v(t,f,\omega)}{h},\\
X(t,f,g,h)&=U(t,f,g,h)-z(t,f)(g),
\end{aligned}
$$

for $t\in[0,T]$ and $h\in\mathbf{R}\backslash\{0\}$. Then,

$$
\begin{aligned}
X(t,f,g,h)=&-\int_0^t AX(s,f,g,h)\,ds-\int_0^t Q(s)\big(v(s,f)\cdot\nabla\big)X(s,f,g,h)\,ds\\
&-\int_0^t Q(s)\big(X(s,f,g,h)\cdot\nabla\big)v(s,f+hg)\,ds\\
&+\int_0^t Q(s)\big(z(s,f)(g)\cdot\nabla\big)\big(v(s,f)-v(s,f+hg)\big)\,ds,
\end{aligned}
\tag{3.44}
$$

for all $t \in [0, T]$. By the chain rule,

$$\begin{aligned}\left|X(t,f,g,h)\right|_H^2 = &-2\nu \int_0^t \left\|X(s,f,g,h)\right\|_V^2 ds\\ &- 2\int_0^t Q(s) b\big(X(s,f,g,h), v(s,f+hg), X(s,f,g,h)\big)\, ds\\ &+ 2\int_0^t Q(s) b\big(z(s,f)(g), v(s,f) - v(s,f+hg), X(s,f,g,h)\big)\, ds,\end{aligned} \tag{3.45}$$

where $b(u,v,v) = 0$ has been used. Hence,

$$\begin{aligned}\left|X(t,f,g,h)\right|_H^2 \leqslant &-2\nu \int_0^t \left\|X(s,f,g,h)\right\|_V^2 ds\\ &+ c\int_0^t Q(s)\left|X(s,f,g,h)\right|_H \left\|v(s,f+hg)\right\|_V \left\|X(s,f,g,h)\right\|_V ds\\ &+ c\int_0^t Q(s)\left\|z(s,f)(g)\right\|_V^{\frac{1}{2}} \left|z(s,f)(g)\right|_H^{\frac{1}{2}} \left\|v(s,f+hg) - v(s,f)\right\|_V^{\frac{1}{2}}\\ &\times \left|v(s,f+hg) - v(s,f)\right|_H^{\frac{1}{2}} \left\|X(s,f,g,h)\right\|_V ds,\end{aligned} \tag{3.46}$$

for $t \in [0, T]$. We next prove the following estimate

$$\begin{aligned}\left|X(t,f,g,h)\right|_H^2 \leqslant &-\nu \int_0^t \left\|X(s,f,g,h)\right\|_V^2 ds + c\int_0^t Q(s)\left|X(s,f,g,h)\right|_H^2 \left\|v(s,f+hg)\right\|_V^2 ds\\ &+ c\int_0^t Q(s)\left\|z(s,f)(g)\right\|_V^2 \left|v(s,f+hg) - v(s,f)\right|_H^2 ds\\ &+ c\int_0^t Q(s)\left|z(s,f)(g)\right|_H^2 \left\|v(s,f+hg) - v(s,f)\right\|_V^2 ds,\end{aligned} \tag{3.47}$$

for $t \in [0, T]$. Use (3.46) and "Young's inequality with $\epsilon$" to see that

$$
\begin{aligned}
\left|X(t,f,g,h)\right|_H^2 \leqslant{} & -2\nu\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds \\
& + c\int_0^t Q(s)\left|X(s,f,g,h)\right|_H \left\|v(s,f+hg)\right\|_V \left\|X(s,f,g,h)\right\|_V\,ds \\
& + c\int_0^t Q(s)\left\|z(s,f)(g)\right\|_V^{1/2}\left|z(s,f)(g)\right|_H^{1/2}\left\|v(s,f+hg)-v(s,f)\right\|_V^{1/2} \\
& \times \left|v(s,f+hg)-v(s,f)\right|_H^{1/2}\left\|X(s,f,g,h)\right\|_V\,ds \\
\leqslant{} & -2\nu\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds \\
& + c\epsilon^{-1}\int_0^t Q(s)\left|X(s,f,g,h)\right|_H^2\left\|v(s,f+hg)\right\|_V^2\,ds \\
& + \epsilon c\|Q\|_\infty\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds + \epsilon c\|Q\|_\infty\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds \\
& + \epsilon^{-1}c\int_0^t Q(s)\left\|z(s,f)(g)\right\|_V\left|z(s,f)(g)\right|_H\left\|v(s,f+hg)-v(s,f)\right\|_V \\
& \times \left|v(s,f+hg)-v(s,f)\right|_H\,ds \\
\leqslant{} & -2\nu\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds \\
& + c\epsilon^{-1}\int_0^t Q(s)\left|X(s,f,g,h)\right|_H^2\left\|v(s,f+hg)\right\|_V^2\,ds \\
& + \epsilon c\|Q\|_\infty\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds + \epsilon c\|Q\|_\infty\int_0^t \left\|X(s,f,g,h)\right\|_V^2\,ds \\
& + \frac{1}{2}\epsilon^{-1}c\int_0^t Q(s)\left\|z(s,f)(g)\right\|_V^2\left|v(s,f+hg)-v(s,f)\right|_H^2\,ds \\
& + \frac{1}{2}\epsilon^{-1}c\int_0^t Q(s)\left|z(s,f)(g)\right|_H^2\left\|v(s,f+hg)-v(s,f)\right\|_V^2\,ds, \qquad (3.48)
\end{aligned}
$$

for $t \in [0, T]$. Now choose $\epsilon$ small enough such that $2\epsilon c\|Q\|_\infty < \nu$. This gives

$$\begin{aligned} \left|X(t, f, g, h)\right|_H^2 \leqslant{}& -\nu \int_0^t \left\|X(s, f, g, h)\right\|_V^2 ds \\ &+ c\epsilon^{-1} \int_0^t Q(s)\left|X(s, f, g, h)\right|_H^2 \left\|v(s, f + hg)\right\|_V^2 ds \\ &+ \frac{1}{2}\epsilon^{-1} c \int_0^t Q(s)\left\|z(s, f)(g)\right\|_V^2 \left|v(s, f + hg) - v(s, f)\right|_H^2 ds \\ &+ \frac{1}{2}\epsilon^{-1} c \int_0^t Q(s)\left|z(s, f)(g)\right|_H^2 \left\|v(s, f + hg) - v(s, f)\right\|_V^2 ds \end{aligned} \tag{3.49}$$

for all $t \in [0, T]$, which implies (3.47).

By (3.47) and Gronwall's inequality (Lemma 5.1), it follows that

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} \left|X(t, f, g, h)\right|_H^2 + \nu \int_0^T \left\|X(s, f, g, h)\right\|_V^2 ds \\ &\quad\leqslant c\Bigg[ \|Q\|_\infty \sup_{0\leqslant t\leqslant T} \left|v(t, f + hg) - v(t, f)\right|_H^2 \int_0^T \left\|z(s, f)(g)\right\|_V^2 ds \\ &\qquad + \|Q\|_\infty \sup_{0\leqslant t\leqslant T} \left|z(t, f)(g)\right|_H^2 \int_0^T \left\|v(t, f + hg) - v(s, f)\right\|_V^2 ds \Bigg] \\ &\qquad \times \exp\left( c\|Q\|_\infty \int_0^T \left\|v(s, f + hg)\right\|_V^2 ds \right) \end{aligned} \tag{3.50}$$

for all $f, g \in H$ and $h \in \mathbf{R}\backslash\{0\}$. By virtue of (3.41) and (3.38), (3.50) implies that

$$\lim_{h\to 0} \sup_{|g|_H \leqslant 1} \left\{ \sup_{0\leqslant t\leqslant T} \left|X(t, f, g, h)\right|_H^2 + \nu \int_0^T \left\|X(s, f, g, h)\right\|_V^2 ds \right\} = 0. \tag{3.51}$$

The equality (3.40) follows immediately from the above relation.

To complete the proof that the map $H \ni f \mapsto v(t, f) \in H$ is $C^{1,1}$ (Fréchet), observe first that the Gateaux derivative $z(t, f) \in L(H)$. So for the map $H \ni f \mapsto v(t, f) \in H$ to be Fréchet continuously differentiable, it is sufficient to prove that the map $H \ni f \mapsto z(t, f) \in L(H)$ is Lipschitz continuous on bounded sets.

In what follows, let $g, f_1, f_2 \in H$ be such that $|f_i|_H \leqslant M$, $i = 1, 2$, $|g|_H \leqslant 1$ and $t \in [0, T]$. From (3.36), we obtain

$$\begin{aligned}
&z(t, f_1)(g) - z(t, f_2)(g) \\
&\quad = -\int_0^t A\big[z(s, f_1)(g) - z(s, f_2)(g)\big]\,ds - \int_0^t Q(s)\big(z(s, f_1)(g)\cdot\nabla\big)v(s, f_1)\,ds \\
&\qquad + \int_0^t Q(s)\big(z(s, f_2)(g)\cdot\nabla\big)v(s, f_2)\,ds - \int_0^t Q(s)\big(v(s, f_1)\cdot\nabla\big)z(s, f_1)(g)\,ds \\
&\qquad + \int_0^t Q(s)\big(v(s, f_2)\cdot\nabla\big)z(s, f_2)(g)\,ds \\
&\quad = -\int_0^t A\big[z(s, f_1)(g) - z(s, f_2)(g)\big]\,ds \\
&\qquad - \int_0^t Q(s)\big\{\big[z(s, f_1)(g) - z(s, f_2)(g)\big]\cdot\nabla\big\}v(s, f_1)\,ds \\
&\qquad - \int_0^t Q(s)\big[z(s, f_2)(g)\cdot\nabla\big]\big\{v(s, f_1) - v(s, f_2)\big\}\,ds \\
&\qquad - \int_0^t Q(s)\big(v(s, f_1)\cdot\nabla\big)\big[z(s, f_1)(g) - z(s, f_2)(g)\big]\,ds \\
&\qquad - \int_0^t Q(s)\big[\big\{v(s, f_1) - v(s, f_2)\big\}\cdot\nabla\big]z(s, f_2)(g)\,ds.
\end{aligned} \tag{3.52}$$

