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The substitution theorem for semilinear stochastic partial differential equations

Salah-Eldin A. Mohammed [a,*,1], Tusheng Zhang [b,c,2]

[a] *Department of Mathematics, Southern Illinois University at Carbondale, Carbondale, IL 62901, USA*
[b] *Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK*
[c] *CMA, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway*

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Abstract

In this article we establish a substitution theorem for semilinear stochastic evolution equations (see's) depending on the initial condition as an infinite-dimensional parameter. Due to the infinite-dimensionality of the initial conditions and of the stochastic dynamics, existing finite-dimensional results do not apply. The substitution theorem is proved using Malliavin calculus techniques together with new estimates on the underlying stochastic semiflow. Applications of the theorem include dynamic characterizations of solutions of stochastic partial differential equations (spde's) with anticipating initial conditions and non-ergodic stationary solutions. In particular, our result gives a new existence theorem for solutions of semilinear Stratonovich spde's with anticipating initial conditions.

Keywords: Malliavin calculus; Stochastic semiflow; C^k cocycle; Stochastic evolution equation (see); Anticipating stochastic partial differential equation (spde)

* Corresponding author.
E-mail addresses: salah@sfde.math.siu.edu (S.-E.A. Mohammed), tzhang@maths.man.ac.uk (T. Zhang).
URL: http://sfde.math.siu.edu.
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1. Introduction. Statement of the substitution theorem

The main objective of this article is to answer the following simple (but basic) question:

Given a non-anticipating stochastic partial differential equation with its initial condition as an infinite-dimensional parameter, is it justified to replace the initial condition/parameter by an arbitrary random variable?

An answer to the affirmative for the above question is well known for a wide class of finite-dimensional sde's via the substitution theorems in [9,12,13]. However, the existing substitution theorems in [9,12,13] do not apply to infinite-dimensional systems. There are two serious obstructions to this approach:

- The substitution theorems are based largely on finite-dimensional selection techniques that are known to fail in infinite-dimensional settings, as indicated by the failure of Kolmogorov's continuity theorem for infinite-dimensional random fields [7,8,11] and the failure of Sobolev inequalities in infinite dimensions.
- The infinite-dimensionality of the dynamics renders the conditions of the substitution theorems in [12,13] inapplicable (cf. [12, Theorem 3.2.6], [13, Theorem 5.3.4]).

Both obstructions are resolved using ideas and techniques of the Malliavin calculus [6,14] together with new global estimates on the semiflow generated by the spde (Section 2). The use of Malliavin calculus techniques in this context seems to be necessitated by the infinite-dimensionality of the underlying stochastic dynamics.

The difficulty in proving the substitution theorem for stochastic systems with memory was pointed out by M. Scheutzow and one of the authors in [10, Part II]; but no rigorous proof or counterexamples are known. The purpose of the discussion in [10] is to provide a dynamic characterization of stable/unstable manifolds for stochastic systems with memory near hyperbolic stationary states.

In work by Grorud, Nualart and Sanz-Solé [5] a substitution theorem for Stratonovich integrals in Hilbert space is developed under the restriction that the substituting random variable takes values in a relatively compact set in the Hilbert space. The substitution result in [5] is obtained within the context of Hilbert space-valued stochastic ordinary differential equations, using metric entropy techniques.

In this article we establish a substitution theorem for semilinear spde's for a large class of infinite-dimensional Malliavin smooth random variables. We strongly believe that the techniques developed in this article will yield a similar substitution theorem for semiflows induced by sfde's.

We expect the results in this article to be useful in establishing regularity in distribution of the invariant manifolds for semilinear spde's.

In order to formulate our results, consider the following semilinear Itô stochastic evolution equation (see):

$$\begin{cases} du(t,x) = -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt + Bu(t,x)\,dW(t), & t > 0, \\ u(0,x) = x \in H \end{cases} \tag{1.1}$$

in a separable real Hilbert space H.

In the above equation $A : D(A) \subset H \to H$ is a closed linear operator on the Hilbert space H. Assume that A has a complete orthonormal system of eigenvectors $\{e_n\colon n \geqslant 1\}$ with corresponding positive eigenvalues $\{\mu_n, n \geqslant 1\}$; i.e., $Ae_n = \mu_n e_n$, $n \geqslant 1$. Suppose $-A$ generates a strongly continuous semigroup of bounded linear operators $T_t : H \to H$, $t \geqslant 0$. Furthermore, we let $F : H \to H$ be a (Fréchet) C_{b}^1 non-linear map, that is F has a globally bounded continuous Fréchet derivative $DF : H \to L(H)$.

Let E be a separable Hilbert space and $W(t), t \geqslant 0$, be an E-valued Brownian motion defined on the canonical filtered Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P)$ and with a separable covariance Hilbert space K. In particular, $K \subset E$ is a Hilbert–Schmidt embedding. Furthermore, Ω is the space of all continuous paths $\omega : \mathbf{R} \to E$ such that $\omega(0) = 0$ with the compact open topology, $\mathcal{F}$ is its Borel σ-field, $\mathcal{F}_t$ is the sub-σ-field of $\mathcal{F}$ generated by all evaluations $\Omega \ni \omega \mapsto \omega(u) \in E$, $u \leqslant t$, and P is Wiener measure on Ω. The Brownian motion is given by

$$W(t, \omega) := \omega(t), \quad \omega \in \Omega,\ t \in \mathbf{R},$$

and may be represented by

$$W(t) = \sum_{k=1}^{\infty} W^k(t) f_k, \quad t \in \mathbf{R}, \tag{1.2}$$

where $\{f_k\colon k \geqslant 1\}$ is a complete orthonormal basis of K, and W^k, $k \geqslant 1$, are standard independent one-dimensional Wiener processes [1, Chapter 4]. Note that, in general, the above series converges absolutely in E but not in K.

Denote by $L_2(K, H) \subset L(K, H)$ the Hilbert space of all Hilbert–Schmidt operators S: $K \to H$, given the norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} \big|S(f_k)\big|_H^2 \right]^{1/2},$$

where $|\cdot|_H$ is the norm on H. Suppose $B : H \to L_2(K, H)$ is a bounded linear operator. The stochastic integral in (1.1) is defined in the following sense [1, Chapter 4].

Let $\psi : [0, a] \times \Omega \to L_2(K, H)$ be $(\mathcal{B}([0, a]) \otimes \mathcal{F}, \mathcal{B}(L_2(K, H)))$-measurable, $(\mathcal{F}_t)_{t\geqslant 0}$-adapted and such that $\int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt < \infty$. Define

$$\int_0^a \psi(t)\, dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k)\, dW^k(t),$$

where the H-valued stochastic integrals on the right-hand side are with respect to the one-dimensional Wiener processes W^k, $k \geqslant 1$. Note that the above series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E\left| \int_0^a \psi(t)(f_k)\, dW^k(t) \right|^2 = \int_0^a E\big\|\psi(t)\big\|^2_{L_2(K,H)}\, dt < \infty.$$

Throughout the rest of the article, we will denote by $\theta : \mathbf{R} \times \Omega \to \Omega$ the standard P-preserving ergodic Wiener shift on Ω:

$$\theta(t,\omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbf{R}.$$

Hence (W, θ) is a *helix*:

$$W(t_1+t_2,\omega) - W(t_1,\omega) = W\big(t_2, \theta(t_1,\omega)\big), \quad t_1, t_2 \in \mathbf{R},\ \omega \in \Omega.$$

As usual, we let $L(H)$ be the Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$. Denote by $L_2(H) \subset L(H)$ the Hilbert space of all Hilbert–Schmidt operators $S : H \to H$, furnished with the Hilbert–Schmidt norm:

$$\|S\|_2 := \left[\sum_{n=1}^{\infty} \big|S(e_n)\big|_H^2\right]^{1/2}.$$

A *mild solution* of (1.1) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t \geqslant 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H$, $x \in H$, satisfying the following stochastic integral equation:

$$u(t,x,\cdot) = T_t x + \int_0^t T_{t-s} F\big(u(s,x,\cdot)\big)\, ds + \int_0^t T_{t-s} B u(s,x,\cdot)\, dW(s), \quad t \geqslant 0, \tag{1.3}$$

[1,2].

The see (1.1) has the equivalent Stratonovich form

$$\begin{cases} du(t,x) = -Au(t,x)\, dt + F\big(u(t,x)\big)\, dt - \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} B_k^2 u(t,x)\, dt + Bu(t,x) \circ dW(t), \\ u(0,x) = x \in H, \end{cases} \tag{1.4}$$

where $B_k \in L(H)$ are given by $B_k(x) := B(x)(f_k)$, $x \in H$, $k \geqslant 1$.

Condition (A$_1$).

$$\sum_{n=1}^{\infty} \mu_n^{-1} \big\|B(e_n)\big\|_{L_2(K,H)}^2 < \infty.$$

Conditions (A$_2$).

(i) A^{-1} is a trace class operator, i.e., $\sum_{n=1}^{\infty} \mu_n^{-1} < \infty$.
(ii) $T_t \in L(H)$, $t \geqslant 0$, is a strongly continuous contraction semigroup.

Condition (B).

(i) The operator $B : H \to L_2(K,H)$ can be extended to a bounded linear operator $H \to L(E,H)$, which will also be denoted by B.

(ii) The series $\sum_{k=1}^{\infty} \|B_k\|_{L(H)}^2$ converges, where the bounded linear operators $B_k : H \to H$, $k \geqslant 1$, are defined as in (1.4).

Remarks.

(i) Note that Condition (A_1) is implied by the following two requirements:
 (a) The operator $B : H \to L_2(K, H)$ is Hilbert–Schmidt.
 (b) $\liminf_{n\to\infty} \mu_n > 0$.
(ii) Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.
(iii) Suppose $A = -\Delta$ where Δ is the Laplacian on a compact smooth d-dimensional manifold with Dirichlet boundary condition. Then Condition (A_2) implies that $d = 1$. This follows easily from the fact that $\mu_n = O(n^{2/d})$ for large n [16, Theorem 3.1, p. 89].
(iv) Unlike Condition (A_2), note that Condition (A_1) does not entail any restriction on the spatial dimension of the underlying spde.

Under Condition (B) together with either (A_1) or (A_2), the see (1.1) (or (1.4)) admits a perfect $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{B}(H) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable cocycle (U, θ), $U : \mathbf{R}^+ \times H \times \Omega \to H$, with the following properties:

(i) For each $\omega \in \Omega$, the map $\mathbf{R}^+ \times H \ni (t, x) \mapsto U(t, x, \omega) \in H$ is continuous; and for fixed $(t, \omega) \in \mathbf{R}^+ \times \Omega$, the map $H \ni x \mapsto U(t, x, \omega) \in H$ is C^1.
(ii) $U(t + s, \cdot, \omega) = U(t, \cdot, \theta(s, \omega)) \circ U(s, \cdot, \omega)$ for all $s, t \in \mathbf{R}^+$ and all $\omega \in \Omega$.
(iii) $U(0, x, \omega) = x$ for all $x \in H$, $\omega \in \Omega$.

For proofs of the above properties see [11, Theorem 1.2.6]; cf. [3,4].

An $\mathcal{F}$-measurable random variable $Y : \Omega \to H$ is said be a *stationary point* for the cocycle (U, θ) if

$$U\big(t, Y(\omega), \omega\big) = Y\big(\theta(t, \omega)\big)$$

for all $(t, \omega) \in \mathbf{R}^+ \times \Omega$.

For any integer $p \geqslant 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are p-integrable together with their Malliavin derivatives $\mathcal{D}Y$ [12,13].

We now state the main substitution theorem in this article.

