SUMMARY OF RESEARCH

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Numbers in [ ] refer to List of Publications in C.V. November 30, 2003.

1. Deterministic Functional Differential Equations on Manifolds.

In my research monograph *Retarded Functional Differential Equations: A Global Point of View* ([A 1], I laid the foundations of a geometric theory of retarded functional differential equations on manifolds [A 1]. This work was the first book to be published on differential-geometric aspects of deterministic hereditary systems on Riemannian manifolds. In [A 1] I deal with the following general framework:

Let X be a manifold. Typically we take X to be a Riemannian manifold (finite or infintedimensional) or a Banach manifold with a sufficiently smooth linear connection. Consider a manifold of paths $P([-r,0],X)$ which inherits its differentiable structure from the ambient manifold X. A retarded functional differential equation (RFDE) on X is a continuous map $F:[0,a)\times P([-r,0],X)\to TX$ such that for each $(t,\theta)\in[0,a)\times P([-r,0],X)$ the vector $F(t,\theta)\in T_{\theta(0)}X$, the tangent space to X at $\theta(0)$. A trajectory of F is a C^1 path $x:[-r,a)\to X$ such that

$$\begin{aligned} x'(t) &= F(t,x_t), \qquad t\in[0,a) \\ x_0 &= \theta\in P([-r,0],X). \end{aligned}$$

In the above equation x_t stands for the segment $x|[t-r,t]$ of the solution x. In Chapter 1, I develop a localization technique (Lemma 1.1, [A 1]) in order to obtain a unique local trajectory for the above initial value problem ([A 1], Theorem 1.2, Chapter 1, p.22). This is done under mild regularity conditions on F, assuming that X is a Banach manifold which admits a linear connection and $P=L_1^2$, the space of all Sobolev paths θ on X with square integrable derivatives. If X carries a Finsler and F satisfies suitable growth conditions, I prove that global trajectories of the hereditary equation exist for all time. Hence one gets a semiflow $\mathbf{R}^+\times L_1^2\to L_1^2$ on the space of initial paths $L_1^2=L_1^2([-r,0],X)$ ([A 1], Theorems (1.3)-(1.5)).

The main objective of Chapter 2 ([A 1]) is to characterize the topological structure of the critical set $\{\theta:F(\theta)=0\}$ when F is autonomous and X is n-dimensional, smooth Riemannian. It is first proved that solutions of the hereditary equation may reach equilibrium by converging asymptotically to a constant critical path, just as for vector fields (Theorem 2.1, [A 1]). The key idea here is to use parallel transport to show that a smooth hereditary coefficient F pulls back into a a smooth vector field ξ^F on L_1^2 ([A 1], Theorem 2.2 and corollary). In spite of the infinite degeneracy of the critical set $C(F)$, we are able to isolate a class of gradient-like hereditary equations for which the critical set is a closed smooth submanifold of L_1^2 with codimension n ([A 1], Proposition 2.4, p. 59). A Morse index exists for this class of hereditary equations ([A 1], Proposition 2.5, p.60). The index is constant on each connected component of $C(F)$. When X is compact, one can count the number of *critical components* in $C(F)$. This way I prove Morse inequalities for such hereditary equations ([A 1], Theorem (2.4) and corollaries). In particular it follows from these inequalities that F has only a finite number of critical components. The number of critical components with index m is always greater than or equal to the m-th Betti number of X, the rank of its m-th singular homology group ([A 1], Corollary 2.4.2).

In Chapter 3 ([A 1]), I linearize the semiflow of F by differentiating the canonical vector field ξ^F covariantly in $L_1^2([-r,0],X)$. This linearization defines a compacting semiflow on the tangent bundle $T\,L_1^2([-r,0],X)$ ([A 1], Theorem 3.3, p.79). Using semigroup techniques along the fibers of $T\,L_1^2([-r,0],X)$ we construct a Whitney direct sum splitting of the tangent bundle into two subbundles: the *unstable* and the *stable* one. Cf. classical results of Hale in the flat case $X = \mathbf{R}^n$. This splitting is invariant under the linearized semiflow. The unstable subbundle is *finite-dimensional*, and on it the linearized semiflow can be continued backwards to give a genuine *flow* which is defined for all time. Within the stable subbundle the linearized semiflow decays exponentially fast in the Sobolev Riemannian metric along each fiber in $T\,L_1^2([-r,0],X)$. This is the *Stable Bundle Theorem* ([A 1], Theorem 3.6, p.100).