Differentiating both sides of (3.52) with respect to $t$, taking inner products of the resulting differential equation with $z(t, f_1)(g) - z(t, f_2)(g)$, using the chain rule and integrating over $t$ gives

$$\begin{aligned}
&\big|z(t, f_1)(g) - z(t, f_2)(g)\big|_H^2 \\
&\quad = -2\nu\int_0^t \big\|z(s, f_1)(g) - z(s, f_2)(g)\big\|_V^2\,ds \\
&\qquad - 2\int_0^t Q(s)b\big(z(s, f_1)(g) - z(s, f_2)(g), v(s, f_1), z(s, f_1)(g) - z(s, f_2)(g)\big)\,ds
\end{aligned}$$

$$-2\int_0^t Q(s)b\big(z(s,f_2)(g),v(s,f_1)-v(s,f_2),z(s,f_1)(g)-z(s,f_2)(g)\big)\,ds$$

$$-2\int_0^t Q(s)b\big(v(s,f_1),z(s,f_1)(g)-z(s,f_2)(g),z(s,f_1)(g)-z(s,f_2)(g)\big)\,ds$$

$$-2\int_0^t Q(s)b\big(v(s,f_1)-v(s,f_2),z(s,f_2)(g),z(s,f_1)(g)-z(s,f_2)(g)\big)\,ds$$

$$=-2\nu\int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds+I_1+I_2+I_3, \tag{3.53}$$

where we have used the fact that $b(u,w,w)=0$, and where

$$I_1:=-2\int_0^t Q(s)b\big(z(s,f_1)(g)-z(s,f_2)(g),v(s,f_1),z(s,f_1)(g)-z(s,f_2)(g)\big)\,ds, \tag{3.54}$$

$$I_2:=-2\int_0^t Q(s)b\big(z(s,f_2)(g),v(s,f_1)-v(s,f_2),z(s,f_1)(g)-z(s,f_2)(g)\big)\,ds, \tag{3.55}$$

$$I_3:=-2\int_0^t Q(s)b\big(v(s,f_1)-v(s,f_2),z(s,f_2)(g),z(s,f_1)(g)-z(s,f_2)(g)\big)\,ds. \tag{3.56}$$

We now estimate each of the terms on the right-hand side of (3.54), (3.55) and (3.56). Using "Young's inequality with $\epsilon$" in (3.54), we obtain

$$\begin{aligned}|I_1|:&\leqslant 4\|Q\|_\infty\int_0^t \big\|v(s,f_1)\big\|_V\big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H\big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V\,ds\\ &\leqslant 4\epsilon\|Q\|_\infty\int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds\\ &\quad+4\epsilon^{-1}\|Q\|_\infty\int_0^t \big\|v(s,f_1)\big\|_V^2\big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^2\,ds.\end{aligned} \tag{3.57}$$

From (3.55), (3.38), "Young's inequality with $\epsilon$", Hölder's inequality and (3.43), it follows that

$$
\begin{aligned}
|I_2| :\leqslant{}& 4\|Q\|_\infty \int_0^t \big\|z(s,f_2)(g)\big\|_V^{1/2}\big|z(s,f_2)(g)\big|_H^{1/2}\big\|v(s,f_1)-v(s,f_2)\big\|_V \\
&\times \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^{1/2}\big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^{1/2}\,ds \\
\leqslant{}& 2c\|Q\|_\infty \int_0^t \big|z(s,f_2)(g)\big|_H\big\|v(s,f_1)-v(s,f_2)\big\|_V^2\,ds \\
&+ 2c\|Q\|_\infty \int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V\big\|z(s,f_2)(g)\big\|_V \\
&\times \big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H\,ds \\
\leqslant{}& 2c\|Q\|_\infty \sup_{0\leqslant s\leqslant T}\big|z(s,f_2)(g)\big|_H\cdot\left\{\int_0^t \big\|v(s,f_1)-v(s,f_2)\big\|_V^2\,ds\right\} \\
&+ 2c\|Q\|_\infty\epsilon \int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds \\
&+ 2c\epsilon^{-1}\|Q\|_\infty \int_0^t \big\|z(s,f_2)(g)\big\|_V^2\big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^2\,ds \\
\leqslant{}& 2c\|Q\|_\infty|g|_H^2\exp\left\{c\|Q\|_\infty^2\frac{1}{2\nu}|f_2|_H^2\right\}\frac{1}{\sqrt{\nu}}|f_1-f_2|_H\exp\left\{\frac{1}{\nu}\|Q\|_\infty^2 c\right\} \\
&+ 2c\|Q\|_\infty\epsilon \int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds \\
&+ 2c\epsilon^{-1}\|Q\|_\infty \int_0^t \big\|z(s,f_2)(g)\big\|_V^2\big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^2\,ds. 
\end{aligned}
\tag{3.58}
$$

In (3.56), we use Hölder inequality and "Young's inequality with $\epsilon$", together with (3.38), (3.43) and (3.8), to get the following estimates

$$
\begin{aligned}
|I_3| \leqslant{}& 4\|Q\|_\infty \int_0^t \big\|v(s,f_1)-v(s,f_2)\big\|_V^{1/2}\big|v(s,f_1)-v(s,f_2)\big|_H^{1/2} \\
&\times \big\|z(s,f_2)(g)\big\|_V\big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^{1/2}\big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^{1/2}\,ds \\
\leqslant{}& 2c\|Q\|_\infty \int_0^t \big\|v(s,f_1)-v(s,f_2)\big\|_V\big|v(s,f_1)-v(s,f_2)\big|_H\big\|z(s,f_2)(g)\big\|_V\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+2\|Q\|_\infty \int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V \\
&\quad\times \big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H \big\|z(s,f_2)(g)\big\|_V\,ds \\
&\leqslant 2c\|Q\|_\infty \left\{\int_0^t \big\|z(s,f_2)(g)\big\|_V^2\,ds\right\}^{1/2} \cdot \left\{\int_0^t \big\|v(s,f_1)-v(s,f_2)\big\|_V^2\,ds\right\}^{1/2} \\
&\quad\times \sup_{0\leqslant s\leqslant T} \big|v(s,f_1)-v(s,f_2)\big|_H + 2\epsilon\|Q\|_\infty \int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds \\
&\quad+2\epsilon^{-1}\|Q\|_\infty \int_0^t \big\|z(s,f_2)(g)\big\|_V^2 \big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^2\,ds \\
&\leqslant 2c\|Q\|_\infty |g|_H \exp\left\{c\|Q\|_\infty^2 \frac{1}{2\nu}|f_2|_H^2\right\} \frac{1}{\sqrt{\nu}}|f_1-f_2|_H^2 \exp\left\{\frac{1}{\nu}c\|Q\|_\infty^2\right\} \\
&\quad+2\epsilon\|Q\|_\infty \int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds \\
&\quad+2\epsilon^{-1}\|Q\|_\infty \int_0^t \big\|z(s,f_2)(g)\big\|_V^2 \big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^2\,ds. 
\end{aligned}
\tag{3.59}
$$

Choose $\epsilon > 0$ sufficiently small such that

$$
6\epsilon\|Q\|_\infty + 2c\epsilon\|Q\|_\infty < \nu. \tag{3.60}
$$

Using (3.57), (3.58), (3.59), and (3.60) in (3.53), we get

$$
\begin{aligned}
&\big|z(t,f_1)(g)-z(t,f_2)(g)\big|_H^2 + \nu\int_0^t \big\|z(s,f_1)(g)-z(s,f_2)(g)\big\|_V^2\,ds \\
&\quad\leqslant c|f_1-f_2|_H^2 + c\int_0^t \big[\big\|v(s,f_1)\big\|_V^2 + \big\|z(s,f_2)(g)\big\|_V^2\big] \\
&\qquad\times \big|z(s,f_1)(g)-z(s,f_2)(g)\big|_H^2\,ds,
\end{aligned}
\tag{3.61}
$$

for all $f_1, f_2, g \in H$ such that $|f_i|_H \leqslant M$, $i = 1, 2$, $|g|_H \leqslant 1$, with $c$ a random constant (dependent on $M$, $\nu$ and $T$). Applying Gronwall's lemma (Lemma 5.1) to (3.61) and using (3.8) and (3.38), we get

$$
\begin{aligned}
&\bigl|z(t,f_1)(g)-z(t,f_2)(g)\bigr|_H^2+\nu\int_0^t \bigl\|z(s,f_1)(g)-z(s,f_2)(g)\bigr\|_V^2\,ds \\
&\leqslant c|f_1-f_2|_H^2\exp\left\{\int_0^t\bigl[\bigl\|v(s,f_1)\bigr\|_V^2+\bigl\|z(s,f_2)(g)\bigr\|_V^2\bigr]\,ds\right\} \\
&\leqslant c|f_1-f_2|_H^2\exp\left[\frac{1}{2\nu}\bigl(|f_1|_H^2+|f_2|_H^2\bigr)\times\frac{|g|_H^2}{\nu}\exp\bigl\{c\|Q\|_\infty^2|f_2|_H^2\bigr\}\right] \\
&\leqslant c|f_1-f_2|_H^2\exp\left[\frac{1}{2\nu}\bigl(2M^2\bigr)\cdot\frac{1}{\nu}\exp\bigl\{c\|Q\|_\infty^2M^2\bigr\}\right] \\
&\leqslant c|f_1-f_2|_H^2.
\end{aligned}
\tag{3.62}
$$

Therefore,

$$
\bigl\|z(t,f_1)-z(t,f_2)\bigr\|_{L(H)}\leqslant c|f_1-f_2|_H^{1/2}
\tag{3.63}
$$

for all $f_1, f_2 \in H$ with $|f_1|_H \leqslant M$, $|f_2|_H \leqslant M$ and all $t \in [0,T]$. This proves that the map $H \ni f \mapsto v(t,f,\omega) \in H$ is $C^1$ for each $\omega \in \Omega$ and each $t \in [0,T]$. Furthermore, its Fréchet derivative $H \ni f \mapsto Dv(t,f,\omega) = z(t,f,\omega) \in L(H)$ is Lipschitz continuous on bounded sets in $H$.

The compactness of the Fréchet derivative $Dv(t,f,\omega)$; $H \to H$, $t > 0$, follows immediately from the fact that the map $H \ni f \mapsto v(t,f,\omega) \in H$, $t > 0$, is $C^1$ and carries bounded sets into relatively compact ones (Proposition 3.3). See the proof of Theorem 3.1 and Lemma 3.1 in [13, Part I]. This completes the proof of Theorem 3.1. □

We are now ready to state the main result in this section.