Theorem 1.1. *Assume that the see* (1.1) *satisfies Condition* (B) *together with either* (A_1) *or* (A_2). *Suppose F is C_b^1. Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable, and $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the C^1 cocycle generated by all mild solutions of the Stratonovich see* (1.4). *Then $U(t, Y)$, $t \geqslant 0$, is a mild solution of the* (*anticipating*) *Stratonovich see*

$$\begin{cases} dU(t,Y) = -AU(t,Y)\,dt + F\big(U(t,Y)\big)\,dt \\ \qquad\qquad - \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} B_k^2 U(t,Y)\,dt + BU(t,Y) \circ dW(t), \quad t > 0, \\ U(0,Y) = Y. \end{cases} \tag{1.5}$$

In particular, if $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is a stationary point of the see (1.4)*, then $U(t, Y) = Y(\theta(t))$, $t \geqslant 0$, is a stationary solution of the* (*anticipating*) *Stratonovich see*

$$\begin{cases} dY\big(\theta(t)\big) = -AY\big(\theta(t)\big)\,dt + F\big(Y\big(\theta(t)\big)\big)\,dt \\ \qquad\qquad - \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} B_k^2 Y\big(\theta(t)\big)\,dt + BY\big(\theta(t)\big) \circ dW(t), \quad t > 0, \\ Y\big(\theta(0)\big) = Y. \end{cases} \tag{1.6}$$

Furthermore, assume that F is C_b^2. Then the linearized cocycle $DU(t, Y)$ is a mild solution of the linearized anticipating see

$$\begin{cases} dDU(t, Y) = -ADU(t, Y)\,dt + DF\big(U(t, Y)\big)DU(t, Y)\,dt \\ \qquad\qquad - \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} B_k^2 DU(t, Y)\,dt + \big\{B \circ DU(t, Y)\big\} \circ dW(t), \quad t > 0, \\ DU(0, Y) = id_{L(H)}. \end{cases} \tag{1.7}$$

In the subsequent sections we will detail the proof of the above theorem. In Section 2, we begin by offering a series of estimates on the cocycle $U(t, x, \cdot)$, its Fréchet derivative $DU(t, x, \cdot)$ and its Malliavin derivative $\mathcal{D}U(t, x, \cdot)$. These estimates-interesting in their own right-will be used in the proofs of the substitution theorem (Theorem 1.1) and its finite-dimensional version (Theorem 3.1). In Section 3, we prove a special case of Theorem 1.1 in case the random variable Y is finite-dimensional (Theorem 3.1). This result is then used to give a detailed proof of Theorem 1.1 in Section 4. Section 5 contains an alternative proof of one of the estimates in Section 2, using a chaos-type expansion in the Hilbert space $L_2(H)$. In Section 6, we show existence and regularity of solutions to semilinear spde's with anticipating initial conditions.

2. Moment estimates of the cocycle

In this section, we develop new estimates on the non-linear cocycle $U : \mathbf{R}^+ \times H \times \Omega \to H$, its spatial Fréchet derivative $DU(t, x, \cdot)$ and its Malliavin derivatives $\mathcal{D}_u U(t, x, \cdot)$ for $u, t \in [0, a]$ and $x \in H$. The derivations are based on results in [11], Gronwall's Lemma and the fact that W has independent increments.

As before, assume the notation and hypotheses of Section 1. Let $\Phi(t, \omega)$, $\omega \in \Omega$, $t \geqslant 0$, be the linear cocycle associated with the see (1.1). That is for each $x \in H$, $\Phi(t, \cdot)(x)$, $t \geqslant 0$, is a mild solution of the linear see

$$\begin{cases} d\Phi(t, \cdot)(x) = -A\Phi(t, \cdot)(x)\,dt + B\Phi(t, \cdot)(x)\,dW(t), \quad t > 0, \\ \Phi(0, \cdot)(x) = x \in H. \end{cases} \tag{2.1}$$

Recall that $(\mathcal{F}_t)_{t \geqslant 0}$ is the filtration generated by Brownian motion W. Define

$$V(t, \omega) := \Phi(t, \omega) - T_t, \quad t \geqslant 0,\ \omega \in \Omega.$$

Then $V(t, \cdot)$, $t \geqslant 0$, is the continuous $L_2(H)$-valued solution of the following stochastic integral equation in $L_2(H)$:

$$V(t,\cdot)=\int_0^t T_{t-s}BV(s,\cdot)\,dW(s)+\int_0^t T_{t-s}BT_s\,dW(s),\quad t\geqslant 0. \tag{2.2}$$

Fix $s\geqslant 0$, and denote

$$\hat{V}(t,\omega):=V\big(t-s,\theta(s,\omega)\big),\quad t\geqslant s.$$

Then $\hat{V}$ is a solution of the following integral equation:

$$\hat{V}(t,\cdot)=\int_s^t T_{t-u}B\hat{V}(u,\cdot)\,dW(u)+\int_s^t T_{t-u}BT_{u-s}\,dW(u),\quad t\geqslant s, \tag{2.3}$$

in $L_2(H)$. See the proof of Theorem 1.2.4 in [11].

We will need the following Gronwall-type lemma.

Lemma 2.1. *Fix $a\in(0,\infty)$. Let $f,g:[0,a]\times\Omega\to\mathbf{R}^+$ be non-negative $(\mathcal{B}([0,a])\otimes\mathcal{F},\mathcal{B}(\mathbf{R}^+))$-measurable processes and $h:[0,a]\times[0,a]\times\Omega\to\mathbf{R}^+$ an $(\mathcal{B}([0,a]\times[0,a])\otimes\mathcal{F},\mathcal{B}(\mathbf{R}^+))$-measurable random field satisfying the following hypotheses*:

- (i) *For a.a. $\omega\in\Omega$ and all $s\in[0,a]$, the paths $f(\cdot,\omega),g(\cdot,\omega),h(\cdot,s,\omega)$ are continuous on $[0,a]$.*
- (ii) *The process f is $(\mathcal{F}_t)_{t\in[0,a]}$-adapted*; *and whenever $0<s<t\leqslant a$, the random variables $h(t-s,s,\cdot)$ are measurable with respect to the σ-algebra generated by the Brownian increments $W(s_2)-W(s_1)$, $s\leqslant s_1\leqslant s_2\leqslant t$.*
- (iii) $E\sup_{0\leqslant t\leqslant a}g(t,\cdot)+\sup_{0\leqslant s\leqslant a}E\sup_{0\leqslant t\leqslant a}h(t,s,\cdot)<\infty$.

Suppose that

$$f(t,\cdot)\leqslant g(t,\cdot)+\int_0^t h(t-s,s,\cdot)\big[1+f(s,\cdot)\big]\,ds \tag{2.4}$$

a.s. for all $t\in[0,a]$. Then $\sup_{0\leqslant t\leqslant a}f(t,\cdot)$ is integrable and there exist positive constants K_1, K_2 such that

$$E\sup_{0\leqslant s\leqslant t}f(s,\cdot)\leqslant K_1e^{K_2t} \tag{2.5}$$

for all $t\in[0,a]$.

Proof. Use conditions (i), (iii), put $t=t'$ in (2.4), and take $\sup_{0\leqslant t'\leqslant t}$ to obtain

$$\sup_{\substack{0\leqslant t'\leqslant t\\ x\in H}} f(t',\cdot)\leqslant\sup_{0\leqslant t'\leqslant a}g(t',\cdot)+\int_0^t\sup_{0\leqslant u\leqslant a}h(u,s,\cdot)\,ds$$

$$+\int_0^t \sup_{0\leqslant u\leqslant a} h(u,s,\cdot)\cdot \sup_{0\leqslant s'\leqslant s} f(s',\cdot)\,ds \tag{2.6}$$

a.s. for all $t\in[0,a]$.

For each integer $N\geqslant 1$, and any $s\in[0,a]$, define the events

$$\Omega_{s,N}:=\Big(\sup_{0\leqslant s'\leqslant s} f(s',\cdot)<N\Big).$$

Since f is $(\mathcal{F}_t)_{t\in[0,a]}$-adapted, then $\Omega_{s,N}\in\mathcal{F}_s$ for all $s\in[0,a]$, $N\geqslant 1$. Furthermore,

$$\Omega_{t,N}\subseteq\Omega_{s,N},\quad s\leqslant t,\ N\geqslant 1,$$

and

$$1_{\Omega_{t,N}}\leqslant 1_{\Omega_{s,N}},\quad s\leqslant t,\ N\geqslant 1. \tag{2.7}$$

Since f has a.a. sample-paths bounded on $[0,a]$ (actually continuous), we have

$$\bigcup_{N\geqslant 1}\Omega_{s,N}=\Omega \tag{2.8}$$

for each $s\in[0,a]$. Define

$$f_N(t,\cdot):=\sup_{0\leqslant t'\leqslant t} f(t',\cdot)\cdot 1_{\Omega_{t,N}},\quad 0\leqslant t\leqslant a,\ N\geqslant 1.$$

Clearly $|f_N(t,\cdot)|\leqslant N$ a.s. and $Ef_N(t,\cdot)|\leqslant N$ for all $t\in[0,a]$ and all $N\geqslant 1$.

Now multiply both sides of (2.6) by $1_{\Omega_{t,N}}$, use (2.7), take expectations, use hypothesis (iii) together with the independence of $\sup_{0\leqslant s'\leqslant s} f(s',\cdot)\cdot 1_{\Omega_{s,N}}$ and $\sup_{0\leqslant u\leqslant a} h(u,s,\cdot)$, to obtain

$$Ef_N(t,\cdot)\leqslant K_1+K_2\int_0^t Ef_N(s,\cdot)\,ds,\quad 0\leqslant t\leqslant a, \tag{2.9}$$

for all $N\geqslant 1$. The positive constants K_1, K_2 in (2.9) are *independent* of N. By Gronwall's Lemma, (2.9) gives

$$Ef_N(t,\cdot)\leqslant K_1e^{K_2t},\quad 0\leqslant t\leqslant a, \tag{2.10}$$

for all $N\geqslant 1$. Letting $N\to\infty$ in (2.10), using the fact that

$$\lim_{N\to\infty} f_N(t,\cdot)=\sup_{0\leqslant t'\leqslant t} f(t',\cdot)$$

a.s., and applying the Monotone Convergence Theorem, we get

$$E\sup_{0\leqslant t'\leqslant t} f(t',\cdot)\leqslant K_1e^{K_2t}$$

for all $0\leqslant t\leqslant a$. This proves (2.5). □

Now consider the random integral equation

$$U(t,x,\cdot)=\Phi(t,\cdot)(x)+\int_0^t \Phi\bigl(t-s,\theta(s,\cdot)\bigr)F\bigl(U(s,x,\cdot)\bigr)\,ds,\quad t\geqslant 0,\ x\in H, \tag{2.11}$$

where $F:H\to H$ is C_{b}^1 (as in Section 1).

Theorem 2.2. *Adopt the set-up of Section* 1. *Assume hypotheses* (B) *and* (A_1) *or* (A_2). *Let* $U:\mathbf{R}^+\times H\times\Omega\to H$ *be the cocycle generated by the mild solutions of the see* (1.1). *Fix any* $a\in(0,\infty)$. *Then the following assertions hold*:

(i) *The estimate*

$$E\sup_{\substack{0\leqslant t\leqslant a\\ x\in H}}\frac{|U(t,x,\cdot)|^{2p}}{(1+|x|_H^{2p})}<\infty \tag{2.12}$$

holds for all $p\geqslant 1$.