Vector fields on the ambient manifold X are used in Chapter 4 ([A 1]) to generate examples of FDE's on the manifold. These include classical vector fields, *differential-delay equations* (DDE's), the *delayed development* and the *Levin-Nohel equation*. It is shown in Theorem 4.1 ([A 1], p.109) that a gradient Levin-Nohel equation on a Riemannian manifold may not admit non-trivial periodic solutions. I also give a detailed study of the *Functional Heat equation* (FHE) in this chapter of the monograph. The FHE is shown to correspond to a discontinuous but closed FDE on the Fréchet space of smooth functions on a compact Riemannian manifold. It is interesting to note here that despite the discontinuity of the equation and the infinite-dimensionality of the function space, the FHE still displays dynamical properties very similar to those of continuous finite-dimensional FDE's. In general, however, the FHE can be solved forward in time only along a closed Fréchet subspace of the state space. *Backward* solutions of the FHE do exist on the complementary subspace in the hyperbolic case. See [A 1], Chapter 4§5, pp. 113-133.

2. Stochastic Differential Systems with Memory.

A significant part of my contribution to the theory of stochastic hereditary systems was published in book form in 1984 by Pitman Research Notes in Mathematics under the title *Stochastic Functional Differential Equations* ([A 2]). It is the first book to be published on the theory of stochastic hereditary systems. The main impetus for developing this theory was the fact that such systems arise as mathematical models of physical phenomena which involve the past history of the state variables as well as the effects of random internal or external fluctuations or “noise”. Some specific examples of stochastic hereditary equations are provided in [A 2], Chapter VI; e.g. the stochastic hereditary Ornstein-Uhlenbeck-type model which was originally proposed by R. Kubo to describe the random motion of a molecule in a gaseous “heat bath.” I studied the large-time asymptotic stability of the heat bath in ([A 2], pp. 223-226). In particular one gets that all solutions of the heat bath model converge in L^2 asymptotically to a stationary Gausian solution with a *uniform* exponential speed ([A 2], Theorem (2.1), pp. 223-226). In fact there is enormous potential for applications of stochastic hereditary models in physics, chemistry, economics and control engineering. One way to arrive at such models is to start with a deterministic hereditary model and either consider random perturbations of the parameters or the effect of white or colored external noise. See the work of Kolmanovski. Stochastic delay equations were used by Scheutzow to model the logistic growth of a population under a negative feedback in its growth rate per capita and allowing for random migration (or removal) from the population.

In [A 2] I study the stochastic hereditary system

$$\begin{aligned} dx(t) &= H(t, x_t)dt + G(t, x_t)dW(t), \quad t > 0 \\ x_0 &= \eta \in C. \end{aligned}$$

where $C = C([-r, 0], \mathbf{R}^n)$ and W is multidimensional Brownian motion. The coefficients H, G are non-linear functionals on the state space C. In [A 2], it is shown that – under sufficiently general local Lipschitz and global linear growth conditions on the coefficients H and G - the above equation admits a unique continuous solution in $L^2(\Omega, C([-r, a], \mathbf{R}^n))$ for every $a > 0$, ([A 2], Chapter II, Theorem (2.1)). Furthermore, the trajectory field $\{^\eta x_t : t \geq 0, \eta \in C\}$ is a continuous map $\mathbf{R}^+ \times C \to L^2(\Omega, C)$ and gives a C-valued Feller process ([B 2], [A 2], Theorems III(1.1), p. 51, III(2.1), p. 64, Theorem III(3.1), p. 67). When the delay r is positive, I showed that the semi-group P_t

