Retarded Functional Differential Equations

A global point of view

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Technical Summary.

This book lays the foundations of a geometric theory of retarded functional differential equations on manifolds. It was the first book to be published on differential-geometric aspects of deterministic hereditary systems on Riemannian manifolds. In this monograph I deal with the following general framework:

Let $X$ be a manifold. Typically we take $X$ to be a Riemannian manifold (finite or infinite-dimensional) or a Banach manifold with a sufficiently smooth linear connection. Consider a manifold of paths $P([-r,0], X)$ which inherits its differentiable structure from the ambient manifold $X$. A retarded functional differential equation (RFDE) on $X$ is a continuous map $F : [0,a) \times P([-r,0], X) \to TX$ such that for each $(t, \theta) \in [0,a) \times P([-r,0], X)$ the vector $F(t, \theta) \in T_{\theta(0)}X$, the tangent space to $X$ at $\theta(0)$. A trajectory of $F$ is a $C^1$ path $x : [-r,a) \to X$ such that

$$x'(t) = F(t, x_t), \quad t \in [0,a)$$
$$x_0 = \theta \in P([-r,0], X).$$

In the above equation $x_t$ stands for the segment $x|[t-r,t]$ of the solution $x$. In Chapter 1, I develop a localization technique (Lemma 1.1) in order to obtain a unique local trajectory for the above initial value problem (Theorem 1.2, Chapter 1, p.22). This is done under mild regularity conditions on $F$, assuming that $X$ is a Banach manifold which admits a linear connection and $P = L^2_1$, the space of all Sobolev paths $\theta$ on $X$ with square integrable derivatives. If $X$ carries a Finsler and $F$ satisfies suitable growth conditions, I prove that global trajectories of the hereditary equation exist for all time. Hence one gets a semiflow $R^+ \times L^2_1 \to L^2_1$ on the space of initial paths $L^2_1([-r,0], X)$ (Theorems (1.3)-(1.5)).
The main objective of Chapter 2 is to characterize the topological structure of the critical set \( \{ \theta : F(\theta) = 0 \} \) when \( F \) is autonomous and \( X \) is \( n \)-dimensional, smooth Riemannian. It is first proved that solutions of the hereditary equation may reach equilibrium by converging asymptotically to a constant critical path, just as for vector fields (Theorem 2.1). The key idea here is to use parallel transport to show that a smooth hereditary coefficient \( F \) pulls back into a a smooth vector field \( \xi^F \) on \( L_1^2 \) (Theorem 2.2 and corollary). In spite of the infinite degeneracy of the critical set \( C(F) \), we are able to isolate a class of gradient-like hereditary equations for which the critical set is a closed smooth submanifold of \( L_1^2 \) with codimension \( n \) (Proposition 2.4, p. 59). A Morse index exists for this class of hereditary equations (Proposition 2.5, p.60). The index is constant on each connected component of \( C(F) \). When \( X \) is compact, one can count the number of critical components in \( C(F) \). This way I prove Morse inequalities for such hereditary equations (Theorem (2.4) and corollaries). In particular it follows from these inequalities that \( F \) has only a finite number of critical components. The number of critical components with index \( m \) is always greater than or equal to the \( m \)-th Betti number of \( X \), the rank of its \( m \)-th singular homology group (Corollary 2.4.2).

In Chapter 3, I linearize the semiflow of \( F \) by differentiating the canonical vector field \( \xi^F \) covariantly in \( L_1^2([-r, 0], X) \). This linearization defines a compacting semiflow on the tangent bundle \( TL_1^2([-r, 0], X) \) (Theorem 3.3, p.79). Using semigroup techniques along the fibers of \( TL_1^2([-r, 0], X) \) we construct a Whitney direct sum splitting of the tangent bundle into two subbundles: the unstable and the stable one. Cf. classical results of Hale in the flat case \( X = \mathbb{R}^n \). This splitting is invariant under the linearized semiflow. The unstable subbundle is finite-dimensional, and on it the linearized semiflow can be continued backwards to give a genuine flow which is defined for all time. Within the stable subbundle the linearized semiflow decays exponentially fast in the Sobolev Riemannian metric along each fiber in \( TL_1^2([-r, 0], X) \). This is the Stable Bundle Theorem (Theorem 3.6, p.100).

Vector fields on the ambient manifold \( X \) are used in Chapter 4 to generate examples of FDE's on the manifold. These include classical vector fields, differential-delay equations (DDE's), the delayed development and the Levin-Nohel equation. It is shown in Theorem 4.1 (p.109) that a gradient Levin-Nohel equation on a Riemannian manifold may not admit non-trivial periodic solutions. I also give a detailed study of the Functional Heat equation (FHE) in this chapter of the monograph. The FHE is shown to correspond to a discontinuous but closed FDE on the Fréchet space of smooth functions on a compact Riemannian manifold. It is interesting to note here that despite the discontinuity of the equation and the infinite-dimensionality of the function space, the FHE still displays dynamical properties very similar to those of continuous finite-dimensional FDE's. In general, however, the
FHE can be solved forward in time only along a closed Fréchet subspace of the state space. *Backward* solutions of the FHE do exist on the complementary subspace in the hyperbolic case. See Chapter 4§5, pp. 113-133.