**Theorem 3.2** *(The cocycle). Let $u(t,f,\cdot)$ be the unique global solution of the stochastic Navier–Stokes equation* (3.1) *for $t \geqslant 0$ and $f \in H$. Denote by $\theta : \mathbf{R}^+ \times \Omega \to \Omega$ the standard Brownian shift*

$$
\theta(t,\omega)(s) := \omega(t+s)-\omega(t), \quad t,s \geqslant 0,\ \omega \in \Omega,
\tag{3.64}
$$

*on Wiener space $(\Omega,\mathcal{F},P)$. Then there is a version $u : \mathbf{R}^+ \times H \times \Omega \to H$ of the solution of* (3.1) *with the following properties*:

- (i) *The map $u : \mathbf{R}^+ \times H \times \Omega \to H$ is jointly measurable, and for each $f \in H$, the process $u(\cdot,f,\cdot) : \mathbf{R}^+ \times \Omega \to H$ is $(\mathcal{F}_t)_{t\geqslant 0}$-adapted.*
- (ii) *For each $t > 0$ and $\omega \in \Omega$, the map $u(t,\cdot,\omega) : H \to H$ takes bounded sets into relatively compact sets.*
- (iii) *$(u,\theta)$ is a $C^{1,1}$ perfect cocycle*; *viz.*

$$
u\bigl(t_2,u(t_1,f,\omega),\theta(t_1,\omega)\bigr) = u(t_1+t_2,f,\omega)
\tag{3.65}
$$

*for all $t_1,t_2 \geqslant 0$, $f \in H$, $\omega \in \Omega$.*

(iv) *For each* $(t, f, \omega) \in \mathbf{R}^+ \times H \times \Omega$, *the Fréchet derivative* $Du(t, f, \omega) \in L(H)$ *of the map* $u(t, \cdot, \omega)$ *is compact linear, and the map*

$$\mathbf{R}^+ \times H \times \Omega \to L(H),$$
$$(t, f, \omega) \mapsto Du(t, f, \omega)$$

*is strongly measurable.*

(v) *For fixed* $\rho, a > 0$,

$$E \log^+ \sup_{\substack{0 \leqslant t_1, t_2 \leqslant a \\ |f|_H \leqslant \rho}} \left\{ \left| u\big(t_2, f, \theta(t_1, \cdot)\big)\right|_H + \left\| Du\big(t_2, f, \theta(t_1, \cdot)\big)\right\|_{L(H)} \right\} < \infty. \tag{3.66}$$

**Proof.** To prove assertion (i) of the theorem, define the required version $u : \mathbf{R}^+ \times H \times \Omega \to H$ by setting

$$u(t, f, \omega) := Q(t, \omega)v(t, f, \omega), \quad t \geqslant 0, \ \omega \in \Omega, \ f \in H. \tag{3.67}$$

Note first that $Q$ is jointly measurable and $(\mathcal{F}_t)_{t \geqslant 0}$-adapted. In view of Proposition 3.2, it follows from (3.67) that $u$ satisfies assertion (i).

Assertion (ii) of the theorem follows immediately from (3.67) and Proposition 3.3.

Next, we establish the perfect cocycle property (iii). To see this, observe that $Q$ has the cocycle property

$$Q(t_1 + t_2, \omega) = Q\big(t_2, \theta(t_1, \omega)\big)Q(t_1, \omega), \quad t_1, t_2 \geqslant 0, \ \omega \in \Omega. \tag{3.68}$$

This follows directly from (3.4). Thus, (3.65) will follow if we prove the following identity

$$v(t_1 + t_2, f, \omega) = Q(t_1, \omega)^{-1} v\big(t_2, Q(t_1, \omega)v(t_1, f, \omega), \theta(t_1, \omega)\big) \tag{3.69}$$

for $t_1, t_2 \geqslant 0$, $\omega \in \Omega$, $f \in H$. Indeed, assume that (3.69) holds. Fix $\omega \in \Omega$ and $t_1 \geqslant 0$ throughout this proof. Then, for $t \geqslant 0$, we have

$$\begin{aligned} u\big(t, u(t_1, f, \omega), \theta(t_1, \omega)\big) &= Q\big(t, \theta(t_1, \omega)\big) v\big(t, Q(t_1, \omega)v(t_1, f, \omega), \theta(t_1, \omega)\big) \\ &= Q(t_1 + t, \omega)Q(t_1, \omega)^{-1} v\big(t, Q(t_1, \omega)v(t_1, f, \omega), \theta(t_1, \omega)\big) \\ &= Q(t_1 + t, \omega)v(t_1 + t, f, \omega) \\ &= u(t_1 + t, f, \omega). \end{aligned}$$

Hence the perfect cocycle property (3.65) holds.

We now show (3.69). To do this, define the processes

$$\left. \begin{aligned} z(t, \omega) &:= Q(t_1, \omega)^{-1} v\big(t, Q(t_1, \omega)v(t_1, f, \omega), \theta(t_1, \omega)\big), \\ \bar{z}(t, \omega) &:= v(t + t_1, f, \omega), \end{aligned} \right\} \tag{3.70}$$

for all $t \geqslant 0$ and all $\omega \in \Omega$. Shifting the time-variable $t$ by $t_1$ in the integral equation for $v$, it is easy to see that $\bar{z}(t,\omega)$ satisfies the following equation

$$\bar{z}(t,\omega) = v(t_1, f, \omega) - \int_0^t A\bar{z}(s,\omega)\,ds - \int_0^t Q(t_1+s,\omega)B\big(\bar{z}(s,\omega)\big)\,ds \tag{3.71}$$

for all $t \geqslant 0$, $\omega \in \Omega$. On the other hand,

$$\begin{aligned} &v\big(t, Q(t_1,\omega)v(t_1,f,\omega), \theta(t_1,\omega)\big) \\ &\quad = Q(t_1,\omega)v(t_1,f,\omega) - \int_0^t Av\big(s, Q(t_1,\omega)v(t_1,f,\omega), \theta(t_1,\omega)\big)\,ds \\ &\qquad - \int_0^t Q\big(s,\theta(t_1,\omega)\big)B\big(v\big(s, Q(t_1,\omega)v(t_1,f,\omega), \theta(t_1,\omega)\big)\big)\,ds \end{aligned} \tag{3.72}$$

for all $t \geqslant 0$, $\omega \in \Omega$. Multiplying the above equation by $Q(t_1,\omega)^{-1}$ and using the bilinear property of $B$, we easily see that

$$z(t,\omega) = v(t_1, f, \omega) - \int_0^t Az(s,\omega)\,ds - \int_0^t Q(t_1+s,\omega)B\big(z(s,\omega)\big)\,ds \tag{3.73}$$

for all $t \geqslant 0$, $\omega \in \Omega$. Subtract $\bar{z}(t,\omega)$ from $z(t,\omega)$ and use a similar calculation as for (3.12) to deduce that $\bar{z}(t,\omega) = z(t,\omega)$ for all $t \geqslant 0$ and $\omega \in \Omega$. This proves the identity (3.69), and so the cocycle property (iii) holds for the solution map $u : \mathbf{R}^+ \times H \times \Omega \to H$ of the SNSE (3.1).

Assertion (iv) of the theorem is a consequence of Theorem 3.1, Proposition 3.2 and relation (3.67). (The strong measurability of $Du(t, f, \omega)$ follows from Eq. (3.40).)

Let us now prove the integrability estimate in assertion (v) of the theorem.

Let $0 \leqslant t_1, t_2 \leqslant a$ and $f \in H$ with $|f|_H \leqslant \rho$. It follows from (3.7) that

$$\begin{aligned} \big|u\big(t_2, f, \theta(t_1,\omega)\big)\big|_H &= Q\big(t_2, \theta(t_1,\omega)\big)\big|v\big(t_2, f, \theta(t_1,\omega)\big)\big|_H \\ &\leqslant Q\big(t_2,\theta(t_1,\omega)\big)|f|_H = Q(t_1+t_2,\omega)Q^{-1}(t_1,\omega)|f|_H \\ &\leqslant \rho\|Q\|_\infty\|Q^{-1}\|_\infty, \end{aligned} \tag{3.74}$$

where $\|Q^{-1}\|_\infty := \sup_{0\leqslant t\leqslant 2a}\|Q^{-1}(t)\|$. Using (3.39) we have

$$\begin{aligned} \big\|Du\big(t_2, f, \theta(t_1,\omega)\big)\big\|_{L(H)} &= Q\big(t_2,\theta(t_1,\omega)\big)\big\|Dv\big(t_2, f, \theta(t_1,\omega)\big)\big\|_{L(H)} \\ &\leqslant \|Q\|_\infty\big\|Q^{-1}\big\|_\infty \exp\left\{c\|Q\|_\infty^2\big\|Q^{-1}\big\|_\infty^2\frac{1}{2\nu}|f|_H^2\right\} \\ &\leqslant \|Q\|_\infty\big\|Q^{-1}\big\|_\infty \exp\left\{c\|Q\|_\infty^2\big\|Q^{-1}\big\|_\infty^2\frac{1}{2\nu}\rho^2\right\}. \end{aligned} \tag{3.75}$$

Combining (3.74) and (3.75), we get

$$\begin{aligned} &E\log^+ \sup_{\substack{0\leqslant t_1,t_2\leqslant a\\ |f|_H\leqslant\rho}} \big|u\big(t_2,f,\theta(t_1,\cdot)\big)\big|_H + E\log^+ \sup_{\substack{0\leqslant t_1,t_2\leqslant a\\ |f|_H\leqslant\rho}} \big\|Du\big(t_2,f,\theta(t_1,\cdot)\big)\big\|_{L(H)} \\ &\quad\leqslant E\bigg[2\log^+\big(\|Q\|_\infty\|Q^{-1}\|_\infty\big) + c\|Q\|_\infty^2\|Q^{-1}\|_\infty^2\frac{1}{2\nu}\rho^2\bigg] + \log^+\rho < \infty. \end{aligned} \tag{3.76}$$

The above relation implies the integrability condition (3.66). The proof of Theorem 3.2 is now complete. □

It is easy to see from (3.76) and (3.63) that the following stronger estimate holds

$$E\log^+ \sup_{0\leqslant t_1,t_2\leqslant a} \big\|u\big(t_2,\cdot,\theta(t_1,\cdot)\big)\big\|_{C^{1,1}} < \infty,$$

where $\|\cdot\|_{C^{1,1}}$ denotes the $C^{1,1}$ norm on the ball $B(0,\rho)$ in $H$.

## 4. The multiplicative ergodic theory

Our objective in this section is to characterize the local behavior of solutions of the SNSE (3.1) near an *equilibrium* or a *stationary point/solution*.

We next describe the concepts of equilibrium or a stationary point for the SNSE (3.1).

**Definition 4.1** *(Equilibrium/stationary point).* An $\mathcal{F}$-measurable random variable $Y:\Omega\to H$ is said be an **equilibrium** or a **stationary random point** for the cocycle $(u,\theta)$ if

$$u\big(t,Y(\omega),\omega\big) = Y\big(\theta(t,\omega)\big) \tag{4.1}$$

perfectly in $\omega\in\Omega$ for all $t\in\mathbf{R}^+$.

A trivial equilibrium or stationary solution of the SNSE (3.1) is $u(t,0,\omega)\equiv 0$ corresponding to the zero initial function $f\equiv 0\in H$.