$$P_t(\phi)(\eta) = E\phi\left[^\eta x_t\right], \; \eta \in C, \; t \geq 0$$

is *never* strongly continuous on the Banach space C_b of all bounded uniformly continuous functions $\phi : C \to \mathbf{R}$ with the sup norm (Mohammed [A 2], Chapter (IV), Theorems (2.1), (2.2), (3.2), pp. 70-97). The semi-group is however weakly continuous and an explicit formula for the weak infinitesimal generator $A : D(A) \subset C_b \to C_b$ is established in ([A 2], Theorems IV(3.2), IV(4.1), IV(4.2), IV(4.3)). Due to the absence of non-trivial differentiable functions with bounded support in C and due to the fact that most tame functions lie outside the domain of the weak infinitesimal generator, I introduced the class of *quasi-tame functions* on the state space. This class consists of smooth functions, is weakly dense and is rich enough to generate the Borel σ-field of C. Furthermore the generator A assumes a simple and concrete form on quasi-tame functions (Definition 4.2, p.105, Theorems IV (4.2), (4.3) in [A 2]). For each fixed (v, η) in the state space $M_2 := \mathbf{R}^n \times L^2([-r, 0], \mathbf{R}^n)$, the "one-point motion" $\{X(t, ., (v, \eta)) : t \geq 0\}$ in M_2 – though a Markov process – is *not* a semi-martingale (c.f. the finite-dimensional stochastic ODE case). This presents some difficulties in dealing with non-linear transforms $\phi[X(t, ., (v, \eta))]$ under smooth functions $\phi : M_2 \to \mathbf{R}$. On the other hand the above analysis shows the existence of the class of smooth quasitame functions $\phi : M^2 \to \mathbf{R}$ for which $\phi[X(t, ., (v, \eta))]$ satisfies an Itô-type formula. It is clear that this fact will play a significant role in the computation of *Lyapunov exponents* for specific cases of *singular* hereditary systems.

The above results point the way towards the need for a workable theory of parabolic P.D.E.'s in infinite dimensions (c.f. work by Daletskii). If such a theory is established, one hopes to obtain in the long term some results on the existence of invariant probability measures on the Hilbert space M_2 for the trajectory process $\{(x(t), x_t) : T \geq 0\}$ (c.f. the finite-dimensional stochastic ODE case $r = 0$).

The *almost sure* dependence of the trajectory ${}^{\eta}x_t$ on the initial path η is a delicate problem. For the one-dimensional *linear* stochastic delay equation

$$dx(t) = x(t - r)dW(t), \qquad t > 0$$

with a *positive* delay r, I obtained the surprising result that the trajectory random field $\{X(t, \eta) \equiv^{\eta} x_t : t \geq 0,\ \eta \in C\}$ has no progressively measurable versions which are a.s. *linear* when viewed as operators $C \to C$ ([A 2], [B 7]). To my knowledge, this fact has never been observed within the context of linear ordinary differential equations (without memory) whether stochastic or deterministic. In contrast, one should note the diffeomorphism property for stochastic o.d.e.'s. See the works of Kunita, Bismut, Baxendale, Ikeda & Watanabe and Elworthy. Note also the continuous (linear) dependence on the initial paths for solutions of linear FDE's. Consult work by J. K. Hale and others. The a.s. non-linearity of the sample functions of the random field is attributed to local unboundedness of these sample functions due to the presence of "delayed diffusion" terms $x(t-r)dW(t)$ $(r > 0)$. These terms seem to occur in examples and applications where the effect of noise in the negative feedback loop cannot be neglected. Because of this highly pathological behavior of the trajectory random field, I introduced a new classification of stochastic differential equations into *regular* and *singular* types [B 14]. On the other hand, when the diffusion coefficients do not look into the past, one gets extremely regular almost sure dependence of the trajectory field on the initial paths ([A 2], Theorem V (2.1) and its corollaries, pp. 121-142). Under a Frobenius type condition on the diffusion matrix, I prove the existence of sufficiently smooth and a.s. locally compactifying versions of the trajectory field for $t \geq r$. In general the compactifying nature of the trajectory field is shown to persist in a *distributional sense* even in the singular case ([A 2], Theorems V (4.6) and (4.7)).

In Chapter VI [A 2], I study several classes of stochastic hereditary systems. Stochastic equations with several random delays are analyzed in VI §3 (pp. 167-186). The trajectory field in this case is *not* a Markov process in C. However, if the delays are essentially bounded and independent of the driving white noise, then one can show that the transition measures of the trajectory field correspond in the mean to a measure-valued process whose values are genuine transition probabilities of a Markov process in C. See [A 2], Theorem VI (3.1) and Lemma VI (3.3). The latter result asserts the stability of the trajectory with respect to random perturbations in the delay processes. As a consequence of these results one gets a sufficient condition for global asymptotic stability in distribution in terms of the corresponding property for the associated system with arbitrary *fixed deterministic delays* ([A 2], Corollary VI 3.1.2, p. 184).

3. Regular Linear and Affine Stochastic Hereditary Systems.

In [B 11], I studied autonomous stochastic hereditary linear systems of the form

$$dx(t) = H(x(t-r_1), x(t-r_2), \ldots, x(t-r_k), x(t), x_t)dt + g(x(t))dW(t)$$

where $H : (\mathbf{R}^n)^k \times M_2 \to M_2$ is continuous linear and $g : \mathbf{R}^n \to \mathbf{R}^{n \times m}$ is linear.