(ii) *Let* F *be of class* C_{b}^1. *Then*

$$E\sup_{\substack{0\leqslant t\leqslant a\\ x\in H}}\bigl\|DU(t,x,\cdot)\bigr\|^{2p}<\infty \tag{2.13}$$

for all $p\geqslant 1$. *In the above estimate, D stands for the Fréchet derivative of U in the spatial variable x.*

(iii) *Let* F *be* C_{b}^2. *Then*

$$E\sup_{\substack{0\leqslant t\leqslant a\\ x\in H}}\bigl\|D^2U(t,x,\cdot)\bigr\|^{2p}<\infty \tag{2.13'}$$

for all $p\geqslant 1$.

Proof. Assume hypotheses (B) and (A_1) or (A_2).

We will first prove the estimate (2.12). Fix any $p\geqslant 1$. By a simple application of Gronwall's Lemma, (2.3) gives

$$E\Bigl[\sup_{s\leqslant t\leqslant a}\bigl\|V\bigl(t-s,\theta(s,\cdot)\bigr)\bigr\|_{L_2(H)}^{2p}\Bigr]<\infty \tag{2.14}$$

for any fixed $s\in[0,a]$; and hence,

$$E\Bigl[\sup_{0\leqslant u\leqslant a}\bigl\|\Phi\bigl(u,\theta(s,\cdot)\bigr)\bigr\|_{L(H)}^{2p}\Bigr]<\infty \tag{2.15}$$

for each $s\in[0,a]$.

By (2.11) and the linear growth property of F, we get

$$\big|U(t,x,\cdot)\big|^{2p} \leqslant \big\|\Phi(t,\cdot)\big\|^{p}_{L(H)}|x|^{2p} + C\int_0^t \big\|\Phi\big(t-s,\theta(s,\cdot)\big)\big\|^{2p}_{L(H)}\big(1+\big|U(s,x,\cdot)\big|^{2p}\big)\,ds \tag{2.16}$$

a.s. for $0 \leqslant t \leqslant a$, $x \in H$, and C is a deterministic positive constant depending only on a. In (2.16), divide both sides of the inequality by $(1+|x|^{2p}_H)$ and take $\sup_{x\in H}$ to obtain

$$\sup_{x\in H}\frac{|U(t,x,\cdot)|^{2p}}{(1+|x|^{2p}_H)} \leqslant \big\|\Phi(t,\cdot)\big\|^{2p}_{L(H)} + C\int_0^t \big\|\Phi\big(t-s,\theta(s,\cdot)\big)\big\|^{2p}_{L(H)}\,ds + C\int_0^t \big\|\Phi\big(t-s,\theta(s,\cdot)\big)\big\|^{2p}_{L(H)}\cdot\sup_{x\in H}\frac{|U(s,x,\cdot)|^{2p}}{(1+|x|^{2p}_H)}\,ds \tag{2.17}$$

a.s. for $0 \leqslant t \leqslant a$. Now set

$$f(t,\cdot) := \sup_{x\in H}\frac{|U(t,x,\cdot)|^{2p}}{(1+|x|^{2p}_H)},\quad g(t,\cdot) := \big\|\Phi(t,\cdot)\big\|^{2p}_{L(H)},\quad h(t,s,\cdot) := \big\|\Phi\big(t,\theta(s,\cdot)\big)\big\|^{2p}_{L(H)},$$

a.s. for $0 \leqslant s \leqslant t \leqslant a$. Then (2.17) becomes

$$f(t,\cdot) \leqslant g(t,\cdot) + \int_0^t h(t-s,s,\cdot)\big[1+f(s,\cdot)\big]\,ds \tag{2.18}$$

a.s. for all $t \in [0,a]$.

We will now verify that the processes f, g, h satisfy all the conditions of Lemma 2.1. First, note that f, g, h are finite a.s. [11, Theorems 1.2.4, 1.2.6]. Secondly, the processes $f, g, h(\cdot, s)$ are sample-continuous for each $s \in [0,a]$ [11, Theorems 1.2.1–1.2.3, 1.2.6]. Thirdly, the process f is $(\mathcal{F}_t)_{t\in[0,a]}$-adapted. Fourthly, from (2.3), it follows that $\hat{V}(t,\cdot) = V(t-s,\theta(s,\cdot)) = \Phi(t-s,\theta(s,\cdot)) - T_{t-s}$ is measurable with respect to the σ-algebra generated by the Brownian increments $W(s_2)-W(s_1)$, $s \leqslant s_1 \leqslant s_2 \leqslant t$, and hence so is $h(t,s,\cdot)$. Finally, hypothesis (iii) of Lemma 2.1 is satisfied because of (2.15) and the measure-preserving property of θ. Hence the conditions of Lemma 2.1 are satisfied; thus (2.12) follows from (2.18). In fact, one gets

$$E\sup_{\substack{0\leqslant t'\leqslant t\\ x\in H}}\frac{|U(t',x,\cdot)|^{2p}}{(1+|x|^{2p}_H)} \leqslant K_1e^{K_2t} \tag{2.19}$$

for all $t \in [0,a]$ and some positive constants K_1, K_2 (depending possibly on a).

To prove part (ii) of the theorem, assume hypotheses (B) and (A_2) or (A_1); and let F be of class C^1_b. Fix any $p \geqslant 1$. Take Fréchet derivatives with respect to $x \in H$ on both sides of the random integral equation

$$U(t,x,\omega) = \Phi(t,\omega)(x) + \int_0^t \Phi\big(t-s,\theta(s,\omega)\big) F\big(U(s,x,\omega)\big)\,ds, \quad t \geqslant 0,\ x \in H,\ \omega \in \Omega.$$

This gives

$$DU(t,x,\cdot) = \Phi(t,\cdot) + \int_0^t \Phi\big(t-s,\theta(s,\cdot)\big)\big(DF\big(U(s,x,\cdot)\big)\big)\big(DU(s,x,\cdot)\big)\,ds, \quad t \geqslant 0.$$

As in the proof of part (i), observe that $\Phi(t-s,\theta(s,\cdot))$ is measurable with respect to the σ-algebra generated by the Brownian increments $W(s_2)-W(s_1)$, $s \leqslant s_1 \leqslant s_2 \leqslant t$, while $DU(\cdot,x,\cdot)$ is $(\mathcal{F}_t)_{t\in[0,a]}$-adapted (and DF is bounded). Using this observation together with the above equation and Lemma 2.1, one obtains

$$E \sup_{\substack{0\leqslant t\leqslant a \\ x\in H}} \big\| DU(t,x,\cdot)\big\|_{L(H)}^{2p} < \infty.$$

This proves the first assertion in (ii) for all $p \geqslant 1$. The proof of the second assertion in (ii) follows by a similar argument.

If F is C_{b}^2, assertion (iii) of the theorem may be proved by an argument similar to the above. □

The next theorem gives global spatial estimates on the Malliavin derivatives of the stochastic semiflow $U : \mathbf{R}^+ \times H \times \Omega \to H$ and its linearization.

Theorem 2.3. *Assume the setting of Section* 1. *In the see* (1.1)*, assume hypotheses* (B) *and* (A_1) *or* (A_2). *Then the following assertions hold*:

(i) *Let* $u, t \in [0,a]$. *Then* $V(t,\cdot) \in \mathbb{D}^{1,2p}(\Omega, L_2(H))$ *and*

$$E\Big[\sup_{u\leqslant t\leqslant a} \big\|\mathcal{D}_u V(t,\cdot)\big\|_{L_2(H)}^{2p}\Big] < \infty \tag{2.20}$$

for all $p \geqslant 1$.

(ii) *Suppose* F *is* C_{b}^1. *Then for all* $p \geqslant 1$*, we have*

$$E\Bigg[\sup_{\substack{0\leqslant t\leqslant a \\ x\in H}} \frac{|\mathcal{D}U(t,x,\cdot)|_H^{2p}}{(1+|x|_H^{2p})}\Bigg] < \infty, \tag{2.21}$$

where $\mathcal{D}$ *stands for the Malliavin derivative.*

(iii) *Let* F *be* C_{b}^2. *Then*

$$E\Bigg[\sup_{\substack{0< u, t\leqslant a \\ x\in H}} \frac{\|\mathcal{D}_u DU(t,x,\cdot)\|_{L_2(H)}^{2p}}{(1+|x|^{2p})}\Bigg] < \infty \tag{2.21$'$}$$

for all $p \geqslant 1$.

Proof. Assume hypotheses (B) and (A_1) or (A_2) throughout this proof. We prove the first assertion in part (i) of the theorem. Let $p \geqslant 1$. Fix $u \in [0, a]$ and take Malliavin derivatives in (2.2) to get the following stochastic integral equation in $L_2(H)$:

$$\mathcal{D}_u V(t, \cdot) = T_{t-u} B V(u, \cdot) + T_{t-u} B T_u + \int_u^t T_{t-s} B \mathcal{D}_u V(s, \cdot)\, dW(s), \quad t \geqslant u. \tag{2.22}$$

Define the sequence of events

$$\tilde{\Omega}_{s,N} := \Bigl(\sup_{u \leqslant s' \leqslant s} \bigl\| \mathcal{D}_u V(s', \cdot) \bigr\|_{L_2(H)}^{2p} < N \Bigr)$$

for $u \leqslant s \leqslant a$, $N \geqslant 1$. Now, from (2.22) and [1, Proposition 7.3], we obtain

$$E\Bigl[\sup_{u \leqslant t' \leqslant t} \bigl\| \mathcal{D}_u V(t', \cdot) \bigr\|_{L_2(H)}^{2p} \cdot 1_{\tilde{\Omega}_{t,N}} \Bigr] \leqslant K_1 E \bigl\| V(u, \cdot) \bigr\|_{L_2(H)}^{2p} + K_2 + K_3 \int_u^t E\Bigl[\sup_{u \leqslant s' \leqslant s} \bigl\| \mathcal{D}_u V(s', \cdot) \bigr\|_{L_2(H)}^{2p} \cdot 1_{\tilde{\Omega}_{s,N}} \Bigr] ds$$

for all $t \geqslant u$.

Using (2.14) together with Gronwall's Lemma and the Monotone Convergence Theorem, the above inequality implies (2.20).

To prove the first assertion in (ii) of the theorem, let F be C_b^1. Rewrite the random integral equation (2.11),

$$U(t, x, \omega) = \Phi(t, \omega)(x) + \int_0^t \Phi\bigl(t - s, \theta(s, \omega)\bigr) F\bigl(U(s, x, \omega)\bigr)\, ds, \quad t \geqslant 0,\ x \in H,\ \omega \in \Omega,$$

in the form

$$U(t, x, \omega) = V(t, \omega)(x) + T_t(x) + \int_0^t \bigl[V\bigl(t - s, \theta(s, \omega)\bigr) + T_{t-s} \bigr] F\bigl(U(s, x, \omega)\bigr)\, ds, \tag{2.23}$$

for $t \geqslant 0$, $x \in H$, $\omega \in \Omega$.

Taking the Malliavin derivative on both sides of (2.23), we get

$$\mathcal{D}_u U(t, x, \cdot) = \mathcal{D}_u V(t, \cdot)(x) + \int_0^t \mathcal{D}_u V\bigl(t - s, \theta(s, \cdot)\bigr) \bigl(F\bigl(U(s, x, \cdot)\bigr)\bigr)\, ds + \int_0^t \bigl[V\bigl(t - s, \theta(s, \cdot)\bigr) + T_{t-s} \bigr] \bigl(DF\bigl(U(s, x, \cdot)\bigr)\bigr) \bigl(\mathcal{D}_u U(s, x, \cdot)\bigr)\, ds, \quad t \geqslant 0. \tag{2.24}$$

As in the proof of Theorem 2.2, observe that $V(t-s,\theta(s,\cdot))$, $\mathcal{D}_u V(t-s,\theta(s,\cdot))$ are measurable with respect to the σ-algebra generated by the Brownian increments $W(s_2)-W(s_1)$, $s\leqslant s_1\leqslant s_2\leqslant t$, while $U(\cdot,x,\cdot)$, $\mathcal{D}_u U(\cdot,x,\cdot)$ are $(\mathcal{F}_t)_{t\in[0,a]}$-adapted. Using this observation together with (2.24) and Lemma 2.1, one obtains

$$E\left[\sup_{\substack{0\leqslant t\leqslant a\\ x\in H}}\frac{|\mathcal{D}_u U(t,x,\cdot)|_H^{2p}}{(1+|x|_H^{2p})}\right]<\infty, \tag{2.25}$$

for all $u\in[0,a]$ and all $p\geqslant 1$. This implies (2.21).