### *4.1. Dynamics near a general equilibrium*

In order to analyze the dynamics of the SNSE (3.1) near a general equilibrium or stationary point $Y:\Omega\to H$, we linearize the $C^{1,1}$ cocycle $u:\mathbf{R}^+\times H\times\Omega\to H$ at $Y$. This gives a linear cocycle of Fréchet derivatives $Du(t,Y(\omega),\omega)\in L(H)$ satisfying the following random equations

$$Du(t,Y) = Q(t,\cdot)Dv(t,Y),\qquad t\geqslant 0, \tag{4.2}$$

and

$$Dv(t,Y)(g) = g - \int_0^t ADv(s,Y)(g)\,ds - \int_0^t Q(s)\big(Dv(s,Y)(g)\cdot\nabla\big)v(s,Y)\,ds$$
$$- \int_0^t Q(s)\big(v(s,f)\cdot\nabla\big)Dv(s,Y)(g)\,ds, \quad t \geqslant 0,\ g \in H. \tag{4.3}$$

We next apply the Oseledec–Ruelle spectral theorem to the compact linear cocycle $(Du(t,Y(\omega),\omega),\theta(t,\omega))$, $t \geqslant 0$, $\omega \in \Omega$ ([16, Theorem 2.1.1], [14]). This gives

**Theorem 4.1** *(The Lyapunov spectrum: General equilibrium). Let $(u(t,\cdot,\omega),\theta(t,\omega))$ be the $C^{1,1}$ cocycle on $H$ generated by the stochastic Navier–Stokes equation* (3.1)*. Suppose that $Y:\Omega\to H$ is a stationary random point for the cocycle $(u,\theta)$ of the SNSE* (3.1) *with $E\log^+|Y|<\infty$. Then the following limit*

$$\Lambda(\omega) := \lim_{t\to\infty}\big\{\big[Du\big(t,Y(\omega),\omega\big)\big]^* \circ \big[Du\big(t,Y(\omega),\omega\big)\big]\big\}^{1/2t} \tag{4.4}$$

*exists in the uniform operator norm in $L(H)$, perfectly in ω. The Oseledec operator $\Lambda(\omega)$ in* (4.4) *is compact, self-adjoint and non-negative with discrete non-random spectrum*

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_n} > \cdots. \tag{4.5}$$

*The Lyapunov exponents $\{\lambda_n\}_{n=1}^\infty$ correspond to values of the limit*

$$\lim_{t\to\infty}\frac{1}{t}\log\big|Du\big(t,Y(\omega),\omega\big)(g)\big|_H \in \{\lambda_n\}_{n=1}^\infty$$

*for any $g\in H$, perfectly in ω. Each eigenvalue $e^{\lambda_j}$ has a fixed finite multiplicity $m_j$ with a corresponding finite-dimensional eigenspace $F_j(\omega)$ such that $m_j := \dim F_j(\omega)$, $j\geqslant 1$, $\omega\in\Omega$. If we set*

$$E_1(\omega) := H, \qquad E_n(\omega) := \left[\bigoplus_{j=1}^{n-1} F_j(\omega)\right]^\perp, \quad n > 1,$$

*then for each $n \geqslant 1$, $\operatorname{codim} E_n(\omega) = \sum_{j=1}^{n-1} m_j < \infty$, and the following assertions are true*:

$$E_n(\omega) \subset E_{n-1}(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = H, \quad n > 1;$$
$$\lim_{t\to\infty}\frac{1}{t}\log\big|Du\big(t,Y(\omega),\omega\big)(g)\big|_H = \lambda_n \tag{4.6}$$

*for $g \in E_n(\omega)\backslash E_{n+1}(\omega)$;*

$$\lim_{t\to\infty}\frac{1}{t}\log\big\|Du\big(t,Y(\omega),\omega\big)\big\|_{L(H)} = \lambda_1; \tag{4.7}$$

*and*

$$Du\big(t, Y(\omega), \omega\big)\big(E_n(\omega)\big) \subseteq E_n\big(\theta(t, \omega)\big) \tag{4.8}$$

*for all* $t \geqslant 0$*, perfectly in* $\omega \in \Omega$*, for all* $n \geqslant 1$.

**Proof.** Recall the Oseledec integrability condition

$$E \log^+ \sup_{0 \leqslant t_1, t_2 \leqslant a} \big\| Du\big(t_2, Y\big(\theta(t_1, \cdot)\big), \theta(t_1, \cdot)\big)\big\|_{L(H)} < \infty \tag{4.9}$$

for any $0 < a < \infty$, which follows directly from (3.66) in Theorem 3.2. Using the above integrability condition and the Ruelle–Oseledec theorem [14, Theorem 2.1.1], there is a random family of compact self-adjoint positive operators $\Lambda(\omega) \in L(H)$, defined perfectly in $\omega$, and satisfies

$$\Lambda(\omega) := \lim_{t \to \infty} \big\{\big[Du\big(t, Y(\omega), \omega\big)\big]^* \circ \big[Du\big(t, Y(\omega), \omega\big)\big]\big\}^{1/2t}. \tag{4.10}$$

The above almost sure limit exists in the uniform operator norm in $L(H)$, perfectly in $\omega$. The operator $\Lambda(\omega)$ has a discrete non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_n} > \cdots \tag{4.11}$$

due to the ergodicity of the Brownian shift $\theta$.

The assertions (4.7) and (4.8) of the theorem follow from the Oseledec–Ruelle spectral theorem [14, Theorem 2.1.1]. □

**Remark.** If the cocycle is linearized at the zero equilibrium $Y \equiv 0$, the Oseledec–Ruelle operator $\Lambda(\omega)$ is non-random. Consequently, the Oseledec spaces $\{E_n\colon n \geqslant 1\}$ are also non-random. This will be shown later in the section.

**Definition 4.2** *(Hyperbolicity).* A stationary point $Y : \Omega \to H$ for the SNSE (3.1) is *hyperbolic* if the linearized cocycle $(Du(t, Y(\omega), \omega), \theta(t, \omega))$ has a non-vanishing Lyapunov spectrum: $\lambda_i \neq 0$ for all $i \geqslant 1$.

Theorem 4.2 below is a consequence of the nonlinear multiplicative ergodic theorem [14, Theorem 2.2.1]. It describes the saddle-point behavior of the random flow of the SNSE (3.1) in the neighborhood of any equilibrium.

For any $\rho > 0$ and any $f \in H$, we will denote by $\bar{B}(f, \rho)$ the closed ball in $H$ center $f$ and radius $\rho$.

**Theorem 4.2** *(The local stable manifold theorem: General equilibrium). Assume that* $Y : \Omega \to H$ *is a hyperbolic stationary random point for the cocycle* $(u, \theta)$ *of the SNSE* (3.1) *with* $E \log^+ |Y| < \infty$*. Denote by* $\{\cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$ *the Lyapunov spectrum of the linearized cocycle* $(Du(t, Y(\omega), \omega), \theta(t, \omega), t \geqslant 0)$ *as given in Theorem* 4.1*. Define* $i_0 := \min\{i\colon \lambda_i < 0\}$.

*Fix* $\epsilon_1 \in (0, -\lambda_{i_0})$ *and* $\epsilon_2 \in (0, \lambda_{i_0 - 1})$*. Then there exist*

(i) *a sure event* $\Omega^* \in \mathcal{F}$ *with* $\theta(t, \cdot)(\Omega^*) = \Omega^*$ *for all* $t \in \mathbf{R}$;

(ii) *$\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true*:
*There are $C^{1,1}$ submanifolds $\mathcal{S}(\omega)$, $\mathcal{U}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties*:

(a) *For $\lambda_{i_0} > -\infty$, $\mathcal{S}(\omega)$ is the set of all $f \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that*

$$\left|u(n, f, \omega) - Y\big(\theta(n, \omega)\big)\right|_H \leqslant \beta_1(\omega) \exp\big\{(\lambda_{i_0} + \epsilon_1)n\big\}$$

*for all integers $n \geqslant 0$. If $\lambda_{i_0} = -\infty$, then $\mathcal{S}(\omega)$ is the set of all $f \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that*

$$\left|u(n, f, \omega) - Y\big(\theta(n, \omega)\big)\right|_H \leqslant \beta_1(\omega) e^{\lambda n}$$

*for all integers $n \geqslant 0$ and any $\lambda \in (-\infty, 0)$. Furthermore,*

$$\limsup_{t \to \infty} \frac{1}{t} \log \left|u(t, f, \omega) - Y\big(\theta(t, \omega)\big)\right|_H \leqslant \lambda_{i_0} \tag{4.12}$$

*for all $f \in \mathcal{S}(\omega)$. The stable subspace $\mathcal{S}^0(\omega)$ of the linearized cocycle $(Du(t, Y(\omega), \cdot), \theta(t, \cdot))$ is tangent at $Y(\omega)$ to the submanifold $\mathcal{S}(\omega)$, viz. $T_{Y(\omega)}\mathcal{S}(\omega) = \mathcal{S}^0(\omega)$. In particular,* $\operatorname{codim} \mathcal{S}(\omega) = \operatorname{codim} \mathcal{S}^0(\omega) = \sum_{j=1}^{i_0 - 1} \dim F_j(\omega)$ *is fixed and finite.*

(b) $$\limsup_{t \to \infty} \frac{1}{t} \log\left[\sup\left\{\frac{|u(t, f_1, \omega) - u(t, f_2, \omega)|_H}{|f_1 - f_2|_H}\colon\ f_1 \neq f_2,\ f_1, f_2 \in \mathcal{S}(\omega)\right\}\right] \leqslant \lambda_{i_0}.$$

(c) *(Cocycle-invariance of the stable manifolds)*:
*There exists $\tau_1(\omega) \geqslant 0$ such that*

$$u(t, \cdot, \omega)\big(\mathcal{S}(\omega)\big) \subseteq \mathcal{S}\big(\theta(t, \omega)\big) \tag{4.13}$$

*for all $t \geqslant \tau_1(\omega)$. Also*

$$Du\big(t, Y(\omega), \omega\big)\big(\mathcal{S}^0(\omega)\big) \subseteq \mathcal{S}^0\big(\theta(t, \omega)\big), \quad t \geqslant 0. \tag{4.14}$$

(d) *$\mathcal{U}(\omega)$ is the set of all $f \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that there is a discrete-time "history" process $y(\cdot, \omega) : \{-n\colon n \geqslant 0\} \to H$ such that $y(0, \omega) = f$ and for each integer $n \geqslant 1$, one has $u(1, y(-n, \omega), \theta(-n, \omega)) = y(-(n-1), \omega)$ and*

$$\left|y(-n, \omega) - Y\big(\theta(-n, \omega)\big)\right|_H \leqslant \beta_2(\omega) \exp\big\{-(\lambda_{i_0 - 1} - \epsilon_2)n\big\}.$$

*If $\lambda_{i_0 - 1} = \infty$, $\mathcal{U}(\omega)$ is the set of all $f \in \bar{B}(Y(\omega), \rho_2(\omega))$ with the property that there is a discrete-time "history" process $y(\cdot, \omega) : \{-n\colon n \geqslant 0\} \to H$ such that $y(0, \omega) = f$ and for each integer $n \geqslant 1$,*

$$\left|y(-n, \omega) - Y\big(\theta(-n, \omega)\big)\right|_H \leqslant \beta_2(\omega) \exp\{-\lambda n\},$$

*for any $\lambda \in (0, \infty)$. Furthermore, for each $f \in \mathcal{U}(\omega)$, there is a unique continuous-time "history" process also denoted by $y(\cdot, \omega) : (-\infty, 0] \to H$ such that $y(0, \omega) = f$, $u(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ for all $s \leqslant 0$, $0 \leqslant t \leqslant -s$, and*

$$\limsup_{t\to\infty} \frac{1}{t} \log \bigl|y(-t, \omega) - Y\bigl(\theta(-t, \omega)\bigr)\bigr|_H \leqslant -\lambda_{i_0-1}.$$