It is proved in [B 11] that the above system is regular in M_2 and its trajectory random field admits a jointly measurable version $X : \mathbf{R}^+ \times \Omega \times M_2 \to M_2$ such that, for each $t \in \mathbf{R}^+$ and a.a. $\omega \in \Omega$, the map $X(t, \omega, .)$ is a continuous linear operator on M_2. In fact I showed that X is a linear cocycle over the standard Brownian shift θ on path space ([B 11], Theorem 3 §3). In ([B 11], Theorem 4) I proved an Oseledec-type multiplicative ergodic theorem which gives a countable almost surely non-random Lyapunov spectrum for the stochastic flow X. The almost sure Lyapunov spectrum is bounded above and has no finite accumulation points ([B 11], Theorem 4, §4). I proved the *Stable Manifold Theorem* for regular linear hyperbolic systems in [B 11], Theorem 4, §4, Corollary 2, pp. 117-130). The proof of this theorem uses deep infinite-dimensional multiplicative ergodic theory methods ([B 11], cf. work of Ruelle, Mañé, Thieullen, Flandoli and Schaumlöffel).

In joint work with M. Scheutzow ([B 14], Theorem 4.2), we proved the existence of a cocycle in M_2 for the trajectory field of the much more general class of regular stochastic linear hereditary equations:

$$dx(t) = \left\{ \int_{[-r,0]} \mu(t)(ds)x(t+s) \right\} dt + dN(t) \int_{-r}^{0} K(t)(s)x(t+s)ds + dL(t)x(t-), \quad t > 0.$$

In the above hereditary equation, $\mu(t)$ is a stationary (ergodic) measure-valued process, $N(t)$, $L(t)$ are jump semi-martingales with stationary (ergodic) increments. The process $K(t)(s)$ is matrix-valued and stationary in t. The increments of L and N may depend on $\mu(t)$ and $K(t)$. The non-delay case was studied by L. Arnold and W. Kliemann when the stationary coefficients are assumed to be independent of the increments of the driving noise. Under fairly general assumptions, we prove that that the above equation has a stochastic flow with a countable set of Lyapunov exponents and a flow-invariant exponential dichotomy in the hyperbolic case ([B 14], Theorems 5.2 and 5.3). In the course of proving these results we develop a new technique for constructing stochastic flows for (linear) stochastic ODE's driven by continuous semimartingales. See [B 14], Theorem 3.1, §3.

In joint work with M. Scheutzow ([B 10]), we studied affine linear stochastic hereditary systems of the form

$$dx(t) = \int_{-r}^{0} \mu(ds)x(t+s)dt + dQ(t).$$

We proved the regularity of the stochastic flow and gave a detailed study of the Lyapunov spectrum and the existence of stationary solutions of the affine hereditary equation. A summary of these results may be found in my survey article ([B 14] §3 C, Theorems 10, 11, 12, 13). Details are given in [B 10]. Under suitable growth conditions on the driving noise Q, the existence of the pth moment Lyapunov exponent

$$g(p) := \lim_{t\to\infty} \frac{1}{t} \log E \parallel x_t \parallel_\infty^p, \quad p \geq 1$$

is proved in ([B 10]). See also [B 14], §3, C, Theorem 14. It is interesting to note here that the above result asserts the existence of *only one* p-th moment Lyapunov exponent under mild non-degeneracy conditions. Furthermore the p-th moment exponent is independent of all random (possibly anticipating) initial paths in the Skorohod space $D([-r,0], \mathbf{R}^n)$. This is surprising if we view the affine system as a *finite-dimensional* stochastic perturbation of the *infinitely degenerate* deterministic homogeneous system ($Q \equiv 0$):

$$dy(t) = \left\{ \int_{[-r,0]} \mu(ds)y(t+s) \right\} dt$$

with a *countably infinite* Lyapunov spectrum. The latter spectrum coincides with the set of real parts $\{\cdots < \beta_3 < \beta_2 < \beta_1\}$ of all roots of the characteristic equation

$$\det(\lambda I - \int_{[-r,0]} e^{\lambda s}\mu(ds)) = 0.$$

For the affine hereditary system one generically has

$$\lim_{t\to\infty} \frac{1}{t} \log E \,\|^{\eta}\, x_t\, \|_{\infty}^{p} = p\beta_1$$

for all random (possibly anticipating) initial conditions $\eta \in D([-r,0],\mathbf{R}^n)$ ([B 10], [B 14], §3 C, Theorem 14). Estimates on the second-moment exponent ($p = 2$) were previously obtained by Mohammed, Scheutzow and Weizsäcker [B 6] and Mohammed [B 5].