Let F be C_b^2. Then assertion (iii) of the theorem follows by a similar argument to the above. □

3. Finite-dimensional substitutions

Assume the notation and hypotheses of Section 1. In this section, we will prove assertion (1.5) of Theorem 1.1 in the special case when the random variable $Y\in\mathbb{D}^{1,4}(\Omega,H)$ is replaced by its finite-dimensional projections on H. The proof of (1.7) (in this special case) is analogous to that of (1.5) and is left to the reader. Relation (1.6) follows immediately from (1.5).

Recall that $\{e_n\colon n\geqslant 1\}$ is a complete orthonormal system of eigenvectors of A. For each integer $n\geqslant 1$, denote by $H_n:=L\{e_i\colon 1\leqslant i\leqslant n\}$, the n-dimensional linear subspace of H spanned by $\{e_i\colon 1\leqslant i\leqslant n\}$. Define the sequence of projections $P_n:H\to H_n$, $n\geqslant 1$, by

$$P_n(x):=\sum_{k=1}^{n}\langle x,e_k\rangle e_k,\quad x\in H. \tag{3.1}$$

Define $Y_n:\Omega\to H_n$ by

$$Y_n:=P_n\circ Y,\quad n\geqslant 1. \tag{3.2}$$

Note that $Y_n\to Y$ a.s.

The main result in this section is the following finite-dimensional substitution theorem (Theorem 3.1). Note that the proof of this theorem still requires Malliavin calculus techniques, largely due to the underlying *infinite-dimensional* semigroup dynamics in $\{T_t\}_{t\geqslant 0}$.

Theorem 3.1. *Assume all the conditions of Theorem* 1.1. *Then for each integer $n\geqslant 1$,* (1.5) *and* (1.7) *hold when $Y\in\mathbb{D}^{1,4}(\Omega,H)$ is replaced by Y_n. In particular,*

$$\begin{cases} dU(t,Y_n)=-AU(t,Y_n)\,dt+F\big(U(t,Y_n)\big)\,dt\\ \qquad\qquad -\dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}B_k^2U(t,Y_n)\,dt+BU(t,Y_n)\circ dW(t),\quad t>0,\\ U(0,Y_n)=Y_n\end{cases} \tag{3.3}$$

for each $n\geqslant 1$.

Proof. In this proof, C_i, $i=1,2,3,\dots$, denote positive deterministic constants. Rewrite (1.4) in its mild form

$$U(t,x)=T_t(x)+\int_0^t T_{t-s}F\big(U(s,x)\big)\,ds-\frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2U(s,x)\,ds$$
$$+\int_0^t T_{t-s}BU(s,x)\circ dW(s),\quad t>0. \tag{3.4}$$

Using the fact that each $Y_n\in\mathbb{D}^{1,4}(\Omega,H_n)$ is a finite-dimensional random variable, we will show that x in (3.4) can be replaced by Y_n to get

$$U(t,Y_n)=T_t(Y_n)+\int_0^t T_{t-s}F\big(U(s,Y_n)\big)\,ds-\frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2U(s,Y_n)\,ds$$
$$+\int_0^t T_{t-s}BU(s,Y_n)\circ dW(s),\quad t>0,\ n\geqslant 1, \tag{3.5}$$

for each $n\geqslant 1$ (cf. [12, Section 3.3.2], [9]). To justify (3.5), it is sufficient to prove that the random field

$$\int_0^t T_{t-s}BU(s,x)\circ dW(s),\quad x\in H_n,$$

has a version $H_n\times\Omega\to H$ satisfying

$$\int_0^t T_{t-s}BU(s,x)\circ dW(s)\Big|_{x=Y_n}=\int_0^t T_{t-s}BU(s,Y_n)\circ dW(s) \tag{3.6}$$

a.s. for fixed $t>0$. To prove (3.6), we will first establish some estimates on $U(t,x)$, $t\geqslant 0$, $x\in H$. Let $x,x'\in H$ and $t\in[0,a]$. Then (3.4) implies

$$E\big|U(t,x)-U(t,x')\big|^{2p}\leqslant C_1\big|T_t(x)-T_t(x')\big|^{2p}$$
$$+C_2E\left|\int_0^t\big\{T_{t-s}F\big(U(s,x)\big)-T_{t-s}F\big(U(s,x')\big)\big\}\,ds\right|^{2p}$$
$$+C_3E\left|\int_0^t T_{t-s}B\big(U(s,x)-U(s,x')\big)\,dW(s)\right|^{2p}$$

$$\leqslant C_4|x-x'|^{2p} + C_5\int_0^t E\big|U(s,x)-U(s,x')\big|^{2p}\,ds$$

$$+ C_6\left\{\int_0^t E\big|U(s,x)-U(s,x')\big|^2\,ds\right\}^p$$

$$\leqslant C_4|x-x'|^{2p} + C_7\int_0^t E\big|U(s,x)-U(s,x')\big|^{2p}\,ds.$$

Gronwall's Lemma implies

$$E\big|U(t,x)-U(t,x')\big|^{2p} \leqslant C_8|x-x'|^{2p}, \quad x,x'\in H,\ t\in[0,a]. \tag{3.7}$$

Fix $0 \leqslant t \leqslant a < \infty$, and define

$$\begin{aligned} S_m(x) &:= \int_0^t T_{t-s}P_m BU(s,x)\circ dW(s) \\ &= \int_0^t T_{t-s}P_m BU(s,x)\,dW(s) + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}(P_m B_k)^2 U(s,x)\,ds \end{aligned} \tag{3.8}$$

for all $x \in H$ and any integer $m \geqslant 1$. Since each H_m is invariant under T_t, $t \in [0,a]$, then $S_m(x) \in H_m$ for all $x \in H$.

Claim. *Assume Condition* (B) *of Section* 1. *Fix $t \in [0,a]$ in* (3.8). *Then there is a constant $C_9 > 0$ independent of m and $t \in [0,a]$ such that*

$$E\big|S_m(x)-S_m(x')\big|^{2p} \leqslant C_9|x-x'|^{2p} \tag{3.9}$$

for all $x, x' \in H$ and all $m \geqslant 1$.

Proof of Claim. Let $x, x' \in H$ and fix $t \in [0,a]$. Assume Condition (B) of Section 1. Then

$$E\left|\int_0^t T_{t-s}P_m BU(s,x)\,dW(s) - \int_0^t T_{t-s}P_m BU(s,x')\,dW(s)\right|^{2p}$$

$$\leqslant E\left|\int_0^t T_{t-s}P_m B\big[U(s,x)-U(s,x')\big]\,dW(s)\right|^{2p}$$

$$\leqslant C_{10}\int_0^t \|T_{t-s}P_m B\| E\big|U(s,x)-U(s,x')\big|^{2p}\,ds$$

$$\leqslant C_{11} \int_0^t E\big|U(s,x) - U(s,x')\big|^{2p}\,ds \leqslant C_{12}|x-x'|^{2p}. \tag{3.10}$$

The series on the right-hand side of (3.8) is absolutely convergent (uniformly in m) because of Condition (B) and the a.s. estimates

$$\begin{aligned}\sum_{k=1}^{\infty}\int_0^t \big\|T_{t-s}(P_mB_k)^2U(s,x)\big\|\,ds &\leqslant C_{13}\sum_{k=1}^{\infty}\|P_mB_k\|^2\int_0^t\big\|U(s,x)\big\|\,ds\\ &\leqslant C_{13}\int_0^t\big\|U(s,x)\big\|\,ds\cdot\sum_{k=1}^{\infty}\|B_k\|^2\\ &< \infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}&E\left|\sum_{k=1}^{\infty}\int_0^t T_{t-s}(P_mB_k)^2U(s,x)\,ds - \sum_{k=1}^{\infty}\int_0^t T_{t-s}(P_mB_k)^2U(s,x')\,ds\right|^{2p}\\ &\quad\leqslant \left\{\sum_{k=1}^{\infty}\int_0^t \big\{E\big|T_{t-s}(P_mB_k)^2\big[U(s,x)-U(s,x')\big]\big|^{2p}\big\}^{1/(2p)}\,ds\right\}^{2p}\\ &\quad\leqslant C_{14}\left\{\sum_{k=1}^{\infty}\int_0^t \|B_k\|^2\big\{E\big|U(s,x)-U(s,x')\big|^{2p}\big\}^{1/(2p)}\,ds\right\}^{2p}\\ &\quad\leqslant C_{14}\cdot C_8\left\{\sum_{k=1}^{\infty}\|B_k\|^2\right\}|x-x'|^{2p} = C_{15}|x-x'|^{2p}.\end{aligned} \tag{3.11}$$

Therefore (3.10) and (3.11) imply (3.9). This proves our claim. □

We next show that in (3.6), one can replace B by P_mB:

$$\int_0^t T_{t-s}P_mBU(s,x)\circ dW(s)\Big|_{x=Y_n} = \int_0^t T_{t-s}P_mBU(s,Y_n)\circ dW(s) \tag{3.12}$$

a.s. for all $m,n \geqslant 1$.

To prove (3.12), write

$$\int_0^t T_{t-s}P_mBU(s,x)\circ dW(s) = \sum_{k=1}^{\infty}\int_0^t T_{t-s}P_mB_kU(s,x)\circ dW^k(s), \quad x\in H. \tag{3.13}$$

Let

$$R_N(x) := \sum_{k=1}^{N} \int_0^t T_{t-s} P_m B_k U(s,x) \circ dW^k(s), \quad N \geqslant 1,\ x \in H_n \tag{3.14}$$

(for fixed $m \geqslant 1$). Then

$$\lim_{N\to\infty} R_N(x) = \sum_{k=1}^{\infty} \int_0^t T_{t-s} P_m B_k U(s,x) \circ dW^k(s) \tag{3.15}$$

in L^2, because the series on the right-hand side of (3.15) converges absolutely in $L^2(\Omega, H_m)$. Also for $x, x' \in H$,

$$\begin{aligned} E\big|R_N(x) - R_N(x')\big|^{2p} &\leqslant \left(\sum_{k=1}^{N} \left\{ E\left| \int_0^t T_{t-s} P_m B_k\big[U(s,x) - U(s,x')\big] \circ dW^k(s)\right|^{2p} \right\}^{1/(2p)} \right)^{2p} \\ &\leqslant C_{15}|x - x'|^{2p} \end{aligned}$$

where C_{15} is independent of m, N (by the proof of the Claim and Condition (B)).

Now apply Lemma 4.1 ([9], or [13, Lemma 5.3.1]) to the sequence of random fields $\{R_N(x)\colon x \in H_n\}$, $N \geqslant 1$. This implies the following limit in probability:

$$\begin{aligned} \lim_{N\to\infty} \big\{ R_N(x)\big|_{x=Y_n} \big\} &= \left(\sum_{k=1}^{\infty} \int_0^t T_{t-s} P_m B_k U(s,x) \circ dW^k(s) \right)\Bigg|_{x=Y_n} \\ &= \int_0^t T_{t-s} P_m B U(s,x) \circ dW(s)\bigg|_{x=Y_n}. \end{aligned} \tag{3.16}$$

We next observe that, for each $k \geqslant 1$, the following substitution rule holds:

$$\int_0^t T_{t-s} P_m B_k U(s,x) \circ dW^k(s)\bigg|_{x=Y_n} = \int_0^t T_{t-s} P_m B_k U(s,Y_n) \circ dW^k(s) \tag{3.17}$$

a.s. [13, Theorem 5.3.3].