*Each unstable subspace $\mathcal{U}^0(\omega)$ of the linearized cocycle $(Du(t, Y(\cdot), \cdot), \theta(t, \cdot))$ is tangent at $Y(\omega)$ to $\mathcal{U}(\omega)$, viz. $T_{Y(\omega)}\mathcal{U}(\omega) = \mathcal{U}^0(\omega)$. In particular,*

$$\dim \mathcal{U}(\omega) = \sum_{j=1}^{i_0-1} \dim F_j(\omega)$$

*is finite and non-random.*

(e) *Let $y(\cdot, f_i, \omega)$, $i = 1, 2$, be the history processes associated with $f_i = y(0, f_i, \omega) \in \mathcal{U}(\omega)$, $i = 1, 2$. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{\frac{|y(-t, f_1, \omega) - y(-t, f_2, \omega)|_H}{|f_1 - f_2|_H}:\ f_1 \neq f_2,\ f_i \in \mathcal{U}(\omega),\ i = 1, 2\right\}\right] \leqslant -\lambda_{i_0-1}.$$

(f) *(Cocycle-invariance of the unstable manifolds)*:
*There exists $\tau_2(\omega) \geqslant 0$ such that*

$$\mathcal{U}(\omega) \subseteq u\bigl(t, \cdot, \theta(-t, \omega)\bigr)\bigl(\mathcal{U}\bigl(\theta(-t, \omega)\bigr)\bigr) \tag{4.15}$$

*for all $t \geqslant \tau_2(\omega)$. Also*

$$Du\bigl(t, \cdot, \theta(-t, \omega)\bigr)\bigl(\mathcal{U}^0\bigl(\theta(-t, \omega)\bigr)\bigr) = \mathcal{U}^0(\omega), \quad t \geqslant 0;$$

*and the restriction*

$$Du\bigl(t, \cdot, \theta(-t, \omega)\bigr)\big|\mathcal{U}^0\bigl(\theta(-t, \omega)\bigr) : \mathcal{U}^0\bigl(\theta(-t, \omega)\bigr) \to \mathcal{U}^0(\omega), \quad t \geqslant 0,$$

*is a linear homeomorphism onto.*

(g) *The submanifolds $\mathcal{U}(\omega)$ and $\mathcal{S}(\omega)$ are transversal, viz.*

$$H = T_{Y(\omega)}\mathcal{U}(\omega) \oplus T_{Y(\omega)}\mathcal{S}(\omega).$$

We will only give an outline of the proof of Theorem 4.2. Full details of the proof may be obtained by adapting the arguments in [13,14].

**An outline of the proof of Theorem 4.2.**

- Develop perfect continuous-time versions of Kingman's subadditive ergodic theorem as well as the ergodic theorem [14, Lemma 2.3.1(ii), (iii)]. The linearized cocycle $(Du(t, Y(\omega), \cdot), \theta(t, \cdot))$ at the equilibrium $Y$ can be shown to satisfy the hypotheses of these perfect ergodic theorems. As a consequence of the perfect ergodic theorems, one obtains stable/unstable subspaces for the linearized cocycle, which will constitute tangent spaces to the local stable and unstable manifolds of the nonlinear cocycle $(u, \theta)$.
- We use hyperbolicity of the equilibrium $Y$, the continuous-time integrability condition (3.66) on the cocycle and perfect versions of the ergodic and subadditive ergodic theorems to show the existence of local stable/unstable manifolds for the discrete cocycle $(u(n, \cdot, \omega), \theta(n, \omega))$ near $Y(\omega)$ (cf. [16, Theorems 5.1 and 6.1]). These manifolds are random objects and are perfectly defined for $\omega \in \Omega$. Using interpolation between discrete times and the (continuous-time) integrability condition (3.66), it can be shown that the above manifolds for the discrete-time cocycle $(u(n, \cdot, \omega), \theta(n, \omega))$, $n \geqslant 1$, also serve as perfectly defined local stable/unstable manifolds for the *continuous-time* cocycle $(u(t, \cdot, \omega), \theta(t, \omega))$, $t \geqslant 0$, near the equilibrium $Y$ (see [13,14,16]).
- Again, by using the integrability condition (3.66) on the nonlinear cocycle and its Fréchet derivatives, it is possible to control the excursions of the continuous-time cocycle $(u(t, \cdot, \omega), \theta(t, \omega))$, $t \geqslant 0$, between discrete times. In view of the perfect subadditive ergodic theorem, these estimates show that the local stable manifolds are asymptotically invariant under the nonlinear cocycle. The asymptotic invariance of the unstable manifolds is obtained via the concept of a *stochastic history process* for the cocycle. The existence of a stochastic history process is needed because the (locally compact) cocycle is *not invertible*.

This completes the outline of the proof of Theorem 4.2. □

We next discuss the behavior of the cocycle $(u, \theta)$ near the zero equilibrium.

*4.2. Dynamics near the zero equilibrium*

For the rest of the article, we will focus on the dynamics of the SNSE (3.1) relative to its zero equilibrium $Y \equiv 0$. In this special case, we are able to express the Lyapunov spectrum of the linearized cocycle $(Du(t, 0, \omega), \theta(t, \omega))$ explicitly in terms of the parameters $\nu, \gamma, \sigma_i$, $i \geqslant 1$, in the SNSE (3.1).

Let $\{T_t\}_{t \geqslant 0}$ be the strongly continuous semigroup of the operator $-A = \nu\Delta$ with a Dirichlet boundary condition on $\partial D$. The operator $-A$ has a discrete spectrum of eigenvalues $\{\mu_n\colon n \geqslant 1\}$ and a complete orthonormal system of corresponding eigenfunctions $\{e_n\colon n \geqslant 1\}$. We assume $\mu_1 < \mu_2 < \cdots < \mu_n < \cdots$. Let $F_n^0$ be the finite-dimensional eigenspace of $-A$ corresponding to the eigenvalue $\mu_n$ for $n \geqslant 1$.

Next, we linearize the cocycle $u : \mathbf{R}^+ \times H \times \Omega \to H$ at the zero equilibrium. To do this, put $Y \equiv 0$ in Eq. (4.3) to obtain

$$Dv(t, 0, \omega)(g) = g - \int_0^t ADv(s, 0, \omega)(g)\, ds, \tag{4.16}$$

for all $g \in H$, $t \in [0, T]$ and $\omega \in \Omega$. This implies that $Dv(t, 0, \omega) = T_t$, $t \geqslant 0$, $\omega \in \Omega$, and thus

$$Du(t,0,\omega) = Q(t,\omega)Dv(t,0,\omega) = Q(t,\omega)T_t, \quad t \geqslant 0,\ \omega \in \Omega. \tag{4.17}$$

The next result is a special case ($Y \equiv 0$) of the Oseledec–Ruelle spectral theorem (Theorem 4.1).

**Theorem 4.3** *(The Lyapunov spectrum: Zero equilibrium). Let* $(u(t,\cdot,\omega),\theta(t,\omega))$ *be the* $C^{1,1}$ *cocycle on* $H$ *generated by the stochastic Navier–Stokes equation* (3.1). *Then all the assertions of Theorem* 4.1 *hold perfectly in* $\omega \in \Omega$ *subject to the following*:

(i) $Y \equiv 0$;

(ii) $\lambda_n = \lambda_n^0 := -\mu_n + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2, \quad n \geqslant 1$;

(iii) *The Oseledec–Ruelle operator is deterministic and given by* $\Lambda^0(\omega) = e^{\gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2}T_1$;

(iv) *The Oseledec spaces are deterministic and given by*

$$E_1^0 := H, \qquad E_n^0 := \left[\bigoplus_{j=1}^{n-1} F_j^0\right]^{\perp}, \quad n > 1.$$

(v) *The Lyapunov exponents satisfy the relations*

$$\lim_{t\to\infty}\frac{1}{t}\log\left|Du(t,0,\omega)(g)\right|_H = -\mu_n + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2$$

*for* $g \in E_n^0 \backslash E_{n+1}^0$; *and*

$$\lim_{t\to\infty}\frac{1}{t}\log\left\|Du(t,0,\omega)\right\|_{L(H)} = -\mu_1 + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2.$$

**Proof.** In order to evaluate the Lyapunov spectrum $\{\lambda_n^0\colon\ n \geqslant 1\}$ of the linearized cocycle $(Du(t,0,\omega),\theta(t,\omega))$, we first compute the Oseledec–Ruelle operators $\Lambda^0(\omega)$ associated with this cocycle. To do this, use relation (4.17) to obtain

$$\begin{aligned}
\Lambda^0(\omega) &:= \lim_{t\to\infty}\left\{\left[Du(t,0,\omega)\right]^* \circ \left[Du(t,0,\omega)\right]\right\}^{1/2t} \\
&= \lim_{t\to\infty}\left[\exp\left\{2\gamma t + \sum_{k=1}^{\infty}\left(2\sigma_k W_k(t) - \sigma_k^2 t\right)\right\}\left(T_t^* \circ T_t\right)\right]^{1/2t} \\
&= \lim_{t\to\infty}\exp\left\{\gamma + \sum_{k=1}^{\infty}\left(\sigma_k\frac{W_k(t)}{t} - \frac{\sigma_k^2}{2}\right)\right\}\lim_{t\to\infty}\left(T_t^* \circ T_t\right)^{1/2t} \\
&= \exp\left\{\gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2\right\}\lim_{t\to\infty}\left(T_t^* \circ T_t\right)^{1/2t}.
\end{aligned} \tag{4.18}$$

Now, it is easy to see that

$$\left(T_t^* \circ T_t\right)(e_n) = \exp\{2\mu_n t\} e_n$$

for all $n \geqslant 1$. Therefore,

$$\left(T_t^* \circ T_t\right)^{1/2t} = T_1 \tag{4.19}$$

for all $t > 0$. By (4.18) and (4.19), assertion (iii) of the theorem holds. In particular, the Oseledec–Ruelle operator $\Lambda^0(\omega) = e^{\gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2} T_1$ is *non-random*. Consequently, the Oseledec spaces $\{E_n\colon n \geqslant 1\}$ are also non-random.

Assertions (iv) and (v) of the theorem follow directly from Theorem 4.1. □

The next corollary is an immediate consequence of the above theorem: It gives necessary and sufficient conditions for hyperbolicity of the zero equilibrium $Y \equiv 0$.

**Corollary 4.3.1** *(Hyperbolicity of the zero equilibrium). In the SNSE* (3.1)*, the zero equilibrium is hyperbolic if and only if the following conditions hold*

(i) $-\mu_1 + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 > 0$;
(ii) $-\mu_n + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 \neq 0$ *for all* $n \geqslant 2$.