For several examples of one-dimensional linear stochastic hereditary equations, we obtained upper bounds on the top almost sure Lyapunov exponent λ_1 in joint work with M. Scheutzow. This work was funded in part by a collaborative research grant from NATO, with M. Scheutzow. Estimates on λ_1 for the following equations appear in ([B 14], §4, Theorems 15, 16, 17):

$$\begin{aligned} dx(t) &= x((t-1)-)dN(t) \qquad t > 0 \\ x_0 &\in D([-1,0],\mathbf{R}) \end{aligned}$$

where N is a compound Poisson process;

$$dx(t) = \{\nu x(t) + \mu x(t-r)\}dt + \left\{\int_{-r}^{0} x(t+s)\sigma(s)ds\right\} dW(t),$$

$$dx(t) = \{\nu x(t) + \mu x(t-r)\}dt + x(t)\, dM(t), \qquad t > 0,$$

where W is a Wiener process and M is a one-dimensional sample continuous square integrable martingale with stationary ergodic (but not necessarily independent) increments. These estimates can be found in [B 14], §4.

For small noise, I proved global asymptotic L^2-stability of $X(t,(v,\eta))$ for stochastic hereditary systems of the form

$$dx(t) = H(x(t),x_t)dt + \epsilon G(x(t),x_t)\, dW(t), \qquad t > 0$$

where the deterministic drift H is continuous *linear* and globally asymptotically stable ([B 5]).

We investigated the linear stochastic hereditary system

$$dx(t) = H(x_t)dt + GDW(t)$$

with constant diffusion matrix G through joint work with H. von Weizsäcker and M. Scheutzow ([B 6]). In this work, the classical (hyperbolic) splitting of the state space into a stable subspace S and a finite-dimensional unstable subspace U is used to determine the L^2 asymptotic stability of the random field $\{X(t,(v,\eta)) : t \geq 0, (v,\eta) \epsilon M_2\}$. In particular for each $(v,\eta) \in S$, $\lim_{t\to\infty} X(t,(v,\eta))$ exists in $L^2(\Omega, M_2)$ and its distribution is an invariant Gaussian measure on M_2. The convergence has exponential rate which is uniform with respect to the initial state (v,η) ([A 2], Theorem 4.2, pp. 208-216). On the other hand, if $(v,\eta) \in U$, then $\| X(t,(v,\eta)) \|$ goes to infinity exponentially fast in the L^2 sense ([B 8]). Again, the exponential speed of explosion in this case is uniform over U. I applied these results to study the stability of the "heat-bath" physical model ([A 2], pp. 223-226).

For the last two mentioned stochastic linear and affine hereditary systems , I proved the existence of a class of unstable distributions within the set of all probability measures on M_2. This class of unstable distributions is invariant under the adjoint semi-group P_t^* ([B 8]).

In joint work with M. Scheutzow ([B 19], [B 20]), I studied Lyapunov exponents for linear stochastic functional differential equations (sfde's) driven by semimartingales. In [B 20] several examples and case studies of linear sfde's are given. The examples fall into two broad classes: *regular* and *singular,* according to whether an underlying stochastic semiflow exists or not. In the singular case $dx(t) = \sigma x(t-r)\, dW(t)$, upper and lower bounds on the maximal exponential growth rate $\overline{\lambda}_1(\sigma)$ of the trajectories are expressed in terms of the noise variance σ. Roughly speaking, for small σ, $\overline{\lambda}_1(\sigma)$ behaves like $-(\sigma^2/2)$; while for large σ, it grows like $\log \sigma$ ([B 20], Theorems 2.2, 2.3). In the regular case, it is shown that a discrete Oseledec spectrum exists, and upper estimates on the top exponent λ_1 are provided ([B 20], Theorems 4.1-4.3). These estimates are sharp in the sense that they reduce to known estimates in the deterministic or non-delay cases.

4. Non-linear Stochastic Hereditary Systems.

In the joint paper [B 9] with D. Bell, we prove that stochastic ODE's can be approximated by stochastic delay equations with small time lags. In this way we develop a new proof of Itô's classical existence theorem for the solution of a stochastic ODE ([B 9]).

