STOCHASTIC DIFFERENTIAL SYSTEMS

WITH MEMORY

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Preface

The impetus for writing this research monograph is two-fold:

During the week July 29-August 4, 1996, the author delivered a series of six introductory lectures on stochastic differential systems with memory to *The Sixth Workshop on Stochastic Analysis*, held at Geilo, Norway. Due to the expository nature of these lectures, there was an apparent need for a more elaborate text in which the arguments are fleshed out in more detail. The current work is an attempt to do that.

Since the publication of the author's monograph *Stochastic Functional Differential Equations* (sfde's) in 1984, there have been significant and interesting devlopments in the theory of stochastic systems with memory. These developments are concerned in part with the impact of the theory of infinite-dimesnional stochastic flows and Oseledec's multiplicative ergodic theory on the dynamics of linear and non-linear sfde's. Other developments pertain to analysis of the regularity in distribution of solutions to sfde's using the machinery of the stochastic calculus of variations. It seemed appropriate at this time to include these developments in a single text.

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Introduction

The purpose of this monograph is to introduce the reader to certain aspects of stochastic differential systems, whose evolution depends on the past history of the state. We shall frequently refer to such systems as *stochastic functional differential equations (sfde's)*. In the deterministic case, sfde's reduce to what is now known as *retarded functional differential equations (rfde's)*. Such equations have received a great deal of attention by analysts during the last few decades. The reader may refer to fundamnetal works by J. Hale [],[],[], Mallet-Paret [], Mizel []...

Chapter I of this monograph begins with simple motivating examples. These include *the noisy feedback loop*, *the logistic time-lag model with Gaussian noise*, and *the classical "heat-bath" model of R. Kubo*, modeling the motion of a "large" molecule in a viscous fluid. These examples are embedded in a general class of stochastic functional differential equations (sfde's). We then establish pathwise existence and uniqueness of solutions to these classes of sfde's under local Lipschitz and linear growth hypotheses on the coefficients. It is interesting to note that the above class of sfde's is *not* covered by classical results of Protter, Metivier and Pellaumail and Doleans-Dade.

In Chapter II, we prove that the Markov (Feller) property holds for the trajectory random field of a sfde. The trajectory Markov semigroup is *not* strongly continuous for positive delays, and its domain of strong continuity does not contain tame (or cylinder) functions with evaluations away from 0. To overcome this difficulty, we introduce a class of *quasitame functions.* These belong to the domain of the weak infinitesimal generator, are weakly dense in the underlying space of continuous functions and generate the Borel σ-algebra of the state space. This chapter also contains a derivation of a formula for the weak infinitesimal generator of the semigroup for sufficiently regular functions, and for a large class of quasitame functions.

In Chapter III, we study pathwise regularity of the trajectory random field in the time variable and in the initial path. Of note here is the non-existence of the stochastic flow for the singular sdde $dx(t) = x(t-r)\,dW(t)$ and a breakdown of linearity and local boundedness. This phenomenon is peculiar to stochastic delay equations. It leads naturally to a classification of sfde's into *regular* and *singular* types. Necessary and sufficient conditions for regularity are not known. The rest of Chapter III is devoted to results on sufficient conditions for regularity of linear systems driven by white noise or semimartingales, and Sussman-Doss type nonlinear sfde's.

Building on the existence of a compacting stochastic semiflow, we develop a multiplicative ergodic theory for regular linear sfde's driven by white noise, or general helix semimartingales (Chapter IV). In particular, we prove a *Stable Manifold Theorem* for such systems.

In Chapter V, we seek asymptotic stability for various examples of one-dimensional linear sfde's. Our approach is to obtain upper and lower estimates for the top Lyapunov exponent.

A large class of regular non-linear sfde's is described in Chapter VI. Using finite-dimensional stochastic flows of diffeomorphisms, we build a locally compacting non-linear cocycle for the above class of regular sfde's.

In Chapter VII, we introduce a non-linear multiplicative ergodic theory for the class of regular sfde's given in Chapter VI. The central result of this chapter is the *Local Stable Manifold Theorem* (Theorem 3.1, Chapter VII).

Several topics are discussed in Chapter VIII. These include the existence of smooth densities for solutions of sfde's using the Malliavin calculus, an approximation technique for multidimensional diffusions using sdde's with small delays, and affine sfde's.

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