Theorem 4.4 below is a consequence of Theorem 4.2. It describes saddle-point behavior of the random flow of the SNSE (3.1) in the neighborhood of the zero equilibrium.

**Theorem 4.4** *(The local stable manifold theorem: Zero equilibrium). In the SNSE* (3.1)*, assume that the zero equilibrium is hyperbolic. Then all the assertions of the local stable manifold theorem* (4.2) *hold under the same choice of parameters as in Theorem* 4.3.

Our next result gives sufficient conditions on the parameters of the SNSE (3.1) to guarantee that the zero equilibrium is its only stationary point.

**Theorem 4.5** *(Uniqueness of the stationary solution). Suppose that*

$$\mu_1 + \gamma + \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 < 0. \tag{4.20}$$

*Then the zero equilibrium* $Y \equiv 0$ *is the only equilibrium* (*stationary point*) *of the SNSE* (3.1).

**Proof.** Assume that the SNSE (3.1) admits a non-zero stationary solution $u_0(t)$. By stationarity, $a := E[|u_0(t)|_H^2] > 0$ and $b := E[\|u_0(t)\|_V^2] > 0$ are independent of $t$. Suppose $t > s > 0$. Then from (3.1) and Ito's formula, we have

$$\begin{aligned}\left|u_0(t)\right|_H^2 = \left|u_0(s)\right|_H^2 - 2\nu\int_s^t \left\|u_0(r)\right\|_V^2\,dr + 2\sum_{k=1}^{\infty}\int_s^t \sigma_k\left|u_0(r)\right|_H^2\,dW_k(r)\\ + 2\gamma\int_s^t \left|u_0(r)\right|_H^2\,dr + \sum_{k=1}^{\infty}\int_s^t \sigma_k^2\left|u_0(r)\right|_H^2\,dr.\end{aligned} \tag{4.21}$$

Taking expectations on both sides of the above identity, we obtain

$$a = a - 2\nu(t-s)b + \sum_{k=1}^{\infty}\sigma_k^2(t-s)a + 2\gamma(t-s)a. \tag{4.22}$$

Hence

$$2\nu b = \left[\sum_{k=1}^{\infty}\sigma_k^2 + 2\gamma\right]a. \tag{4.23}$$

Combining the above equality with the Poincare inequality:

$$a \leqslant \frac{\nu b}{-\mu_1}, \tag{4.24}$$

it follows that

$$-\mu_1 \leqslant \gamma + \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2. \tag{4.25}$$

This proves the theorem. □

We conclude this section by stating the *Local and Global Invariant Manifold Theorems* for the SNSE (3.1) when $\gamma = 0$ (Theorems 4.6 and 4.7 below). The *Local Invariant Manifold Theorem* (Theorem 4.6) characterizes the almost sure asymptotic stability of the random flow of the SNSE (3.1) in the neighborhood of the zero equilibrium, in the special case when the linear drift vanishes ($\gamma = 0$). On the other hand, the *Global Invariant Manifold Theorem* (Theorem 4.7) gives a random cocycle-invariant foliation of the energy space $H$. The leaves of the foliation are characterized by the Lyapunov exponents $\{\lambda_i^0 = -\mu_i + \gamma - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2\colon i \geqslant 1\}$ of the linearized cocycle $(Du(t,0,\omega), \theta(t,\omega))$.

**Theorem 4.6** *(Local invariant manifolds). Consider the SNSE* (3.1) *with* $\gamma = 0$. *Fix* $\epsilon_1 \in (0, -\mu_1 + \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2)$. *Then there exist*

(i) *a sure event* $\Omega^* \in \mathcal{F}$ *with* $\theta(t,\cdot)(\Omega^*) = \Omega^*$ *for all* $t \in \mathbf{R}$;
(ii) $\mathcal{F}$*-measurable random variables* $\rho_i, \beta_i : \Omega^* \to (0,1)$, $\beta_i > \rho_i \geqslant \rho_{i+1} > 0$, $i \geqslant 1$, *such that for each* $\omega \in \Omega^*$, *the following is true*:
*There are* $C^{1,1}$ *submanifolds* $\mathcal{S}_i(\omega)$, $i \geqslant 1$, *of* $\bar{B}(0, \rho_i(\omega))$ *with the following properties*:

(a) *$\mathcal{S}_i(\omega)$ is the set of all $f \in \bar{B}(0, \rho_i(\omega))$ such that*

$$\big|u(n, f, \omega)\big|_H \leqslant \beta_i(\omega) \exp\left\{\left(\mu_i - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 + \epsilon_1\right)n\right\}$$

*for all integers $n \geqslant 0$. Furthermore,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \big|u(t, f, \omega)\big|_H \leqslant \mu_i - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 \tag{4.26}$$

*for all $f \in \mathcal{S}_i(\omega)$. Each Oseledec space $E_i^0$ of the linearized cocycle $(Du(t, 0, \cdot), \theta(t, \cdot))$ is tangent at 0 to the submanifold $\mathcal{S}_i(\omega)$, viz. $T_0\mathcal{S}_i(\omega) = E_i^0$. In particular, $\operatorname{codim} \mathcal{S}_i(\omega) = \operatorname{codim} E_i^0 = \sum_{j=1}^{i-1} \dim F_j^0$ (fixed and finite).*

(b)

$$\limsup_{t\to\infty} \frac{1}{t} \log\left[\sup\left\{\frac{|u(t, f_1, \omega) - u(t, f_2, \omega)|_H}{|f_1 - f_2|_H}:\ f_1 \neq f_2,\ f_1, f_2 \in \mathcal{S}_i(\omega)\right\}\right] \leqslant \mu_i - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2.$$

(c) *(Cocycle-invariance)*:
*There exists $\tau_i(\omega) \geqslant 0$ such that*

$$u(t, \cdot, \omega)\big(\mathcal{S}_i(\omega)\big) \subseteq \mathcal{S}_i\big(\theta(t, \omega)\big) \tag{4.27}$$

*for all $t \geqslant \tau_i(\omega)$. Also*

$$Du(t, 0, \omega)\big(E_i^0\big) \subseteq E_i^0, \quad t \geqslant 0. \tag{4.28}$$

**Proof.** Let $\epsilon_1 \in (0, -\mu_1 + \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2)$. Then there exist $\Omega^* \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbf{R}$, and $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$ such that $\beta_i(\omega) > \rho_i(\omega) > 0$, and $C^{1,1}$ local stable submanifolds $S_i(\omega) \subset \bar{B}(0, \rho_i(\omega))$ such that

$$S_i(\omega) := \left\{f \in \bar{B}\big(0, \rho_i(\omega)\big):\ \big|u(n, f, \omega)\big|_H \leqslant \beta_i(\omega) e^{(\mu_i - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 + \epsilon_1)n} \text{ for all } n \geqslant 1\right\}. \tag{4.29}$$

Furthermore,

$$\limsup_{t\to\infty} \frac{1}{t} \log \big|u(t, f, \omega)\big|_H \leqslant \mu_i - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2 \tag{4.30}$$

for all $f \in S_i(\omega)$.

Also, $T_0 S_i(\omega) = E_i^0$, the Oseledec space for the linearized cocycle $(Du(t,0,\omega), \theta(t,\omega))$ corresponding to the Lyapunov exponent $\lambda_i := \mu_i - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k^2$, $i \geqslant 1$. Following the argument in [13,16], the random variables $\rho_i(\omega)$, $\beta_i(\omega)$ may be selected such that

$$\rho_i(\omega)e^{(\lambda_i+\epsilon_1)t} \leqslant \rho_i\big(\theta(t,\omega)\big) \tag{4.31}$$

and

$$\beta_i(\omega)e^{(\lambda_i+\epsilon_1)t} \leqslant \beta_i\big(\theta(t,\omega)\big) \tag{4.32}$$

for all $t \geqslant 0$ and $\omega \in \Omega^*$.

We now show that there exists $\tau_i(\omega) > 0$ such that

$$u(t,\cdot,\omega)\big(S_i(\omega)\big) \subseteq S_i\big(\theta(t,\omega)\big) \tag{4.33}$$

for all $t \geqslant \tau_i(\omega)$ and all $\omega \in \Omega^*$.

Let $f \in S_i(\omega)$, $t \geqslant 0$ and let $n \geqslant 0$ be any integer. Then (by the cocycle property),

$$\big|u\big(n, u(t,f,\omega), \theta(t,\omega)\big)\big|_H = \big|u(n+t,f,\omega)\big|_H. \tag{4.34}$$

From [13,16], we have

$$\limsup_{t\to\infty} \frac{1}{t}\log\Bigg[\sup_{\substack{f\in S_i(\omega)\\ f\neq 0}} \frac{|u(t,f,\omega)|_H}{|f|_H}\Bigg] \leqslant \lambda_i. \tag{4.35}$$

From the above estimate, for any $\epsilon' \in (0,\epsilon_1)$, there exists $N_0 = N_0(\epsilon') > 0$ such that

$$\sup_{t\geqslant N} \frac{1}{t}\log\Bigg[\sup_{\substack{f\in S_i(\omega)\\ f\neq 0}} \frac{|u(t,f,\omega)|_H}{|f|_H}\Bigg] \leqslant \lambda_i + \epsilon',$$

for all $N \geqslant N_0$. Thus

$$\sup_{\substack{f\in S_i(\omega)\\ f\neq 0}} \frac{|u(t,f,\omega)|_H}{|f|_H} \leqslant e^{(\lambda_i+\epsilon')t},$$

for all $t \geqslant N_0$. Define

$$\beta_i^{\epsilon'}(\omega) := \sup_{0\leqslant t\leqslant N_0}\ \sup_{\substack{f\in S_i(\omega)\\ f\neq 0}} \frac{|u(t,f,\omega)|_H}{|f|_H}\cdot e^{-(\lambda_i+\epsilon')N_0}.$$

Therefore,

$$\sup_{\substack{f\in S_i(\omega)\\ f\neq 0}} \frac{|u(t,f,\omega)|_H}{|f|_H} \leqslant \beta_i^{\epsilon'}(\omega)\cdot e^{(\lambda_i+\epsilon')t}$$

for all $t \geqslant 0$. Since $f \in S_i(\omega) \subset B(0,1)$, then $|f|_H \leqslant 1$ and

$$\big|u(t, f, \omega)\big|_H \leqslant \beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t} \tag{4.36}$$

for all $t \geqslant 0$. Therefore, from (4.34) and (4.36),

$$\begin{aligned} \big|u\big(n, u(t, f, \omega), \theta(t, \omega)\big)\big|_H &\leqslant \beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')(n+t)} \\ &\leqslant \beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t} \cdot e^{(\lambda_i+\epsilon_1)n} \end{aligned} \tag{4.37}$$

for $t \geqslant 0$ and all integers $n \geqslant 0$.