Part of my recent research efforts are directed towards investigating the a.s. local behavior of stochastic flows of regular non-linear autonomous stochastic hereditary systems along trajectories of a stationary solution. In joint work with M. Scheutzow, we construct an infinite-dimensional non-linear stochastic semiflow with the appropriate regularity properties ([B 26], *Journal of Functional Analysis*, 205, 271-305, 2003). We state and prove a *Local Stable Manifold Theorem* for non-linear stochastic functional differential equations (SFDE's) ([B 27], *Journal of Functional Analysis* (to appear)). We introduce the notion of hyperbolicity for stationary solutions of SFDE's. We then establish the existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary solution. The stable and unstable manifolds are stationary and asymptotically invariant under the stochastic semiflow. The proof uses an interpolation argument between delay periods together with Ruelle's discrete inifinite-dimensional multiplicative ergodic theorem. The proof also covers the (finite-dimensional) non-delay case. This project was supported in part by three NSF grants DMS-9206785 (1992-1995), DMS-9503702 (1995-1997) and DMS-9703596 (1997-2001)

5. Finite-dimensional Stochastic Flows. The Stable Manifold Theorem.

The article [B 21] is joint work with M. Scheutzow. It deals with finite-dimensional stochastic flows $\phi_{s,t}(x)$ on $\mathbf{R}^d$, driven by Kunita-type continuous spatial semimartingales. For this class of flows new spatial estimates for large $|x|$ are obtained under very mild regularity conditions on the driving semimartingale random field. In particular, it is shown that the stochastic flow $\phi_{s,t}(x)$ grows slower than $|x|(\log|x|)^\epsilon$ as $|x|$ goes to infinity, for arbitrarily small positive ϵ ([B 21]). Examples show that this bound is sharp. An example is given of a one-dimensional sde with *sublinear* coefficients but with the underlying stochastic flow growing *superlinearly* for large $|x|$ ([B 21]). In this example the stochastic flow has a.s. unbounded spatial derivatives, even though the driving spatial martingale has local characteristics with all derivatives globally bounded. It is interesting to note that in this example the driving noise is *infinite-dimensional*. However the infinite-dimensionality of the driving noise is *not* the crucial factor: An example is given of a sde driven by *one-dimensional* Brownian motion, has coefficients with *globally bounded* derivatives, while its stochastic flow has a.s. *unbounded* derivatives. This result is surprising since it is in sharp contrast with well-known behavior of deterministic flows driven by vector fields whose derivatives are globally bounded. For one-dimensional sde's, sufficient conditions on the coefficients are given in order for the stochastic flow to have sublinear growth and a.s. bounded derivatives ([B 21]). These results are of interest for the theory of stochastic flows on non-compact manifolds, non-linear filtering, and spde's.

In joint work with M. Scheutzow, we formulate and prove a *Local Stable Manifold Theorem* for stochastic differential equations (sde's) that are driven by spatial Kunita-type semimartingales with stationary ergodic increments ([B 23]). Both Stratonovich and Itô-type equations are treated. Starting with the existence of a stochastic flow for a sde, we introduce the notion of a hyperbolic stationary trajectory. We prove the existence of invariant random stable and unstable manifolds in the neighborhood of the hyperbolic stationary solution. For Stratonovich sde's, the stable and unstable manifolds are dynamically characterized using forward and backward solutions of the anticipating sde. The proof of the stable manifold theorem is based on Ruelle-Oseledec multiplicative ergodic theory. The local stable manifold theorem for sde's is a fundamental result in the theory of finite-dimensional stochastic dynamical systems.

6. The Malliavin Calculus.

According to my earlier work ([A 2]), all measurable versions of the trajectory random field of a singular stochastic linear delay equation are a.s. non-linear in the initial path and locally unbounded for any reasonable choice of the state space (e.g. C or M_2). In order to regularize this pathological behavior, I studied with D. Bell the Malliavin smoothness in ω of the measurable version $x(t, \omega, v, \eta)$ of the solution to the singular non-linear system

$$dx(t) = g(x(t-r))\, dW(t)$$

with a positive delay $r > 0$. Such systems were previously studied by Kusuoka and Stroock in the case when g is smooth and *bounded uniformly away* from zero. Under the above strong non-degeneracy condition, S. Kusuoka and D. Stroock proved that the solution of the above equation has a smooth density with respect to Lebesque measure on Euclidean space. Needless to say the Kusuoka-Stroock result *excludes* the singular *linear* case. In recent joint work with D. Bell, we proved that the solution of the above equation admits a smooth density for fixed t and (v, η), even if the vector field g has several polynomial-type degeneracies ([B 12], [B 13]). This result is obtained using the Malliavin calculus and new probabilistic bounds on the segment x_t of the solution (e.g., [B 12], Lemma 4, §4, [B 17], Theorem 1 §2).