From (3.14)–(3.17), and the finite-dimensional substitution theorem for Stratonovich integrals [13, Theorem 5.3.4], we get the following limits in probability:

$$\int_0^t T_{t-s} P_m B U(s,x) \circ dW(s)\bigg|_{x=Y_n} = \lim_{N\to\infty} \big\{ R_N(x)\big|_{x=Y_n} \big\}$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\sum_{k=1}^{N}\int_0^t T_{t-s}P_mB_kU(s,x)\circ dW^k(s)\Big|_{x=Y_n}\\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\int_0^t T_{t-s}P_mB_kU(s,Y_n)\circ dW^k(s)\\
&= \sum_{k=1}^{\infty}\int_0^t T_{t-s}P_mB_kU(s,Y_n)\circ dW^k(s)\\
&= \int_0^t T_{t-s}P_mBU(s,Y_n)\circ dW(s). \qquad (3.18)
\end{aligned}
$$

The last equality in (3.18) follows from the definition of the Stratonovich integral in infinite-dimensions. This proves (3.12). Recall that

$$
S_m(x) := \int_0^t T_{t-s}P_mBU(s,x)\circ dW(s) + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}(P_mB_k)^2U(s,x)\,ds. \qquad (3.19)
$$

Define

$$
\begin{aligned}
S(x) &:= \int_0^t T_{t-s}BU(s,x)\circ dW(s)\\
&= \int_0^t T_{t-s}BU(s,x)\,dW(s) + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2U(s,x)\,ds. \qquad (3.20)
\end{aligned}
$$

We will show that

$$
\lim_{m\to\infty} S_m(x) = S(x) \qquad (3.21)
$$

in probability for all $x \in H$. Now, for every $x \in H$,

$$
\begin{aligned}
&\lim_{m\to\infty} P_mB\big(U(s,x)\big) = B\big(U(s,x)\big), \quad \text{a.s.},\\
&\big|P_mB\big(U(s,x)\big)\big| \leqslant \big|B\big(U(s,x)\big)\big|,
\end{aligned}
$$

and

$$
E\left|\int_0^t T_{t-s}P_mBU(s,x)\,dW(s) - \int_0^t T_{t-s}BU(s,x)\,dW(s)\right|^{2p}
$$

$$
\begin{aligned}
&= E\left|\int_0^t T_{t-s}(P_m B - B)U(s,x)\,dW(s)\right|^{2p} \\
&= E\left|\int_0^t T_{t-s}\bigl\{P_m B\bigl(U(s,x)\bigr) - B\bigl(U(s,x)\bigr)\bigr\}\,dW(s)\right|^{2p} \\
&\leqslant C_{16}\int_0^t E\bigl|P_m B\bigl(U(s,x)\bigr) - B\bigl(U(s,x)\bigr)\bigr|^{2p}\,ds
\end{aligned} \tag{3.22}
$$

a.s. for all $m \geqslant 1$ and all $x \in H$. Then by the Dominated Convergence Theorem, it follows that

$$
\lim_{m\to\infty} E\left|\int_0^t T_{t-s} P_m B U(s,x)\,dW(s) - \int_0^t T_{t-s} B U(s,x)\,dW(s)\right|^{2p} = 0, \quad x \in H.
$$

Also, for each $m \geqslant 1$ and any $x \in H$, we have

$$
\begin{aligned}
&E\bigl|T_{t-s}(P_m \circ B_k)^2 U(s,x) - T_{t-s} B_k^2 U(s,x)\bigr|^{2p} \\
&\quad = E\bigl|T_{t-s}\bigl\{(P_m \circ B_k)^2\bigl(U(s,x)\bigr) - B_k^2\bigl(U(s,x)\bigr)\bigr\}\bigr|^{2p} \\
&\quad \leqslant C_{17} E\bigl|(P_m \circ B_k \circ P_m \circ B_k)\bigl(U(s,x)\bigr) - (P_m \circ B_k \circ B_k)\bigl(U(s,x)\bigr)\bigr|^{2p} \\
&\quad \leqslant C_{18} E\bigl|(P_m \circ B_k \circ B_k)\bigl(U(s,x)\bigr) - B_k^2\bigl(U(s,x)\bigr)\bigr|^{2p} \\
&\quad \leqslant C_{17}\|B_k\| E\bigl|P_m\bigl(B_k\bigl(U(s,x)\bigr)\bigr) - B_k\bigl(U(s,x)\bigr)\bigr|^{2p} \\
&\qquad + E\bigl|P_m\bigl(B_k^2\bigl(U(s,x)\bigr)\bigr) - B_k^2\bigl(U(s,x)\bigr)\bigr|^{2p}
\end{aligned} \tag{3.23}
$$

and

$$
\begin{aligned}
&\bigl|T_{t-s}(P_m \circ B_k)^2 U(s,x) - T_{t-s} B_k^2 U(s,x)\bigr| \\
&\quad \leqslant C_{19}\bigl[\|P_m \circ B_k\|^2 + \|B_k\|^2\bigr]\bigl|U(s,x)\bigr| \\
&\quad \leqslant 2C_{19}\|B_k\|^2\bigl|U(s,x)\bigr|
\end{aligned} \tag{3.24}
$$

a.s. The right-hand side of (3.23) converges to 0 a.s. as $m \to \infty$ (for each fixed $k \geqslant 1$). Hence, by (3.24), Theorem 2.2, the convergence of $\sum_{k=1}^{\infty}\|B_k\|^2$ and the Dominated Convergence Theorem, we obtain

$$
\lim_{m\to\infty}\sum_{k=1}^{\infty}\int_0^t T_{t-s}(P_m \circ B_k)^2 U(s,x)\,ds = \sum_{k=1}^{\infty}\int_0^t T_{t-s} B_k^2 U(s,x)\,ds \tag{3.25}
$$

in L^{2p} for all $x \in H$.

Using (3.19), (3.20), (3.22), and (3.25), we get (3.21). By (3.21), we may apply Lemma 5.3.1 in [13], [9, Lemma 4.1], to get

$$\lim_{m\to\infty} S_m(Y_n) = S(Y_n) \quad \text{in probability} \tag{3.26}$$

for each $n \geqslant 1$.

Using (3.26), we may let $m \to \infty$ in (3.18) to get

$$\begin{aligned}\lim_{m\to\infty} \int_0^t T_{t-s} P_m BU(s,x) \circ dW(s)\Big|_{x=Y_n} &= \lim_{m\to\infty} \int_0^t T_{t-s} P_m BU(s,Y_n) \circ dW(s) \\ &= \int_0^t T_{t-s} BU(s,x) \circ dW(s)\Big|_{x=Y_n}.\end{aligned} \tag{3.27}$$

Observe that

$$\lim_{m\to\infty} T_{t-s} P_m BU(s,Y_n) = T_{t-s} BU(s,Y_n)$$

in $L^2([0,T]\times\Omega)$.

Using a truncation argument, one can show that the process $[0,t] \ni s \mapsto T_{t-s}BU(s,Y_n) \in L_2(K,H)$ is Stratonovich integrable, and

$$\lim_{m\to\infty} \int_0^t T_{t-s} P_m BU(s,Y_n) \circ dW(s) = \int_0^t T_{t-s} BU(s,Y_n) \circ dW(s). \tag{3.28}$$

Details of the truncation argument are given in Section 3 (replacing B by $P_m B$). (Note that this truncation argument does not depend on (3.28).) Combining (3.27) and (3.28) gives

$$\int_0^t T_{t-s} BU(s,x) \circ dW(s)\Big|_{x=Y_n} = \int_0^t T_{t-s} BU(s,Y_n) \circ dW(s). \tag{3.29}$$

This proves (3.6) and hence (3.5) holds. □

4. Proof of the substitution theorem

In this section, we will complete the proof of the main substitution theorem (Theorem 1.1) in Section 1. Our argument will appeal to the estimates in Section 2 on the cocycle $U(t,x)$, $t \geqslant 0$, $x \in H$, its Frèchet and Malliavin derivatives $DU(t,x)$, $\mathcal{D}U(t,x)$, respectively.

Proof of Theorem 1.1. Assume that $Y \in \mathbb{D}^{1,4}(\Omega,H)$, and the see (1.1) satisfies hypothesis (B) together with either (A_1) or (A_2). We will prove the equality (1.5) in Section 1. Equality (1.6) is a special case of (1.5). The proof of (1.7) is similar to that of (1.5), and is left to the reader. Note that here the proof of (1.7) employs the estimates (2.13′) and (2.21′).

Fix $t > 0$ throughout this proof.

To prove (1.5) in Section 1, we will show that the anticipating process $U(t, Y)$ satisfies the Stratonovich integral equation

$$U(t,Y) = T_t(Y) + \int_0^t T_{t-s}F\big(U(s,Y)\big)\,ds - \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2U(s,Y)\,ds + \int_0^t T_{t-s}BU(s,Y)\circ dW(s). \tag{4.1}$$

We start with the mild Stratonovich form of the see (1.1):

$$U(t,x) = T_t(x) + \int_0^t T_{t-s}F\big(U(s,x)\big)\,ds - \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2U(s,x)\,ds + \int_0^t T_{t-s}BU(s,x)\circ dW(s). \tag{4.2}$$

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t]\times\Omega \to H$ such that $v \in L^2([0,t]\times\Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t\int_0^t \|\mathcal{D}_u v(s,\cdot)\|_H^2\,du\,ds] < \infty$. We say that v belongs to $\mathbb{L}^{1,2}_{\mathrm{loc}}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F}\times\mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \to \infty$,
(ii) $v = v^m$ on Ω_m.

We first show that the Stratonovich integral in (4.1) is well defined. To prove this, it is sufficient to verify that the process $v(s) := T_{t-s}BU(s,Y)$, $s \leqslant t$, belongs to $\mathbb{L}^{1,2}_{\mathrm{loc}}$ [13, Theorem 5.2.3]. For any integer $m \geqslant 1$, let $\phi_m \in C_{\mathrm{b}}^2(\mathbf{R},\mathbf{R})$ be a bump function such that $\phi_m(z) = 1$ for $|z| \leqslant m$ and $\phi_m(z) = 0$ for $|z| > m+1$. Define $v^m(s) := v(s)\phi_m(|Y|_H)$, $s \leqslant t$. Clearly, $v = v^m$ on $\Omega_m := \{\omega\colon |Y(\omega)|_H \leqslant m\}$ for each $m \geqslant 1$. Thus v is Stratonovich integrable if we can show that $v^m \in \mathbb{L}^{1,2}$ for every $m \geqslant 1$. To see this, note first that the estimate

$$\big|v^m(s)\big|_H \leqslant C \sup_{\substack{x\in H\\ |x|_H\leqslant m+1}} \big|U(s,x)\big|_H, \quad s \leqslant t,$$

together with Theorem 2.2(i) imply that $v^m \in L^2([0,t]\times\Omega)$ for each $m \geqslant 1$. On the other hand,

$$\mathcal{D}_u v^m(s) = T_{t-s}B\big[\mathcal{D}_uU(s,Y) + DU(s,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big) + T_{t-s}BU(s,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H$$

for all $u, s \in [0,t]$. Therefore,

$$\big|\mathcal{D}_u v^m(s)\big|_H \leqslant C_m \sup_{|x|_H\leqslant m+1}\big|\mathcal{D}_uU(s,x)\big|_H + C_m \sup_{|x|_H\leqslant m+1}\big\|DU(s,x)\big\|_{L(H)}|\mathcal{D}_uY|_H + C_m \sup_{|x|_H\leqslant m+1}\big|U(s,x)\big|_H\mathcal{D}_u|Y|_H, \quad u, s \in [0,t]. \tag{4.3}$$

Using the fact that $Y \in \mathbb{D}^{1,4}(\Omega, H)$, Theorem 2.3(ii) and Theorem 2.2(i), (ii), it follows from (4.3) that

$$E\left[\int_0^t \int_0^t \left|\mathcal{D}_u v^m(s)\right|_H^2 \, ds\, du\right] < \infty.$$

Hence, $v^m \in \mathbb{L}^{1,2}$ for each $m \geqslant 1$.