Since $\epsilon' < \epsilon_1$, it follows that

$$\lim_{t\to\infty} \frac{\beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t}}{\beta_i(\omega) e^{(\lambda_i+\epsilon_1)t}} = \lim_{t\to\infty} \frac{\beta_i^{\epsilon'}(\omega)}{\beta_i(\omega)} \cdot e^{(\epsilon'-\epsilon_1)t} = 0.$$

Hence there exists $\tilde{\tau}_i(\omega) > 0$ so that

$$\beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t} \leqslant \beta_i(\omega) e^{(\lambda_i+\epsilon_1)t}, \tag{4.38}$$

for all $t \geqslant \tilde{\tau}_i(\omega)$. By (4.37), (4.38) and (4.32), we get

$$\begin{aligned} \big|u\big(n, u(t, f, \omega), \theta(t, \omega)\big)\big|_H &\leqslant \beta_i(\omega) e^{(\lambda_i+\epsilon_1)t} \cdot e^{(\lambda_i+\epsilon_1)n} \\ &\leqslant \beta_i\big(\theta(t,\omega)\big) e^{(\lambda_i+\epsilon_1)n} \end{aligned} \tag{4.39}$$

for all $t \geqslant \tilde{\tau}_i(\omega)$ and all $n \geqslant 1$. Again, because $\epsilon' < \epsilon_1$, we have

$$\lim_{t\to\infty} \frac{\beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t}}{\rho_i(\theta(t,\omega))} \leqslant \lim_{t\to\infty} \frac{\beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t}}{\rho_i(\omega) e^{(\lambda_i+\epsilon_1)t}} = 0. \tag{4.40}$$

Therefore, there exists $\tilde{\tilde{\tau}}_i(\omega) > 0$ such that

$$\beta_i^{\epsilon'}(\omega) e^{(\lambda_i+\epsilon')t} \leqslant \rho_i\big(\theta(t,\omega)\big) \tag{4.41}$$

for all $t \geqslant \tilde{\tilde{\tau}}_i(\omega)$. Hence $u(t, f, \omega) \in \bar{B}(0, \rho_i(\theta(t,\omega)))$ for all $t \geqslant \tilde{\tilde{\tau}}_i(\omega)$.

Set $\tau_i(\omega) := \tilde{\tau}_i(\omega) \vee \tilde{\tilde{\tau}}_i(\omega)$. Then $u(t, f, \omega) \in \bar{B}(0, \rho_i(\theta(t,\omega)))$ and satisfies (4.39) for all $n \geqslant 0$ and all $t \geqslant \tau_i(\omega)$. By definition of $S_i(\theta(t,\omega))$, it follows that $u(t, f, \omega) \in S_i(\theta(t,\omega))$ for all $t \geqslant \tau_i(\omega)$. Thus $u(t, \cdot, \omega)(S_i(\omega)) \subseteq S_i(\theta(t,\omega))$ for all $t \geqslant \tau_i(\omega)$. Note that $\tau_i(\omega)$, $\tilde{\tau}_i(\omega)$, $\tilde{\tilde{\tau}}_i(\omega)$ are all independent $f \in S_i(\omega)$ because $\beta_i(\omega)$, $\beta_i^{\epsilon'}(\omega)$ and $\rho_i(\omega)$ are independent of $f \in S_i(\omega)$. This completes the proof of asymptotic invariance of $S_i(\omega)$, $i \geqslant 1$. □

Our final result (Theorem 4.7 below) gives the existence of a global invariant flag for the cocycle $(u, \theta)$. The foliation is induced by the Lyapunov spectrum $\{\lambda_i\}_{i=1}^{\infty}$ of the linearized cocycle.

**Theorem 4.7** *(Global invariant flag). Consider the SNSE* (3.1) *with* $\gamma = 0$. *Define the random family of sets* $\{M_i(\omega)\colon\ \omega \in \Omega^*,\ i \geqslant 1\}$ *by*

$$M_i(\omega) := \left\{ f \in H\colon\ \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left|u(t, f, \omega)\right|_H \leqslant \lambda_i \right\} \tag{4.42}$$

*for* $i \geqslant 1$, $\omega \in \Omega^*$. *For fixed* $i \geqslant 1$, $\omega \in \Omega^*$, *define the sequence* $\{S_i^n(\omega)\}_{n=1}^{\infty}$, *inductively by*:

$$S_i^1(\omega) := S_i(\omega), \tag{4.43}$$

$$S_i^n(\omega) := \begin{cases} u(n, \cdot, \omega)^{-1}[S_i(\theta(n, \omega))], & \text{if } S_i^{n-1}(\omega) \subseteq u(n, \cdot, \omega)^{-1}[S_i(\theta(n, \omega))], \\ S_i^{n-1}(\omega), & \text{otherwise}, \end{cases} \tag{4.44}$$

*for all* $n \geqslant 2$. *In* (4.43) *and* (4.44), *the* $S_i(\omega)$ *are the local invariant* $C^{1,1}$ *Hilbert submanifolds of* $H$ *constructed in Theorem* 4.6.

*Then the following is true for each* $i \geqslant 1$ *and* $\omega \in \Omega^*$:

(i) *The sets* $\{M_i(\omega)\colon\ \omega \in \Omega^*,\ i \geqslant 1\}$ *are cocycle-invariant*:

$$u(t, \cdot, \omega)\big(M_i(\omega)\big) \subseteq M_i\big(\theta(t, \omega)\big) \tag{4.45}$$

*for all* $t \geqslant 0$.

(ii) $S_i^n(\omega) \subseteq S_i^{n+1}(\omega)$ *for all* $n \geqslant 1$, *and*

$$M_i(\omega) = \bigcup_{n=1}^{\infty} S_i^n(\omega), \quad i \geqslant 1, \tag{4.46}$$

(*perfectly in* $\omega$).

(iii) $M_{i+1}(\omega) \subseteq M_i(\omega)$.

(iv) *For any* $f \in M_i(\omega) \backslash M_{i+1}(\omega)$,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left|u(t, f, \omega)\right|_H \in (\lambda_{i+1}, \lambda_i]. \tag{4.47}$$

**Proof.** Fix $\omega \in \Omega^*$, where $\Omega^*$ is defined as in Theorem 4.6.

(i) To prove the cocycle invariance property (4.45), let $f \in M_i(\omega)$ and $t_1 > 0$. Then by definition (4.42) of $M_i(\omega)$, we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left|u(t, f, \omega)\right|_H \leqslant \lambda_i. \tag{4.48}$$

By the cocycle property of $(u, \theta)$, we have

$$\begin{aligned} \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left|u\big(t, u(t_1, f, \omega), \theta(t_1, \omega)\big)\right|_H &= \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left|u(t + t_1, f, \omega)\right|_H \\ &= \overline{\lim_{t\to\infty}} \frac{1}{t + t_1} \log \left|u(t + t_1, f, \omega)\right|_H \cdot \lim_{t\to\infty} \frac{t + t_1}{t} \end{aligned}$$

$$= \varlimsup_{t\to\infty} \frac{1}{t} \log \left|u(t, f, \omega)\right|_H$$
$$\leqslant \lambda_i.$$

The above inequality implies that $u(t_1, f, \omega) \in M_i(\theta(t_1, \omega))$. Hence $u(t_1, \cdot, \omega)(M_i(\omega)) \subseteq M_i(\theta(t_1, \omega))$ and so (4.45) holds for all $t \geqslant 0$.

(ii) To prove assertion (ii) of the theorem, observe first that (4.44) implies that $S_i^n(\omega) \subseteq S_i^{n+1}(\omega)$ for all $n \geqslant 1$. Next, we show that

$$S_i^n(\omega) \subset M_i(\omega) \tag{4.49}$$

for all $n \geqslant 1$. We prove (4.49) by induction on $n \geqslant 1$.

Let $f \in S_i^1(\omega) = S_i(\omega)$. By Theorem 4.6 and assertion (4.26), it follows that

$$\varlimsup_{t\to\infty} \frac{1}{t} \log \left|u(t, f, \omega)\right|_H \leqslant \lambda_i \tag{4.50}$$

perfectly in $\omega$. Therefore, $f \in M_i(\omega)$. Hence $S_i^1(\omega) = S_i(\omega) \subset M_i(\omega)$. Assume, by induction, that

$$S_i^k(\omega) \subset M_i(\omega)$$

for all $1 \leqslant k \leqslant n$. If $S_i^{n+1}(\omega) \nsubseteq u(n+1, \cdot, \omega)^{-1}[S_i(\theta(n+1, \omega))]$, then $S_i^{n+1}(\omega) = S_i^n(\omega) \subset M_i(\omega)$, by inductive hypothesis. Otherwise, $S_i^{n+1}(\omega) = u(n+1, \cdot, \omega)^{-1}[S_i(\theta(n+1, \omega))]$. Let $f \in S_i^{n+1}(\omega) = u(n+1, \cdot, \omega)^{-1}[S_i(\theta(n+1, \omega))]$. Then by the cocycle property and the definition of $S_i(\theta(n+1, \omega))$, it follows that

$$\left|u\big(n' + n + 1, f, \omega\big)\right|_H \leqslant \beta_i\big(\theta(n+1, \omega)\big) e^{n'\lambda_i} \tag{4.51}$$

for all $n' \geqslant 1$. This implies that

$$\varlimsup_{n'\to\infty} \frac{1}{n'} \log \left|u\big(n' + n + 1, f, \omega\big)\right| \leqslant \lambda_i.$$

Hence

$$\begin{aligned}
\varlimsup_{n''\to\infty} \frac{1}{n''} \log \left|u\big(n'', f, \omega\big)\right|_H &= \varlimsup_{n'\to\infty} \frac{1}{n' + n + 1} \log \left|u\big(n' + n + 1, f, \omega\big)\right|_H \\
&= \varlimsup_{n'\to\infty} \frac{n'}{n' + n + 1} \cdot \varlimsup_{n'\to\infty} \frac{1}{n'} \log \left|u\big(n' + n + 1, f, \omega\big)\right|_H \\
&\leqslant \lambda_i.
\end{aligned}$$

Therefore, $f \in M_i(\omega)$, and $S_i^{n+1}(\omega) \subset M_i(\omega)$. So, by induction, it follows that

$$S_i^n(\omega) \subset M_i(\omega) \tag{4.52}$$

for all $n \geqslant 1$. Thus

$$\bigcup_{n=1}^{\infty} S_i^n(\omega) \subseteq M_i(\omega). \tag{4.53}$$

In order to prove the converse inclusion

$$M_i(\omega) \subseteq \bigcup_{n=1}^{\infty} S_i^n(\omega), \tag{4.54}$$

we establish the following:

**Claim.** *There exist an increasing* (*random*) *sequence of integers* $n^k \uparrow \infty$ *such that*

$$S_i^{n^k}(\omega) = u\big(n^k, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n, \omega\big)\big)\big]$$

*for all* $k \geqslant 1$.