I worked with D. Bell on the relationship between degenerate stochastic ODE's, elliptic parabolic partial differential operators ([B 17]). This work was published in *Duke Mathematical Journal*. We establish the existence of a large new class of smooth hypoelliptic second order parabolic partial differential operators with exponential degeneracies. The proof is based on the Malliavin calculus and involves the derivation of new probabilistic estimates for multidimensional time-dependent degenerate diffusion processes. We allow several *moving* degeneracy hypersurfaces of *infinite* (exponential) order in the diffusion covariance. These degeneracy surfaces are called *non-Hörmander sets (of parabolic type)*. Hörmander's general Lie algebra condition fails for these classes of operators. See Theorems 1.1, 1.2 in §1 [B 17]. In particular Theorem 1.1 ([B 17]) establishes parabolic hypoellipticity of the partial differential operator when the second-order coefficient matrix can have a degeneracies of exponential order p, with $p \in (0,1)$. These degeneracies may occur anywhere on a finite set of moving hypersurfaces in Euclidean space. Furthermore the range $(0,1)$ for p is *optimal.* As far as I know these results cannot be obtained using classical PDE techniques, e.g. weighted Sobolev spaces. In the course of proving the main theorems, we obtain several results concerning the existence of smooth densities for time-dependent degenerate stochastic hereditary and ordinary differential equations ([B 17], [B 18], §2, Theorems 2.1-2.3). This research was funded in part by NSF grants DMS-9206785, DMS-9503702 and DMS-9703596.

In another direction, I plan to examine the existence of smooth densities for *singular* stochastic delay equations and also for general stochastic hereditary systems. This analysis will be carried out assuming exponential degeneracy of the noise covariance near a collection of cylindrical hypersurfaces in the underlying infinite-dimensional state space (cf. [B 16]). By appealing to Skorohod's theory of random linear operators, I hope to establish the existence of a *Lyapunov spectrum* in probability for degenerate *singular linear* stochastic hereditary systems. It is expected that the above regularity analysis will lead to estimates on the Lyapunov spectrum. Numerical computations and simulations will be used to analyze particular case studies arising from applications in mathematical finance and control engineering.

7. Degenerate Boundary-value Problems.

This is the subject of the NSF projects *Degenerate SDE's and PDE's* (DMS-9503702), *Degenerate Stochastic Systems and Related Problems in Analysis* (DMS-9703596) (joint work with Denis Bell, University of North Florida).

I intend to continue my on-going research on problems related to degenerate multi-dimensional diffusions and second-order degenerate linear and quasilinear PDE's ([B 15–17]).

In the article ([B 17]), I have established (joint work with Denis Bell) a "maximal" extension of Hörmander's classical hypoellipticity theorem, whereby Hörmander's general condition is allowed to fail on a collection of hypersurfaces in Euclidean domains. The analysis in [B 17] is probabilistic in nature, using the methods of the Malliavin calculus. *So far, no classical proof is known for this result.* In the light of this development, I intend to take a new look at the classical theory of boundary-value problems for degenerate second-order linear and quasilinear partial differential equations. To this end, a classification of non-Hörmander degeneracy surfaces into *elliptic* and *parabolic* types is given. In particular, one seeks to establish the existence and uniqueness of classical smooth solutions to the Poisson-Dirichlet and the Neumann problems, *in the presence of elliptic non-Hörmander degeneracy surfaces.* In joint work with D. Bell, we established the existence of a unique calssical solution to the Dirichlet problem for superdegenerate differential operators. This work has appeared in *C.R. Acad. Sci. Paris (French Academy of Sciences)* ([B 24]). The proof uses the Malliavin calculus. *At present, there appears to be no proof of this result using classical analytic techniques.*