Next we prove that $U(t, Y)$ satisfies Eq. (4.1). For any integer $n \geqslant 1$, define $Y_n := P_n \circ Y$ as in (3.2). Then by Theorem 3.1, we know that for every $n \geqslant 1$,

$$U(t, Y_n) = T_t(Y_n) + \int_0^t T_{t-s} F\big(U(s, Y_n)\big)\, ds - \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t T_{t-s} B_k^2 U(s, Y_n)\, ds$$
$$+ \int_0^t T_{t-s} B U(s, Y_n) \circ dW(s), \quad t > 0. \tag{4.4}$$

We wish to pass to the limit a.s. as $n \to \infty$ in (4.4). To do this, first note the following easy a.s. limits:

$$\lim_{n\to\infty} U(t, Y_n) = U(t, Y),$$
$$\lim_{n\to\infty} T_t(Y_n) = T_t(Y),$$
$$\lim_{n\to\infty} \int_0^t T_{t-s} F\big(U(s, Y_n)\big)\, ds = \int_0^t T_{t-s} F\big(U(s, Y)\big)\, ds,$$
$$\lim_{n\to\infty} \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t T_{t-s} B_k^2 U(s, Y_n)\, ds = \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t T_{t-s} B_k^2 U(s, Y)\, ds.$$

Therefore, (4.1) will hold provided we show that

$$\lim_{n\to\infty} \int_0^t T_{t-s} B U(s, Y_n) \circ dW(s) = \int_0^t T_{t-s} B U(s, Y) \circ dW(s) \tag{4.5}$$

in probability.

To prove (4.5), we use the following truncation argument. By the local property of the Stratonovich integral [12], we have

$$\int_0^t T_{t-s} B U(s, Y_n) \circ dW(s) = \int_0^t T_{t-s} B U(s, Y_n)\phi_m\big(|Y|_H\big) \circ dW(s)$$

on $\Omega_m := \{\omega\colon\ |Y(\omega)|_H \leqslant m\}$, and

$$\int_0^t T_{t-s} BU(s,Y)\circ dW(s) = \int_0^t T_{t-s} BU(s,Y)\phi_m\big(|Y|_H\big)\circ dW(s)$$

on Ω_m for any fixed $m \geqslant 1$. So, to establish (4.5), it is enough to prove that

$$\lim_{n\to\infty}\int_0^t T_{t-s} BU(s,Y_n)\phi_m\big(|Y|_H\big)\circ dW(s) = \int_0^t T_{t-s} BU(s,Y)\phi_m\big(|Y|_H\big)\circ dW(s) \tag{4.6}$$

in probability for each $m \geqslant 1$. To see this, fix $m \geqslant 1$ and let

$$g_n(s) := T_{t-s} BU(s,Y_n)\phi_m\big(|Y|_H\big), \qquad g(s) := T_{t-s} BU(s,Y)\phi_m\big(|Y|_H\big)$$

for all $s \in [0,t]$. We first show that $g = \lim_{n\to\infty} g_n$ in $\mathbb{L}^{1,2}$. Since both $g_n(s)$ and $g(s)$ are bounded by $C\sup_{|x|_H\leqslant m+1}|U(s,x)|_H$, then by Theorem 2.2 and the Dominated Convergence Theorem, it follows that the sequence $\{g_n\}_{n=1}^{\infty}$ converges to g in $L^2([0,a]\times\Omega, L_2(K,H))$ for each $a \in (0,\infty)$. Notice that

$$\begin{aligned}\mathcal{D}_u g_n(s) = {} & T_{t-s}B\big[\mathcal{D}_u U(s,Y_n) + DU(s,Y_n)\mathcal{D}_u Y_n\big]\phi_m\big(|Y|_H\big)\\ & + T_{t-s} BU(s,Y_n)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H,\end{aligned} \tag{4.7}$$

for all $s \in [0,t]$. Since $|Y_n|_H \leqslant |Y|_H$ and $|\mathcal{D}_u Y_n|_H \leqslant |\mathcal{D}_u Y|_H$, we have

$$\begin{aligned}\big|\mathcal{D}_u g_n(s)\big|_H \leqslant {} & C_m \sup_{|x|_H\leqslant m+1}\big|\mathcal{D}_u U(s,x)\big|_H + C_m \sup_{|x|_H\leqslant m+1}\big\|DU(s,x)\big\|_{L(H)}|\mathcal{D}_u Y|_H\\ & + C_m \sup_{|x|_H\leqslant m+1}\big|U(s,x)\big|_H \mathcal{D}_u|Y|_H.\end{aligned} \tag{4.8}$$

Applying Theorems 2.2(i), (ii), 2.3(ii) and the Dominated Convergence Theorem again, we conclude that

$$\lim_{n\to\infty} E\left[\int_0^T\int_0^T \big|\mathcal{D}_u g_n(s) - \mathcal{D}_u g(s)\big|^2_{L_2(K,H)}\,du\,ds\right] = 0. \tag{4.9}$$

For a given process v, recall the following notations from [15]:

$$\begin{aligned}(\mathcal{D}_+ v)_u &= \lim_{s\to u+} \mathcal{D}_u v(s),\\ (\mathcal{D}_- v)_u &= \lim_{s\to u-} \mathcal{D}_u v(s),\\ (\nabla v)_u &= (\mathcal{D}_+ v)_u + (\mathcal{D}_- v)_u.\end{aligned}$$

We now find the expressions $(\nabla g_n)_u$ and $(\nabla g)_u$. Replacing x by Y_n in (2.24), we obtain

$$\mathcal{D}_u U(s, Y_n) = \mathcal{D}_u V(s, \cdot)(Y_n) + \int_0^s \mathcal{D}_u V\big(s-l, \theta(l, \cdot)\big)\big(F\big(U(l, Y_n, \cdot)\big)\big)\, dl$$
$$+ \int_0^s \big(V\big(s-l, \theta(l, \cdot)\big) + T_{s-l}\big)\big(DF\big(U(l, Y_n, \cdot)\big)\big)\big(\mathcal{D}_u U(l, Y_n, \cdot)\big)\, dl. \quad (4.10)$$

By (2.22), we have

$$(\mathcal{D}_+ V)_u = \lim_{s \to u+} \mathcal{D}_u V(s, \cdot) = BV(u, \cdot) + BT_u \quad \text{a.s.}$$

Similarly, we obtain

$$\big(\mathcal{D}_+ V_{\cdot - l}\big(\theta(l, \omega)\big)\big)_u = \lim_{s \to u+} \mathcal{D}_u V\big(s-l, \theta(l, \omega)\big)$$
$$= BV\big(u-l, \theta(l, \omega)\big) + BT_{u-l}$$

for a.a. $\omega \in \Omega$. Thus, it follows from (4.10) that

$$(\mathcal{D}_+ U)_u(Y_n)$$
$$= [BV_u + BT_u](Y_n) + \int_0^u \big[BV\big(u-l, \theta(l, \cdot)\big) + BT_{u-l}\big]\big(F\big(U(l, Y_n, \cdot)\big)\big)\, dl$$
$$+ \int_0^u \big(V\big(u-l, \theta(l, \cdot)\big) + T_{u-l}\big)\big(DF\big(U(l, Y_n, \cdot)\big)\big)\big(\mathcal{D}_u U(l, Y_n, \cdot)\big)\, dl \quad \text{a.s.} \quad (4.11)$$

Now taking limits as $s \to u+$ in (4.7), we get

$$(\mathcal{D}_+ g_n)_u = T_{t-u} B\big[(\mathcal{D}_+ U)_u(Y_n) + DU(u, Y_n)\mathcal{D}_u Y_n\big]\phi_m\big(|Y|_H\big)$$
$$+ T_{t-u} BU(u, Y_n)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H. \quad (4.12)$$

Note that $\mathcal{D}_u U(s, Y_n) = 0$ when $u > s$. Therefore, letting $s \to u-$ in (4.7) gives

$$(\mathcal{D}_- g_n)_u = T_{t-u} B\big[DU(u, Y_n)\mathcal{D}_u Y_n\big]\phi_m\big(|Y|_H\big)$$
$$+ T_{t-u} BU(u, Y_n)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H. \quad (4.13)$$

Because of the continuity of the functions involved, it is easy to see from (4.12) and (4.13) that

$$\lim_{n \to \infty} (\nabla g_n)_u = \lim_{n \to \infty}\big[(\mathcal{D}_+ g_n)_u + (\mathcal{D}_- g_n)_u\big]$$
$$= (\nabla g)_u = (\mathcal{D}_+ g)_u + (\mathcal{D}_- g)_u, \quad (4.14)$$

where $(\mathcal{D}_+ g)_u$ and $(\mathcal{D}_- g)_u$ are given by

$$(\mathcal{D}_+ g)_u = T_{t-u} B\big[(\mathcal{D}_+ U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) \\ + T_{t-u} BU(u,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H$$

and

$$(\mathcal{D}_- g)_u = T_{t-u} B\big[DU(u,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) + T_{t-u} BU(u,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H.$$

Now, (4.9) implies that

$$\lim_{n\to\infty}\int_0^t g_n(s)\,dW(s) = \int_0^t g(s)\,dW(s)$$

in probability, where the stochastic integral is the Skorohod integral. Therefore, (4.6) (and (4.5)) will hold, and hence the theorem, if we can show that

$$\int_0^t g_n(s)\circ dW(s) = \int_0^t g_n(s)\,dW(s) + \frac{1}{2}\int_0^t (\nabla g_n)_s\,ds, \quad n \geqslant 1, \tag{4.15$'$}$$

and

$$\int_0^t g(s)\circ dW(s) = \int_0^t g(s)\,dW(s) + \frac{1}{2}\int_0^t (\nabla g)_s\,ds \quad \text{a.s.} \tag{4.15}$$

We will prove (4.15). The proof of $(4.15)'$ is very similar. It seems difficult to verify the known sufficient conditions in the literature for proving (4.15) (cf. [12,13]). Instead, we will prove (4.15) from first principles, using approximations by Riemann sums. Following [12], choose any partition $\pi = \{t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ of $[0,t]$, with mesh $|\pi|$, and introduce the following step process:

$$g^\pi(r) := \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\left(\int_{t_i}^{t_{i+1}} g(s)\,ds\right) I_{(t_i,t_{i+1}]}(r), \quad r \in [0,t].$$

Consider the Riemann sums:

$$S^\pi := \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\left(\int_{t_i}^{t_{i+1}} g(s)\,ds\right)\big(W(t_{i+1}) - W(t_i)\big).$$

From the definition of the Stratonovich integral, it follows that

$$\lim_{|\pi|\to 0} S^\pi = \int_0^t g(s)\circ dW(s)$$

whenever the above limit in probability exists. On the other hand, by (3.4) in [12], we have

$$S^\pi = \int_0^t g^\pi(s)\,dW(s) + \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathcal{D}_u g(s)\,du\,ds.$$