**Proof.** Define $n^1 := \inf\{n > 1\colon\ S_i^{n-1}(\omega) \subseteq u(n, \cdot, \omega)^{-1}[S_i(\theta(n, \omega))]\}$. Then

$$S_i^{n^1-1}(\omega) \subseteq u\big(n^1, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^1, \omega\big)\big)\big],$$

and by definition (4.44),

$$S_i^{n_1}(\omega) = u(n_1, \cdot, \omega)^{-1}\big[S_i\big(\theta(n_1, \omega)\big)\big]. \tag{4.55}$$

Furthermore, $S_i^{n-1}(\omega) \nsubseteq u(n, \cdot, \omega)^{-1}[S_i(\theta(n, \omega))]$ for all $1 < n < n^1$, and so by definition (4.44),

$$S_i^n(\omega) = S_i^{n-1}(\omega) = S_i^{n-2}(\omega) = \cdots = S_i^1(\omega) = S_i(\omega)$$

for all $1 < n < n^1$. In particular,

$$S_i(\omega) = S_i^{n^1-1}(\omega) \subseteq u\big(n^1, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^1, \omega\big)\big)\big].$$

Therefore,

$$u\big(n^1, \cdot, \omega\big)\big(S_i(\omega)\big) \subseteq S_i\big(\theta\big(n^1, \omega\big)\big).$$

Hence

$$n^1 = \inf\big\{n > 1\colon\ u(n, \cdot, \omega)\big(S_i(\omega)\big) \subseteq S_i\big(\theta(n, \omega)\big)\big\}. \tag{4.56}$$

Since $S_i(\omega)$ is asymptotically cocycle invariant (Theorem 4.6(c), (4.27)), it follows from (4.56) that $1 < n^1 < \infty$. Next, define $n^2 > n^1$ by

$$n^2 := \inf\big\{n > n^1\colon\ S_i^{n-1}(\omega) \subseteq u(n, \cdot, \omega)^{-1}\big[S_i\big(\theta(n, \omega)\big)\big]\big\}. \tag{4.57}$$

As before, the definition (4.44) implies that

$$S_i^{n^2-1}(\omega) = S_i^{n^1+1}(\omega) = u\big(n^1, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^1, \omega\big)\big)\big] \tag{4.58}$$

and

$$S_i^{n^2}(\omega) = u\big(n^2, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^2, \omega\big)\big)\big]. \tag{4.59}$$

Since

$$S_i^{n^2-1}(\omega) \subseteq u\big(n^2, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^2, \omega\big)\big)\big], \tag{4.60}$$

it follows from (4.58) that

$$u\big(n^1, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^1, \omega\big)\big)\big] \subseteq u\big(n^2, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^2, \omega\big)\big)\big].$$

Therefore,

$$\big\{u\big(n^2, \cdot, \omega\big)^{-1} \circ u\big(n^1, \cdot, \omega\big)^{-1}\big\}\big[S_i\big(\theta\big(n^1, \omega\big)\big)\big] \subseteq S_i\big(\theta\big(n^2, \omega\big)\big). \tag{4.61}$$

Using the cocycle property (Theorem 3.2(iii)), (4.61) implies

$$u\big(n^2 - n^1, \cdot, \theta\big(n^1, \omega\big)\big)\big[S_i\big(\theta\big(n^1, \omega\big)\big)\big] \subseteq S_i\big(\theta\big(n^2 - n^1, \theta\big(n^1, \omega\big)\big)\big). \tag{4.62}$$

By the asymptotic cocycle invariance of $S_i(\theta(n^1, \omega))$, it follows from (4.62) that $n^1 < n^2 < \infty$. Hence by induction, there exists an increasing sequence of integers $\{n^k\}_{k=1}^{\infty}$ such that $n^k \uparrow \infty$ as $k \to \infty$ and

$$S_i^{n^k}(\omega) = u\big(n^k, \cdot, \omega\big)^{-1}\big[S_i\big(\theta\big(n^k, \omega\big)\big)\big] \tag{4.63}$$

for all integers $k \geqslant 1$. This completes the proof of our claim. □

We now proceed to prove the inclusion (4.54). Let $f \in M_i(\omega)$. Then by definition of $M_i(\omega)$, we have

$$\varlimsup_{t \to \infty} \frac{1}{t} \log \big|u(t, f, \omega)\big|_H \leqslant \lambda_i. \tag{4.64}$$

Fix $\epsilon_1 \in (0, -\lambda_1)$ as in Theorem 4.6. Let $0 < \epsilon < \epsilon_1$. Then, using (4.64), there exists a positive integer $n_0$ such that

$$\sup_{t \geqslant n} \frac{1}{t} \log \big|u(t, f, \omega)\big|_H < \lambda_i + \epsilon$$

for all $n \geqslant n_0$. In particular,

$$\big|u(n, f, \omega)\big|_H < e^{n(\lambda_i + \epsilon)}$$

for all $n \geqslant n_0$. Define

$$K(\omega) := \max_{1 \leqslant k < n_0} \left| u(k, f, \omega) \right|_H.$$

Therefore,

$$\left| u(n, f, \omega) \right|_H \leqslant K(\omega) e^{n(\lambda_i + \epsilon)} \tag{4.65}$$

for all $n \geqslant 1$.

Pick $m_0$ sufficiently large such that

$$K(\omega) e^{n(\epsilon - \epsilon_1)} \leqslant \beta_i\big(\theta(n, \omega)\big) \tag{4.66}$$

for all $n \geqslant m_0$. Let $n \geqslant m_0$, $n' \geqslant 0$. Using the cocycle property and (4.65), we obtain

$$\begin{aligned} \left| u\big(n', u(n, f, \omega), \theta(n, \omega)\big) \right|_H &= \left| u\big(n' + n, f, \omega\big) \right|_H \\ &\leqslant K(\omega) e^{(n+n')(\lambda_i + \epsilon)} \\ &\leqslant K(\omega) e^{n(\epsilon - \epsilon_1)} \cdot e^{n'(\lambda_i + \epsilon_1)}. \end{aligned} \tag{4.67}$$

Pick $m_1 \geqslant m_0$ and sufficiently large such that

$$K(\omega) e^{n(\epsilon - \epsilon_1)} \leqslant \beta_i\big(\theta(n, \omega)\big) \tag{4.68}$$

for all $n \geqslant m_1$. From (4.67) and (4.68), we get

$$\left| u\big(n', u(n, f, \omega), \theta(n, \omega)\big) \right|_H \leqslant \beta_i\big(\theta(n, \omega)\big) e^{n'(\lambda_i + \epsilon_1)} \tag{4.69}$$

for all $n' \geqslant 0$ and $n \geqslant m_1$. Since $u(n, f, \omega) \to 0$ as $n \to \infty$, then there exists $m_2 > 0$ such that $u(n, f, \omega) \in \bar{B}(0, \rho_i(\omega))$ for all $n \geqslant m_2$. Thus (4.69) implies that

$$u(n, f, \omega) \in S_i\big(\theta(n, \omega)\big)$$

for all $n \geqslant \max(m_1, m_2)$; i.e. $f \in u(n, \cdot, \omega)^{-1}[S_i(\theta(n, \omega))]$, for all $n \geqslant \max(m_1, m_2)$. Now pick $k$ sufficiently large such that $n^k \geqslant \max(m_1, m_2)$ and $f \in u(n^k, \cdot, \omega)^{-1}[S_i(\theta(n^k, \omega))] = S_i^{n^k}(\omega)$. This proves that $f \in \bigcup_{n=1}^{\infty} S_i^n(\omega)$; and so the inclusion (4.54) holds. The proof of assertion (ii) of the theorem is complete.

Assertions (iii) and (iv) of the theorem follow directly from the definition (4.42) of the flag $M_i(\omega)$, $i \geqslant 1$. □

**Remark.** It is not clear if the $M_i(\omega)$ in Theorem 4.7 are $C^{1,1}$ immersed submanifolds in $H$. This would require transversality of the global semiflow $u(n, \cdot, \omega)$ and the local stable manifold $S_i(\theta(n, \omega))$.

## 5. Appendix

The following version of Gronwall's lemma is used throughout Section 3.

**Lemma 5.1.** *Suppose* $\alpha : [0, T] \to \mathbf{R}$ *is* $C^1$ *and* $h, \psi : [0, T] \to \mathbf{R}^{\geqslant 0}$ *are continuous. Assume that*

$$\alpha(t) + \int_0^t \psi(s)\,ds \leqslant \alpha(0) + \int_0^t h(s)\alpha(s)\,ds \tag{5.1}$$

*for all* $t \in [0, T]$. *Then*

$$\alpha(t) + \int_0^t \psi(s)\,ds \leqslant \alpha(0)\exp\left(\int_0^t h(s)\,ds\right) \tag{5.2}$$

*for all* $t \in [0, T]$.

**Proof.** Suppose (5.1) holds. Since $\alpha$ is $C^1$, then (5.1) implies

$$\int_0^t h(s)\alpha(s)\,ds - \int_0^t \alpha'(s)\,ds - \int_0^t \psi(s)\,ds \geqslant 0$$

for all $t \in [0, T]$. Hence

$$\int_0^t \big[h(s)\alpha(s) - \alpha'(s) - \psi(s)\big]\,ds \geqslant 0 \tag{5.3}$$

for all $t \in [0, T]$. The above relation implies

$$h(t)\alpha(t) - \alpha'(t) - \psi(t) \geqslant 0$$

for all $t \in [0, T]$; i.e.,

$$\alpha'(t) - h(t)\alpha(t) \leqslant -\psi(t) \tag{5.4}$$

for all $t \in [0, T]$. We now multiply both sides of (5.4) by the "integrating factor" $\mu(t) := e^{-\int_0^t h(s)\,ds}$. This gives

$$\frac{d}{dt}\big[\mu(t)\alpha(t)\big] \leqslant -\mu(t)\psi(t) \tag{5.5}$$

for all $t \in [0, T]$. Integrating both sides of (5.5), we get

$$\mu(t)\alpha(t) - \alpha(0) \leqslant -\int_0^t \mu(s)\psi(s)\,ds$$

for all $t \in [0, T]$. Therefore,

$$\alpha(t) + \mu(t)^{-1}\int_0^t \mu(s)\psi(s)\,ds \leqslant \alpha(0)\mu(t)^{-1} \tag{5.6}$$

for all $t \in [0, T]$. So

$$\alpha(t) + e^{\int_0^t h(s)\,ds}\int_0^t e^{-\int_0^s h(u)du}\psi(s)\,ds \leqslant \alpha(0)e^{\int_0^t h(s)\,ds};$$

i.e.,

$$\alpha(t) + \int_0^t e^{\int_s^t h(u)du}\psi(s)\,ds \leqslant \alpha(0)e^{\int_0^t h(s)\,ds} \tag{5.7}$$

for all $t \in [0, T]$. Since $h(u) \geqslant 0$ for all $u \in [0, T]$, (5.7) implies that

$$\alpha(t) + \int_0^t \psi(s)\,ds \leqslant \alpha(t) + \int_0^t e^{\int_s^t h(u)du}\psi(s)\,ds \leqslant \alpha(0)e^{\int_0^t h(s)\,ds}$$

for all $t \in [0, T]$. Therefore (5.2) holds. □

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