The analysis of quasilinear boundary-value problems will use the Malliavin calculus and will require sharp probabilistic estimates quantifying the interaction between diffusion processes and parabolic non-Hörmander surfaces (cf. [B 17]). I will also attempt to establish the hypoellipticity of infinitely degenerate second-order linear operators with elliptic non-Hörmander sets that *may not be enveloped in smooth hypersurfaces in Euclidean space* (cf. [B 17]). It is expected that this analysis will clarify the relationship between the allowable rate of failure of Hörmander's condition and the geometry of the elliptic non-Hörmander surface. It is interesting to note that the *anticipated results of this line of research are currently inaccessible using classical PDE techniques.*

8. Stochastic Numerics and Finance.

In the articles [B 28], [B 30], I developed (jointly with F. Yan and Y. Hu) several numerical schemes for solving stochastic differential systems with memory: strong Euler-Maruyama schemes for stochastic delay differential equations (SDDE's) and stochastic functional differential equations (SFDE's) with continuous memory, and a strong Milstein scheme for SDDE's. The convergence orders of the Euler-Maruyama and Milstein schemes are 0.5 and 1 respectively. In order to establish the Milstein scheme, we prove an infinite-dimensional Itô formula for "tame" functions acting on the segment process of the solution of an SDDE. It is interesting to note that the presence of the memory in the SDDE requires the use of the Malliavin calculus together with the anticipating stochastic analysis of Nualart and Pardoux. Given the *non-anticipating* nature of the SDDE, the use of *anticipating* calculus methods appears to be novel. These results will appear in *The Annals of Probability.*

In [B 29] (joint work with M. Arriojas, Y. Hu and Y. Pap) we develop a *A Delayed Black and Scholes Formula* for pricing European options when the underlying stock price follows a non-linear stochastic functional differential equation. We believe that the proposed model is sufficiently flexible to fit real market data, and is yet simple enough to allow for a closed-form representation of the option price. Furthermore, the model maintains the completeness of the market.

9. Finite and Infinite-Dimensional Stochastic Dynamical Systems.

This is a long-term project currently supported by a *five-year* NSF award DMS-0203368 (2002-2007) titled *Finite and Infinite-Dimensional Stochastic Dynamical Systems*. The research in this project focuses on several dynamical and numerical aspects of stochastic differential equations. Stochastic ordinary differential equations (sodes) on finite-dimensional manifolds generate stochastic flows on the manifold. One objective of the research is to construct invariant manifolds for such flows near stationary solutions, under suitable regularity and growth conditions on the driving vector fields. In particular, this yields the existence of stable, unstable and center manifolds near each stationary point on the manifold. An important class of infinite-dimensional semiflows on Hilbert space is generated by dissipative semilinear stochastic partial differential equations (spdes) on smooth compact manifolds or smooth bounded Euclidean domains. For these semiflows, we construct smooth stable and unstable manifolds in the neighborhood of a hyperbolic stationary solution of the underlying stochastic equation. The stable and unstable manifolds are stationary, live in a stationary tubular neighborhood of the stationary solution and are asymptotically invariant under the stochastic semiflow of the see/spde. The proof uses infinite-dimensional multiplicative ergodic theory techniques and interpolation arguments ([B 33], [B 34], [B 35]). This is joint work with T. S. Zhang and H. Zhao. Important examples of spdes covered by this analysis are Burger's equation, affine linear stochastic evolution equations and stochastic reaction-diffusion equations. The results of the research reveal new features of the stochastic dynamics of these well-studied models.

One encounters models of stochastic systems with memory (sfdes) in many engineering and physical applications. Deterministic smooth constraints on the solutions of such models lead naturally to sfdes on (compact) Riemannian manifolds. The article [B 25] is joint work with R. Léandre. In this article we prove an existence theorem for solutions of stochastic functional differential equations under smooth constraints in Euclidean space. The initial states are semimartingales on a compact Riemannian manifold. It is shown that, under suitable regularity hypotheses on the coefficients, and given an initial semimartingale, a sfde on a compact manifold admits a unique solution living on the manifold for all time. We also study the Chen-Souriau regularity of the solution of the sfde in the initial process.

It would be interesting to study the path-space-valued Markov process generated by trajectories of sfdes on the manifold. I am planning to establish an Itô formula for the infinite-dimensional segment process and will attempt to obtain its infinitesimal generator in terms of geometric invariants of the ambient Riemannian manifold. From the pathwise dynamical point of view, one would like to construct perfect semiflows on the path space which are induced by solutions of sfdes on a compact Riemannian manifold. The existence of invariant manifolds will also be examined.