Since $\lim_{|\pi|\to 0} g^\pi = g$ in $\mathbb{L}^{1,2}$ (see [12]), then

$$\lim_{|\pi|\to 0} \int_0^t g^\pi(s)\,dW(s) = \int_0^t g(s)\,dW(s)$$

in probability. So, to complete the proof of (4.15), it remains to show that

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathcal{D}_u g(s)\,du\,ds = \int_0^t (\nabla g)_s\,ds. \tag{4.16}$$

To simplify the notation, set

$$I^\pi := \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathcal{D}_u g(s)\,du\,ds.$$

Split I^π into two integrals,

$$I^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_{t_i}^{u} \mathcal{D}_u g(s)\,ds + \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_{u}^{t_{i+1}} \mathcal{D}_u g(s)\,ds.$$

Denote the first and second term on the right-hand side of the above equality by II^π and III^π, respectively. Write,

$$\begin{aligned} II^\pi :=& \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_{t_i}^{u} T_{t-s} B\big[DU(s,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\,ds \\ &+ \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_{t_i}^{u} T_{t-s} BU(s,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H\,ds. \end{aligned}$$

We will prove that

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_{t_i}^{u} T_{t-s} B\big[DU(s,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\,ds$$

$$= \frac{1}{2}\int_0^t T_{t-u}B\big[DU(u,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)\,du \tag{4.17}$$

and

$$\lim_{|\pi|\to 0}\sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}du\int_{t_i}^{u}T_{t-s}BU(s,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H\,ds$$

$$= \frac{1}{2}\int_0^t T_{t-u}BU(u,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H\,du. \tag{4.18}$$

We will prove (4.17). The proof of (4.18) is very similar. Rewrite the left-hand side of (4.17) in the form

$$\sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}du\int_{t_i}^{u}T_{t-s}B\big[DU(s,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)\,ds$$

$$= \sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}du\int_{t_i}^{u}\big\{T_{t-s}B\big[DU(s,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)$$

$$- T_{t-u}B\big[DU(u,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)\big\}\,ds$$

$$+ \sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}(u-t_i)T_{t-u}B\big[DU(u,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)\,du. \tag{4.19}$$

Since the sequence of functions

$$[0,t]\ni u\mapsto\sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}(u-t_i)I_{(t_i,t_{i+1}]}(u)\in\mathbf{R},\quad n\geqslant 1,$$

converges weakly to the constant function $\frac{1}{2}$ in $L^2([0,t],\mathbf{R})$, then

$$\lim_{|\pi|\to 0}\sum_{i=0}^{n-1}\frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}(u-t_i)T_{t-u}B\big[DU(u,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)\,du$$

$$= \frac{1}{2}\int_0^t T_{t-u}B\big[DU(u,Y)\mathcal{D}_uY\big]\phi_m\big(|Y|_H\big)\,du. \tag{4.20}$$

For a given partition $\pi := \{t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = t\}$ of $[0, t]$, and any $u \in (t_i, t_{i+1}]$, denote

$$u^{\pi-} := t_i, \qquad u^{\pi+} := t_{i+1}.$$

We now estimate the first term of (4.19) as follows

$$\begin{aligned}
&\sum_{i=0}^{n-1} \frac{1}{t_{i+1} - t_i} \left| \int_{t_i}^{t_{i+1}} du \int_{t_i}^{u} \big\{ T_{t-s} B\big[DU(s, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) \right. \\
&\qquad \left. - T_{t-u} B\big[DU(u, Y) D_u Y\big]\phi_m\big(|Y|_H\big)\big\}\, ds \right| \\
&\leqslant \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sup_{u^{\pi-} \leqslant s \leqslant u} \big\{ \big| T_{t-s} B\big[DU(s, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) \\
&\qquad - T_{t-u} B\big[DU(u, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\big| \big\}\, du \\
&= \int_0^t \sup_{u^{\pi-} \leqslant s \leqslant u} \big\{ \big| T_{t-s} B\big[DU(s, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) \\
&\qquad - T_{t-u} B\big[DU(u, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\big| \big\}\, du. \qquad (4.21)
\end{aligned}$$

By the continuity of $T_{t-s} B DU(s, Y)$ in $s \in [0, t]$, we see that

$$\lim_{|\pi| \to 0} \sup_{u^{\pi-} \leqslant s \leqslant u} \big\{ \big| T_{t-s} B\big[DU(s, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) - T_{t-u} B\big[DU(u, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\big| \big\} = 0$$

for any fixed $u \in (t_i, t_{i+1}]$, $0 \leqslant i \leqslant n - 1$. On the other hand,

$$\begin{aligned}
&\sup_{u^{\pi-} \leqslant s \leqslant u} \big\{ \big| T_{t-s} B\big[DU(s, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big) - T_{t-u} B\big[DU(u, Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\big| \big\} \\
&\quad \leqslant 2 \sup_{0 \leqslant s \leqslant t} \big\| T_{t-s} B\big[DU(s, Y)\big]\big\| |\mathcal{D}_u Y| \phi_m\big(|Y|_H\big)
\end{aligned}$$

a.s. for all $u \in [0, t]$. Applying the Dominated Convergence Theorem, we see that the right-hand side of (4.21) tends to zero as $|\pi|$ tends to 0. Thus, (4.17) follows from (4.19) and (4.20). This gives the a.s. limit

$$\begin{aligned}
\lim_{|\pi| \to 0} II^{\pi} &= \frac{1}{2} \int_0^t T_{t-u} B DU(u, Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u |Y|_H\, du \\
&\quad + \frac{1}{2} \int_0^t T_{t-u} B U(u, Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u |Y|_H\, du. \qquad (4.22)
\end{aligned}$$

To treat III^π, write it in the form

$$III^\pi = \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_u^{t_{i+1}} T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\, ds$$
$$+ \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_u^{t_{i+1}} T_{t-s} BU(s,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H\, ds. \tag{4.23}$$

We will prove that

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_u^{t_{i+1}} T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\, ds$$
$$= \frac{1}{2}\int_0^t T_{t-u} B\big[(\mathcal{D}_+U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\, du \tag{4.24}$$

and

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_u^{t_{i+1}} T_{t-s} BU(s,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H\, ds$$
$$= \frac{1}{2}\int_0^t T_{t-u} BU(u,Y)\phi_m'\big(|Y|_H\big)\mathcal{D}_u|Y|_H\, du. \tag{4.25}$$

The proof of (4.25) is similar to that of (4.24). We will complete the proof of the theorem by proving (4.24). To do this, consider

$$\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \int_u^{t_{i+1}} T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\, du\, ds$$
$$= J_1^\pi + J_2^\pi,$$

where

$$J_1^\pi := \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} du \Bigg[\int_u^{t_{i+1}} \Big\{T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)$$
$$- T_{t-u} B\big[(\mathcal{D}_+U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big]\phi_m\big(|Y|_H\big)\Big\}\, ds\Bigg]$$

and

$$J_2^\pi := \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} (t_{i+1}-u) T_{t-u} B\big[(\mathcal{D}_+ U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big)\, du.$$

As noted before, the sequence of functions

$$[0,t] \ni u \mapsto \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}(t_{i+1}-u) I_{(t_i,t_{i+1}]}(u) \in \mathbf{R}$$

converges weakly to the constant function $\frac{1}{2}$ in $L^2([0,t],\mathbf{R})$. Therefore,

$$\lim_{|\pi|\to 0} J_2^\pi = \frac{1}{2}\int_0^t T_{t-u} B\big[(\mathcal{D}_+ U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big)\, du. \tag{4.26}$$

Now we show that J_1^π tends to zero as $|\pi| \to 0$. First note that

$$\begin{aligned} \big|J_1^\pi\big| \leqslant \int_0^t \Big(\sup_{u \leqslant s \leqslant u^{\pi+}} \big\{ \big| T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big) \\ - T_{t-u} B\big[(\mathcal{D}_+ U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big) \big| \big\} \Big)\, du. \end{aligned} \tag{4.27}$$

Furthermore, there is a positive random constant C such that

$$\begin{aligned} &\sup_{u \leqslant s \leqslant u^{\pi+}} \big\{ \big| T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big) \\ &\quad - T_{t-u} B\big[(\mathcal{D}_+ U)_u(Y) + DU(u,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big) \big| \big\} \\ &\leqslant 2 \sup_{u \leqslant s \leqslant t} \big\{ \big| T_{t-s} B\big[\mathcal{D}_u U(s,Y) + DU(s,Y)\mathcal{D}_u Y\big] \phi_m\big(|Y|_H\big) \big| \big\} \\ &\leqslant C\Big[\sup_{u \leqslant s \leqslant t} \big|\mathcal{D}_u U(s,Y)\big| + |\mathcal{D}_u Y| \Big] \quad \text{a.s.} \end{aligned} \tag{4.28}$$

Let $\hat{h}(u,a) := \sup_{u \leqslant s \leqslant a} |\mathcal{D}_u U(s,Y)|$. Replacing x by $Y(\omega)$ in (2.24), there is a positive random constant c such that

$$\begin{aligned} \hat{h}(u,a) \leqslant c|Y| \Bigg(\sup_{u \leqslant s \leqslant t} \big\|\mathcal{D}_u V(s,\cdot)\big\| + \int_0^t \sup_{u \leqslant s \leqslant t} \big\|\mathcal{D}_u V\big(s-l,\theta(l,\cdot)\big)\big\|\, dl \Bigg) \\ + c\int_0^a \sup_{l \leqslant s \leqslant a} \big\{ \big\|V\big(s-l,\theta(l,\cdot)\big)\big\| + \|T_{s-l}\| \big\} \hat{h}(u,l)\, dl \end{aligned}$$

a.s. for all $a \in [0,t]$. By Gronwall's inequality, the above inequality implies

$$\hat{h}(u,t) \leqslant c|Y| \left(\sup_{u \leqslant s \leqslant t} \left\| \mathcal{D}_u V(s,\cdot) \right\| + \int_0^t \sup_{u \leqslant s \leqslant t} \left\| \mathcal{D}_u V\big(s-l, \theta(l,\cdot)\big) \right\| dl \right)$$
$$\times \exp\left(\int_0^t \sup_{l \leqslant s \leqslant t} \left\{ \left\| V\big(s-l, \theta(l,\cdot)\big) \right\| + \|T_{s-l}\| \right\} dl \right).$$

Using the above estimate and Theorem 2.3(i), it is easy to see that

$$\int_0^t \big(\hat{h}(u,t)\big)^2 \, du < \infty. \tag{4.29}$$

By the definition of $(\mathcal{D}_+ U)$, the integrand in (4.27) approaches zero as $|\pi| \to 0$ for any fixed u. Applying the Dominated Convergence Theorem, it follows from (4.28) and (4.29) that

$$\lim_{|\pi| \to 0} J_1^\pi = 0. \tag{4.30}$$

This together with (4.26) implies (4.24). The proof of equality (4.15) is now complete. □

5. Alternative proof of Theorem 2.3(ii)

In this section we give an alternative proof of the estimate in Theorem 2.3(ii). This proof is based on a chaos-type expansion in the Hilbert space $L_2(H)$. The argument we present is of independent interest.

Proof of Theorem 2.3(ii). In the see (1.1), assume hypotheses (B) and (A_1) or (A_2). Suppose F is C_b^1. In this proof, C will denote a generic positive constant which may change from line to line.

Recall Eq. (2.24):

$$\mathcal{D}_u U(t,x,\cdot) = \mathcal{D}_u V(t,\cdot)(x) + \int_0^t \mathcal{D}_u V\big(t-s, \theta(s,\cdot)\big)\Big(F\big(U(s,x,\cdot)\big)\Big)\, ds$$
$$+ \int_0^t \left[V\big(t-s, \theta(s,\cdot)\big) + T_{t-s} \right] \Big(DF\big(U(s,x,\cdot)\big)\Big)\big(\mathcal{D}_u U(s,x,\cdot)\big)\, ds,$$

for $u \in [0,a]$, $a \in [0,t]$, $t \geqslant 0$. Fix any $p \geqslant 1$. Set

$$h(u,t) := \sup_{x \in H} \frac{|\mathcal{D}_u U(t,x)|_H}{(1 + |x|_H)}, \quad t \geqslant 0.$$

From (2.24) it follows that there are positive constants M, C such that

$$h(u,t) \leqslant M\|\mathcal{D}_u V(t,\cdot)\|_{L_2(H)} + C\int_0^t \|\mathcal{D}_u V(t-s,\theta(s,\cdot))\|_{L_2(H)}\,ds + C\int_0^t \big(\|V(t-s,\theta(s,\cdot))\|_{L_2(H)} + 1\big)h(u,s)\,ds, \quad t \geqslant 0. \tag{5.1}$$

Define

$$g(u,t) := M\|\mathcal{D}_u V(t,\cdot)\|_{L_2(H)} + C\int_0^t \|\mathcal{D}_u V(t-s,\theta(s,\cdot))\|_{L_2(H)}\,ds$$

and

$$L(s,t) := \|V(t-s,\theta(s,\cdot))\|_{L_2(H)} + 1, \quad 0 \leqslant s \leqslant t.$$

Iterating the inequality (5.1) n times, we obtain

$$h(u,t) \leqslant g(u,t) + \sum_{k=1}^{n} C^k \int_0^t L(s_1,t)\,ds_1 \int_0^{s_1} L(s_2,s_1)\,ds_2 \ldots \int_0^{s_{k-1}} L(s_k,s_{k-1})g(u,s_k)\,ds_k + R_{n+1},$$

where

$$R_{n+1} := C^{n+1}\int_0^t L(s_1,t)\,ds_1 \int_0^{s_1} L(s_2,s_1)\,ds_2 \ldots \int_0^{s_n} L(s_{n+1},s_n)h(u,s_{n+1})\,ds_{n+1}$$
$$\leqslant \frac{1}{(n+1)!}C^{n+1}\Big(\sup_{0\leqslant u_2\leqslant u_1\leqslant t} L(u_2,u_1)\Big)^{n+1} \sup_{0\leqslant u\leqslant s\leqslant t} h(u,s) \to 0$$

almost surely as $n \to \infty$. This implies that

$$h(u,t) \leqslant g(u,t) + \sum_{k=1}^{\infty} C^k \int_0^t L(s_1,t)\,ds_1 \int_0^{s_1} L(s_2,s_1)\,ds_2 \ldots \int_0^{s_{k-1}} L(s_k,s_{k-1})g(u,s_k)\,ds_k. \tag{5.2}$$

Next we estimate $E[(h(u,t))^{2p}]$. First observe that

$$E\left[\left(\int_0^t L(s_1,t)\,ds_1\int_0^{s_1} L(s_2,s_1)\,ds_2\ldots\int_0^{s_{k-1}} L(s_k,s_{k-1})g(u,s_k)\,ds_k\right)^{2p}\right]$$
$$\leqslant\left(\int_{0<s_k<\cdots<s_1<t} ds_k\ldots ds_1\right)^{2p-1}$$
$$\times\int_{0<s_k<\cdots<s_1<t} E\big[L(s_1,t)^{2p}L(s_2,s_1)^{2p}\cdots L(s_k,s_{k-1})^{2p}g(u,s_k)^{2p}\big]\,ds_k\ldots ds_1. \quad (5.3)$$

Since $L(s_i,s_{i-1})$ is measurable with respect to the σ-algebra $\mathcal{F}_{s_i,s_{i-1}} := \sigma\{W(l)-W(s_i)\colon l\in[s_i,s_{i-1}]\}$ and W has independent increments, it follows that the random variables

$$L(s_1,t)^{2p},\quad L(s_2,s_1)^{2p},\quad \ldots,\quad L(s_k,s_{k-1})^{2p},\quad g(u,s_k)^{2p}$$

are independent for $0<s_k<\cdots<s_1<t$. Hence,

$$E\big[L(s_1,t)^{2p}L(s_2,s_1)^{2p}\cdots L(s_k,s_{k-1})^{2p}g(u,s_k)^{2p}\big]$$
$$=E\big[L(s_1,t)^{2p}\big]E\big[L(s_2,s_1)^{2p}\big]\cdots E\big[L(s_k,s_{k-1})^{2p}\big]E\big[g(u,s_k)^{2p}\big].$$

Therefore, (5.3) gives

$$E\left(\int_0^t L(s_1,t)\,ds_1\int_0^{s_1} L(s_2,s_1)\,ds_2\ldots\int_0^{s_{k-1}} L(s_k,s_{k-1})g(u,s_k)\,ds_k\right)^{2p}$$
$$\leqslant\frac{t^k}{k!}\frac{t^{k-1}M_t^k}{(k-1)!}\int_0^t E\big[g(u,s)^{2p}\big]\,ds, \quad (5.4)$$

where

$$M_t := \sup_{0\leqslant u_2\leqslant u_1\leqslant t} E\big[L(u_2,u_1)^{2p}\big].$$

Combining (5.2) with (5.4), we arrive at

$$\big(E\big[h(u,t)^{2p}\big]\big)^{\frac{1}{2p}}$$
$$\leqslant\big(E\big[g(u,t)^{2p}\big]\big)^{\frac{1}{2p}}+\sum_{k=1}^{\infty}C^k\left(\frac{t^k}{k!}\frac{t^{k-1}M_t^k}{(k-1)!}\right)^{\frac{1}{2p}}\left(\int_0^t E\big[g(u,s)^{2p}\big]\,ds\right)^{\frac{1}{2p}},\quad t\geqslant 0.$$

Hence,

$$E\left[\sup_{x\in H}\frac{|\mathcal{D}U(t,x)|_H^{2p}}{(1+|x|_H^{2p})}\right]\leqslant C\int_0^t E\big[h(u,t)^{2p}\big]\,du$$
$$\leqslant C\left\{\int_0^t E\big[g(u,t)^{2p}\big]\,du+\int_0^t\int_0^t E\big[g(u,s)^{2p}\big]\,du\,ds\right\},\quad t\geqslant 0. \tag{5.5}$$

We show now that the right-hand side of (5.5) is finite. It is easy to see that

$$V\big(t-s,\theta(s,\cdot)\big)=\int_s^t T_{t-l}BV\big(l-s,\theta(s,\cdot)\big)\,dW(l)+\int_s^t T_{t-l}BT_{l-s}\,dW(l).$$

Thus $D_uV(t-s,\theta(s,\cdot))=0$ for $u\notin[s,t]$; and for $u\in[s,t]$,

$$\mathcal{D}_uV\big(t-s,\theta(s,\cdot)\big)=\int_s^t T_{t-l}B\mathcal{D}_uV\big(l-s,\theta(s,\cdot)\big)\,dW(l)$$
$$+T_{t-u}BV\big(u-s,\theta(s,\cdot)\big)+T_{t-u}BT_{u-s}.$$

By the Itô isometry we get

$$\int_s^t E\big[\big\|\mathcal{D}_uV_{t-s}\big(\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\,du\leqslant C\int_s^t\int_s^l E\big[\big\|T_{t-l}B\mathcal{D}_uV\big(l-s,\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\,du\,dl$$
$$+C\int_s^t E\big[\big\|T_{t-u}BV\big(u-s,\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\,du$$
$$+C\int_s^t E\big[\|T_{t-u}BT_{u-s}\|_{L_2(H)}^{2p}\big]\,du.$$

This implies that, for any $T>0$,

$$\int_s^t E\big[\big\|\mathcal{D}_uV\big(t-s,\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\,du\leqslant C+C\int_s^t\int_s^l E\big[\big\|\mathcal{D}_uV\big(l-s,\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\,du\,dl$$

and for all $t\in[0,T]$. By Gronwall's inequality, we have

$$\int_s^t E\big[\big\|\mathcal{D}_uV\big(t-s,\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\leqslant C \tag{5.6}$$

for all $t \in [0, T]$. Now,

$$\int_0^t E\big[g(u,t)^{2p}\big]\,du \leqslant C\int_0^t E\big[\big\|\mathcal{D}_u V(t,\cdot)\big\|_{L_2(H)}^{2p}\big]\,du$$
$$+C\int_0^t\int_0^t E\big[\big\|\mathcal{D}_u V\big(t-s,\theta(s,\cdot)\big)\big\|_{L_2(H)}^{2p}\big]\,ds\,du.$$

So it follows from Theorem 2.3(i) and (5.6) that the right-hand side of (5.5) is finite, which completes the proof of (2.21). □

6. Anticipating semilinear spde's

As a corollary of Theorem 1.1, we show existence and regularity of solutions to a semilinear Stratonovich see with anticipating initial conditions. The proof is essentially a reformulation of the corresponding argument for Theorem 1.1. It is not clear whether the solution of (6.1) is unique.

Corollary 6.1. *Assume that Condition* (B) *together with either* (A_1) *or* (A_2). *Suppose F is C_b^1 and let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Consider the following anticipating semilinear see*:

$$\begin{cases} dv(t) = -Av(t)\,dt + F\big(v(t)\big)\,dt + Bv(t)\circ dW(t), & t>0,\\ v(0) = Y. \end{cases} \tag{6.1}$$

Then the anticipating semilinear see (6.1) *has a pathwise continuous* $(\mathcal{B}(\mathbf{R}^+)\otimes\mathcal{F},\mathcal{B}(H))$-*measurable mild solution $v:\mathbf{R}^+\times\Omega\to H$ with the following properties*:

(i) $v(t) \in \mathcal{D}^{1,2}(\Omega, H)$ *for all* $t \geqslant 0$.
(ii) $\sup_{t\in[0,a]} E|\mathcal{D}v(t)|_H^2 < \infty$ *for all* $a \in (0,\infty)$.
(iii) $\sup_{t\in[0,a]} |v(t,\omega)|_H \leqslant K(\omega)[1+|Y(\omega)|_H]$ *for a.a.* $\omega\in\Omega$,
where K is a random positive constant such that $K \in L^{2p}(\Omega,\mathbf{R}^+)$ for all integers $p \geqslant 1$.

Proof. Assume all the conditions of the corollary. In (6.1), replace the initial condition Y by a deterministic vector $x \in H$. Then with this replacement, (6.1) is equivalent to the semilinear Itô see

$$\begin{cases} du(t,x) = -Au(t,x)\,dt + F\big(u(t,x)\big)\,dt \\ \qquad\qquad + \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty} B_k^2 U(t,x)\,dt + Bu(t,x)\,dW(t), & t>0,\\ u(0,x) = x. \end{cases} \tag{6.2}$$

Set

$$F_0(u) := F(u) + \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u$$

for all $u \in H$. By Condition (B), it is easy to see that $F_0 : H \to H$ is C_{b}^1. Therefore the (adapted) see (6.2) satisfies all the requirements of Theorem 1.1. In particular, its mild solutions generate a C^1 cocycle $U_0 : \mathbf{R}^+ \times H \times \Omega \to H$. Moreover, the cocycle U_0 satisfies all the estimates in Section 2 (Theorems 2.2(i), (ii), 2.3(ii)). Now set $v(t,\omega) := U_0(t, Y(\omega), \omega)$ for all $t \geqslant 0$, $\omega \in \Omega$. Using the substitution theorem it is not hard to check that v is a mild solution of (6.1) which satisfies all the estimates in Corollary 6.1. □

Remark. A similar result for anticipating stochastic ordinary differential equations in a Hilbert space H is given in [5, Theorem 4.4], under the restriction that the initial random variable takes values in some relatively compact set in H.

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